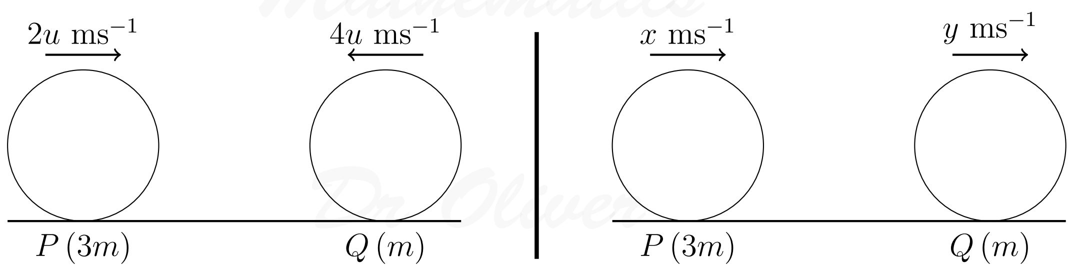
Dr Oliver Mathematics
Advance Level Mathematics
Mechanics 1: Calculator
1 hour 30 minutes

The total number of marks available is 75.
You must write down all the stages in your working.

1. Two particles, P and Q, have masses $3m$ and m respectively. They are moving in opposite directions towards each other along the same straight line on a smooth horizontal plane and collide directly. The speeds of P and Q immediately before the collision are $2u$ and $4u$ respectively. The magnitude of the impulse received by each particle in the collision is $\dfrac{21mu}{4}$.

   (a) Find the speed of P after the collision. (3)

   Solution

   $$(3m \times x) - (3m \times 2u) = -\frac{21mu}{4} \Rightarrow 3m(x - 2u) = -\frac{21mu}{4}$$
   $$\Rightarrow x - 2u = -\frac{7u}{4}$$
   $$\Rightarrow x = \underline{\underline{\frac{u}{4}}}$$

   (b) Find the speed of Q after the collision. (3)

   Solution

$$(m \times y) - [m \times (-4u)] = \frac{21mu}{4} \Rightarrow m(y + 4u) = \frac{21mu}{4}$$
$$\Rightarrow y + 4u = \frac{21u}{4}$$
$$\Rightarrow y = \frac{5u}{4}$$

2. A particle of mass 2 kg lies on a rough plane. The plane is inclined to the horizontal at 30°. The coefficient of friction between the particle and the plane is $\frac{1}{4}$. The particle is held in equilibrium by a force of magnitude P newtons. The force makes an angle of 20° with the horizontal and acts in a vertical plane containing a line of greatest slope of the plane, as shown in Figure 1. (10)

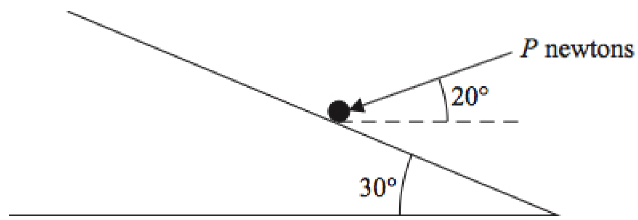


Figure 1: a particle of mass 2 kg lies on a rough plane

Find the least possible value of P.

Solution

The particle is on an inclination of

$$30 + 20 = 50^\circ$$

with the plane. Now,

$$\begin{aligned} \text{Parallel:} \quad & P\cos 50^\circ + F = 2g\sin 30^\circ \\ \text{Perpendicular:} \quad & R = 2g\cos 30^\circ + P\sin 50^\circ \\ F = \mu R: \quad & F = \tfrac{1}{4}R. \end{aligned}$$

$$\begin{aligned}
F = \tfrac{1}{4}R \Rightarrow 2g\sin 30^\circ - P\cos 50^\circ &= \tfrac{1}{4}(2g\cos 30^\circ + P\sin 50^\circ) \\
\Rightarrow 2g\sin 30^\circ - P\cos 50^\circ &= \tfrac{1}{2}g\cos 30^\circ + \tfrac{1}{4}P\sin 50^\circ \\
\Rightarrow 2g\sin 30^\circ - \tfrac{1}{2}g\cos 30^\circ &= \tfrac{1}{4}P\sin 50^\circ + P\cos 50^\circ \\
\Rightarrow g(2\sin 30^\circ - \tfrac{1}{2}\cos 30^\circ) &= P(\tfrac{1}{4}\sin 50^\circ + \cos 50^\circ) \\
\Rightarrow P &= \frac{g(2\sin 30^\circ - \frac{1}{2}\cos 30^\circ)}{\frac{1}{4}\sin 50^\circ + \cos 50^\circ} \\
\Rightarrow P &= 6.660\,055\,188 \text{ (FCD)} \\
\Rightarrow \underline{\underline{P = 6.7 \text{ (2 sf)}}}.
\end{aligned}$$

3. A wooden beam AB, of mass 150 kg and length 9 m, rests in a horizontal position supported by two vertical ropes. The ropes are attached to the beam at C and D, where $AC = 1.5$ m and $BD = 3.5$ m. A gymnast of mass 60 kg stands on the beam at the point P, where $AP = 3$ m, as shown in Figure 2. The beam remains horizontal and in equilibrium.

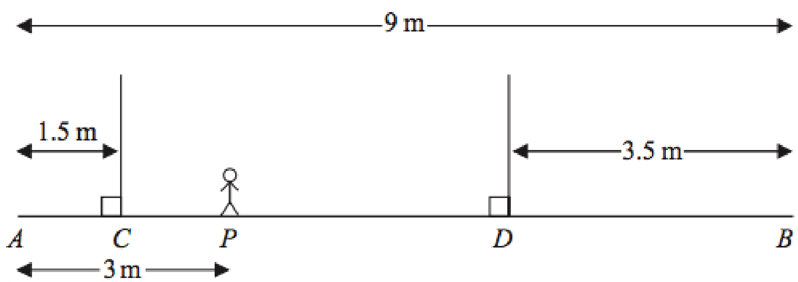


Figure 2: a gymnast of mass 60 kg stands on the beam

By modelling the gymnast as a particle, the beam as a uniform rod and the ropes as light inextensible strings,

(a) find the tension in the rope attached to the beam at C. (3)

Solution

Let the tension in the rope at C be R N. Now, the distance

$$CD = (9 - 3.5) - 1.5 = 4 \text{ m}$$

and

$$DP = 4 - 1.5 = 2.5 \text{ m}.$$

Moments about D:

$$4 \times R = 1 \times 150g + 2.5 \times 60g \Rightarrow 4R = 300g$$
$$\Rightarrow \underline{\underline{R = 75g}}.$$

The gymnast at P remains on the beam at P and another gymnast, who is also modelled as a particle, stands on the beam at B. The beam remains horizontal and in equilibrium. The mass of the gymnast at B is the largest possible for which the beam remains horizontal and in equilibrium.

(b) Find the tension in the rope attached to the beam at D. (4)

Solution

Let the tension in the rope at C be 0 N (why?) and the tension in the rope at D be T N.

Moments about B:

$$3.5 \times T = (4.5 \times 150g) + (6 \times 60g) \Rightarrow 3.5T = 1\,035g$$
$$\Rightarrow \underline{\underline{T = \tfrac{2070}{7}g}}.$$

4. A ball of mass 0.2 kg is projected vertically downwards with speed U ms^{-1} from a point A which is 2.5 m above horizontal ground. The ball hits the ground. Immediately after hitting the ground, the ball rebounds vertically with a speed of 10 ms^{-1}. The ball receives an impulse of magnitude 7 Ns in its impact with the ground. By modelling the ball as a particle and ignoring air resistance, find

(a) the value of U. (6)

Solution

$s = 2.5$, $u = U$, $v =?$, $t =?$, and $a = g$:

$$v^2 = u^2 + 2as \Rightarrow v^2 = U^2 + 2 \times g \times 2.5$$
$$\Rightarrow v^2 = U^2 + 5g.$$

Now,

$$0.2[10 - (-v)] = 7 \Rightarrow 10 + v = 35$$
$$\Rightarrow v = 25.$$

Finally,

$$25^2 = U^2 + 5g \Rightarrow U^2 = 576$$
$$\Rightarrow \underline{\underline{U = 24}}.$$

After hitting the ground, the ball moves vertically upwards and passes through a point B which is 1 m above the ground.

(b) Find the time between the instant when the ball hits the ground and the instant when the ball first passes through B. (4)

Solution

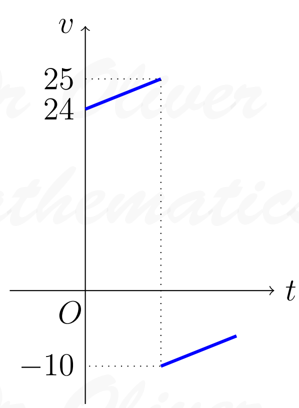
$s = 1$, $u = 10$, $v = ?$, $t = ?$, and $a = -g$:

$$s = ut + \tfrac{1}{2}at^2 \Rightarrow 1 = 10t - 4.9t^2$$
$$\Rightarrow 4.9t^2 - 10t + 1 = 0$$
$$\Rightarrow t = \frac{10 \pm \sqrt{80.4}}{9.8}$$
$$\Rightarrow t = 0.105\,448\,499\,1 \text{ or } t = 1.935\,367\,827 \text{ (FCD)};$$

hence, the instant when the ball first passes through B is $\underline{\underline{t = 0.11 \text{ s}}}$.

(c) Sketch a velocity-time graph for the motion of the ball from when it was projected from A to when it first passes through B. (You need not make any further calculations to draw this sketch.) (3)

Solution



5. A lift of mass 250 kg is being raised by a vertical cable attached to the top of the lift. A

woman of mass 60 kg stands on the horizontal floor inside the lift, as shown in Figure 3.

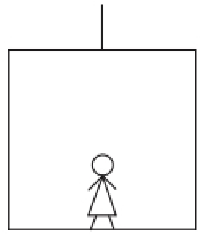


Figure 3: a lift of mass 250 kg is being raised

The lift ascends vertically with constant acceleration 2 ms^{-2}. There is a constant downwards resistance of magnitude 100 N on the lift. By modelling the woman as a particle,

(a) find the magnitude of the normal reaction exerted by the floor of the lift on the woman. (3)

Solution
Let M N be the magnitude of the normal reaction exerted by the floor of the lift on the woman. Now,

$$\begin{aligned} M - 60g = 60 \times 2 &\Rightarrow M = 60g + 120 \\ &\Rightarrow M = 708 \\ &\Rightarrow \underline{\underline{M = 710 \text{ (2 sf)}}}. \end{aligned}$$

The tension in the cable must not exceed 10 000 N for safety reasons, and the maximum upward acceleration of the lift is 3 ms^{-2}. A typical occupant of the lift is modelled as a particle of mass 75 kg and the cable is modelled as a light inextensible string. There is still a constant downwards resistance of magnitude 100 N on the lift.

(b) Find the maximum number of typical occupants that can be safely carried in the lift when it is ascending with an acceleration of 3 ms^{-2}. (7)

Solution

Let the number of persons be n. Now, the mass is $(75n + 250)$ and

$$\begin{aligned} & 10\,000 - (75n + 250) \times g - 100 = 3 \times (75n + 250) \\ \Rightarrow\ & 9\,900 = (75n + 250)(g + 3) \\ \Rightarrow\ & 75n + 250 = \frac{9\,900}{g + 3} \\ \Rightarrow\ & 75n = \frac{9\,900}{g + 3} - 250 \\ \Rightarrow\ & n = \frac{1}{75}\left(\frac{9\,900}{g + 3} - 250\right) \\ \Rightarrow\ & n = 6\tfrac{47}{48}; \end{aligned}$$

so, 6 persons.

6. (*In this question* **i** *and* **j** *are horizontal unit vectors due east and due north respectively.*) Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act on a particle P of mass 0.5 kg.

$\mathbf{F}_1 = (4\mathbf{i} - 6\mathbf{j})$ N and $\mathbf{F}_2 = (p\mathbf{i} + q\mathbf{j})$ N.

Given that the resultant force of $\mathbf{F}_1$ and $\mathbf{F}_2$ is in the same direction as $(-2\mathbf{i} - \mathbf{j})$,

(a) show that $p - 2q = -16$. (5)

Solution

$$(4\mathbf{i} - 6\mathbf{j}) + (p\mathbf{i} + q\mathbf{j}) = (4 + p)\mathbf{i} + (-6 + q)\mathbf{j}$$

and so

$$\begin{aligned} \frac{4 + p}{-6 + q} = \frac{-2}{-1} & \Rightarrow -(4 + p) = -2(-6 + q) \\ & \Rightarrow 4 + p = -12 + 2q \\ & \Rightarrow p - 2q = -16, \end{aligned}$$

as required.

Given that $q = 3$,

(b) find the magnitude of the acceleration of P, (5)

Solution

$$q = 3 \Rightarrow p - 6 = -16 \Rightarrow p = -10$$

and

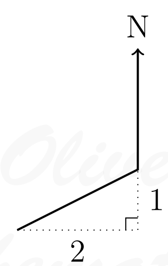
$$(4\mathbf{i} - 6\mathbf{j}) + (-10\mathbf{i} + 3\mathbf{j}) = -6\mathbf{i} - 3\mathbf{j}.$$

Finally,

$$\begin{aligned} F = ma \Rightarrow a &= \frac{F}{m} \\ \Rightarrow a &= \frac{\sqrt{(-6)^2 + (-3)^2}}{0.5} \\ \Rightarrow a &= 6\sqrt{5} \\ \Rightarrow a &= 13.416\,407\,86 \text{ (FCD)} \\ \Rightarrow a &= \underline{\underline{13 \text{ ms}^{-2} \text{ (2 sf)}}}. \end{aligned}$$

(c) find the direction of the acceleration of P, giving your answer as a bearing to the nearest degree. (3)

Solution



The bearing is

$$\begin{aligned} 180 + \tan^{-1}\left(\frac{2}{1}\right) &= 180 + 63.434\,948\,82 \text{ (FCD)} \\ &= 243.434\,948\,82 \text{ (FCD)} \\ &= \underline{\underline{243^\circ \text{ (nearest degree)}}}. \end{aligned}$$

7. A particle P of mass $4m$ is held at rest at the point X on the surface of a rough inclined plane which is fixed to horizontal ground. The point X is a distance h from the bottom of the inclined plane. The plane is inclined to the horizontal at an angle α where $\tan\alpha = \frac{3}{4}$. The coefficient of friction between P and the plane is $\frac{1}{4}$. The particle P is attached to one end of a light inextensible string. The string passes over a small smooth pulley which

is fixed at the top of the plane. The other end of the string is attached to a particle Q of mass m which hangs freely at a distance d, where $d > h$, below the pulley, as shown in Figure 4.

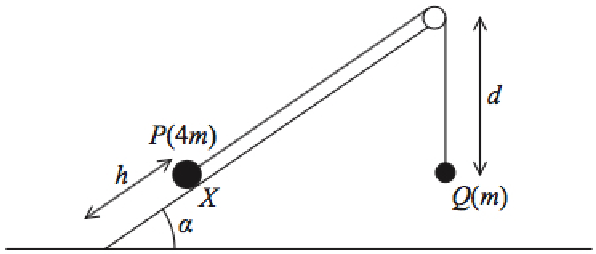


Figure 4: a particle P of mass $4m$

The string lies in a vertical plane through a line of greatest slope of the inclined plane. The system is released from rest with the string taut and P moves down the plane. For the motion of the particles before P hits the ground,

(a) state which of the information given above implies that the magnitudes of the accelerations of the two particles are the same, (1)

> **Solution**
> It is the inextensible string.

(b) write down an equation of motion for each particle, (5)

> **Solution**
> Let T N be the tension and let $a = \text{ms}^{-2}$ be the acceleration. Now,
>
> $$\tan\alpha = \tfrac{3}{4} \Rightarrow \sin\alpha = \tfrac{3}{5}, \cos\alpha = \tfrac{4}{5}.$$
>
> P:
>
> $$\begin{aligned} \text{Parallel:} \quad & 4mg\sin\alpha - T - F = 4ma \\ \text{Perpendicular:} \quad & R = 4mg\cos\alpha \\ F = \mu R: \quad & F = \tfrac{1}{4}R. \end{aligned}$$
>
> Q:
>
> $$T - mg = ma.$$

(c) find the acceleration of each particle. (5)

Solution
For Q,

$$T - mg = ma \Rightarrow T = mg + ma.$$

For P,

$$\begin{aligned} & F = \tfrac{1}{4}R \\ \Rightarrow\ & 4mg\sin\alpha - T - 4ma = \tfrac{1}{4}(4mg\cos\alpha) \\ \Rightarrow\ & 4mg\sin\alpha - T - 4ma = mg\cos\alpha \\ \Rightarrow\ & \tfrac{12}{5}mg - T - 4ma = \tfrac{4}{5}mg \\ \Rightarrow\ & \tfrac{8}{5}mg = T + 4ma \\ \Rightarrow\ & T = \tfrac{8}{5}mg - 4ma. \end{aligned}$$

Set the equations for T equal to each other:

$$\begin{aligned} mg + ma = \tfrac{8}{5}mg - 4ma \Rightarrow g + a &= \tfrac{8}{5}g - 4a \\ \Rightarrow 5a &= \tfrac{3}{5}g \\ \Rightarrow a &= \tfrac{3}{25}g. \end{aligned}$$

When P hits the ground, it immediately comes to rest. Given that Q comes to instantaneous rest before reaching the pulley,

(d) show that (5)

$$d > \frac{28h}{25}.$$

Solution
Let h m be the distance $s = h$, $u = 0$, $v = ?$, $t = ?$, and $a = -\frac{3}{25}g$:

$$\begin{aligned} v^2 = u^2 + 2as \Rightarrow v^2 &= 0 + 2 \times \tfrac{3}{25}g \times h \\ \Rightarrow v^2 &= \tfrac{6}{25}gh \\ \Rightarrow v &= \sqrt{\tfrac{6}{25}gh}. \end{aligned}$$

$s = ?$, $u = \sqrt{\frac{6}{25}gh}$, $v = 0$, $t = ?$, and $a = -g$:

$$\begin{aligned} v^2 = u^2 + 2as \Rightarrow 0 &= \tfrac{6}{25}gh - 2 \times g \times s \\ \Rightarrow 2gs &= \tfrac{6}{25}gh \\ \Rightarrow s &= \tfrac{3}{25}h. \end{aligned}$$

The total distance is

$$h + \tfrac{3}{25}h = \tfrac{28}{25}h$$

which must be less than d. Hence,

$$\underline{\underline{d > \frac{28h}{25}}}.$$