Dr Oliver Mathematics
Further Pure Mathematics
Hyperbolic Equations
Past Examination Questions

This booklet consists of 57 questions across a variety of examination topics.
The total number of marks available is 422.

1. Using the definitions of $\cosh x$ and $\sinh x$ in terms of exponentials,

   (a) prove that $\cosh^2 x - \sinh^2 x \equiv 1$, (3)

   (b) solve (4)

   $$\operatorname{cosech} x - 2\coth x = 2,$$

   giving your answers in the form $k \ln a$, where k and a are integers.

2. Given that $y = \sinh^{n-1} x \cosh x$,

   (a) show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = (n-1)\sinh^{n-2} x + n \sinh^n x$. (3)

   The integral I_n is defined by

   $$I_n = \int_0^{\operatorname{arsinh} 1} \sinh^n x \, \mathrm{d}x, \; n \geqslant 0.$$

   (b) Using the result in part (a), or otherwise, show that (2)

   $$nI_n = \sqrt{2} - (n-1)I_{n-2} \, n \geqslant 2.$$

   (c) Hence find the value of I_4. (4)

3. (a) Show that, for $x = \ln k$, where k is a positive constant, (3)

   $$\cosh 2x = \frac{k^4 + 1}{2k^2}.$$

   Given that $\mathrm{f}(x) = px - \tanh 2x$, where p is a constant,

   (b) find the value of p for which $\mathrm{f}(x)$ has a stationary value at $x = \ln 2$, giving your answer as an exact fraction. (4)

4. Figure 1 shows a sketch of the curve with equation

   $$y = x \operatorname{arcosh} x, \; 1 \leqslant x \leqslant 2.$$

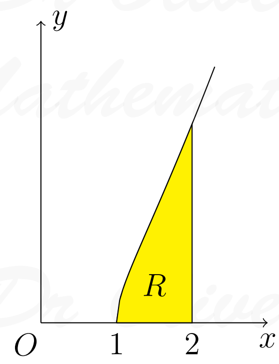

Figure 1: $y = x \operatorname{arcosh} x$

The region R, as shown shaded in the figure, is bounded by the curve, the x-axis, and the $x = 2$. Show that the area of R is

$$\tfrac{7}{4}\ln(2+\sqrt{3}) - \tfrac{\sqrt{3}}{2}.$$

5. (a) Show that, for $0 < x \leqslant 1$, (3)

$$\ln\left(\frac{1-\sqrt{1-x^2}}{x}\right) = -\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right).$$

(b) Using the definitions of $\cosh x$ or $\operatorname{sech} x$ in terms of exponentials, for $0 < x \leqslant 1$, (5)

$$\operatorname{arsech} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right).$$

(c) Solve the equation (5)

$$3\tanh^2 x - 4\operatorname{sech} x + 1 = 0,$$

giving your answers in terms of natural logarithms.

6. Evaluate

$$\int_1^4 \frac{1}{\sqrt{x^2-2x+17}}\,\mathrm{d}x,$$

giving your answer as an exact logarthim.

7. (a) Using the definitions of $\cosh x$ in terms of exponentials, prove that (3)

$$4\cosh^3 x - 3\cosh x \equiv \cosh 3x.$$

(b) Hence, or otherwise, solve the equation (4)

$$\cosh 3x = 5\cosh x,$$

giving your answer as natural logarithms.

8. (a) Show that $\operatorname{artanh}(\sin \frac{\pi}{4}) = \ln(1+\sqrt{2})$. (3)

(b) Given that $y = \operatorname{artanh}(\sin x)$, show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \sec x$. (2)

(c) Find the exact value of $\int_0^{\frac{\pi}{4}} \sin x \operatorname{artanh}(\sin x)\,\mathrm{d}x$. (5)

9. Find the values of x for which (6)

$$5\cosh x - 2\sinh x = 11,$$

giving your answer as natural logarithms.

10. The curve with equation

$$y = -x + \tanh 4x,\ x \geqslant 0,$$

has a maximum turning point A.

(a) Find, in exact logarithmic form, the x-coordinate of A. (4)

(b) Show that the y-coordinate of A is $\frac{1}{4}\left\{2\sqrt{3} - \ln(2+\sqrt{3})\right\}$. (3)

11. Figure 2 shows a sketch of the curve with equation (10)

$$y = x^2 \operatorname{arsinh} x.$$

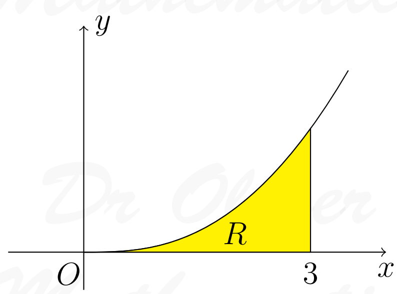

Figure 2: $y = x^2 \operatorname{arsinh} x$

The region R, as shown shaded in the figure, is bounded by the curve, the x-axis, and the $x = 3$. Show that the area of R is

$$9\ln(3+\sqrt{10}) - \tfrac{1}{9}(2+7\sqrt{10}).$$

12.
$$I_n = \int x^n \cosh x \, \mathrm{d}x, \; n \geqslant 0.$$

(a) Show that, for $n \geqslant 2$, (4)

$$I_n = x^n \sinh x - n x^{n-1} \cosh x + n(n-1) I_{n-2}.$$

(b) Hence show that (5)

$$I_4 = \mathrm{f}(x) \sinh x + \mathrm{g}(x) \cosh x + c,$$

where $\mathrm{f}(x)$ and $\mathrm{g}(x)$ are functions of x to be found, and c is an arbitrary constant.

(c) Find the exact value of $\displaystyle\int_0^1 x^4 \cosh x \, \mathrm{d}x$, given your answer in terms of e. (3)

13. Evaluate $\displaystyle\int_1^3 \frac{1}{\sqrt{x^2 + 4x - 5}} \, \mathrm{d}x$, giving your answer as a natural logarithm. (5)

14. (a) Using the definitions of $\cosh x$ and $\sinh x$ in terms of exponentials, prove that (3)

$$\cosh(A - B) \equiv \cosh A \cosh B - \sinh A \sinh B.$$

(b) Hence, or otherwise, given that $\cosh(x - 1) = \sinh x$, show that (4)

$$\tanh x = \frac{\mathrm{e}^2 + 1}{\mathrm{e}^2 + 2\,\mathrm{e} - 1}.$$

15. Figure 3 shows a sketch of the curve with equation (10)

$$y = \operatorname{arsinh} \sqrt{x}, \; x \geqslant 0.$$

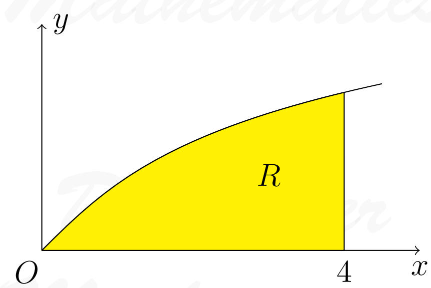

Figure 3: $y = \operatorname{arsinh} \sqrt{x}$

The region R, as shown shaded in the figure, is bounded by the curve, the x-axis, and the $x = 4$. Using the substitution $x = \sinh^2 \theta$, or otherwise, show that the area of R is

$$k \ln(2 + \sqrt{5}) - \sqrt{5},$$

where k is a constant to be found.

16. Show that (4)

$$\frac{\mathrm{d}}{\mathrm{d}x}[\ln(\tanh x)] = 2\,\mathrm{cosech}\,2x,\ x > 0.$$

17. Find the values of x for which (6)

$$8\cosh x - 4\sinh x = 13,$$

giving your answers as a natural logarithms.

18. Show that (7)

$$\int_5^6 \frac{3+x}{\sqrt{x^2-9}}\,\mathrm{d}x = 3\ln\left(\frac{2+\sqrt{3}}{3}\right) + 3\sqrt{3} - 4.$$

19. The curve C has equation $y = \mathrm{arsinh}(x^3)$, $x \geqslant 0$. The point P on C has x-coordinate $\sqrt{2}$. Show that an equation of the tangent to C at P is (5)

$$y = 2x - 2\sqrt{2} + \ln(3 + 2\sqrt{2}).$$

20. Figure 4 shows a sketch of the curve with equation

$$y = \tfrac{1}{10}\cosh x\arctan(\sinh x),\ x \geqslant 0.$$

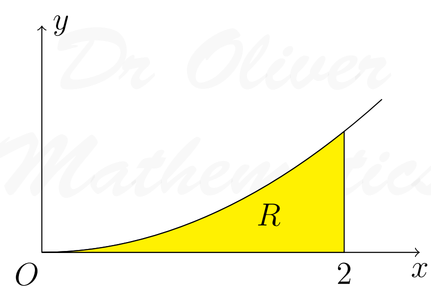

Figure 4: $y = \frac{1}{10}\cosh x\arctan(\sinh x)$

The region R, as shown shaded in the figure, is bounded by the curve, the x-axis, and the $x = 2$.

(a) Find $\int \cosh x\arctan(\sinh x)\,\mathrm{d}x$. (5)

(b) Hence show that, to 2 significant figures, the area of R is 0.34. (2)

21. (a) Starting from the definitions of $\cosh x$ and $\sinh x$ in terms of exponentials, prove that (2)

$$\sinh 2x \equiv 2\sinh x\cosh x.$$

(b) Hence find the exact values of x for which (7)

$$\sinh 2x = 6\sinh^2 x + 7\sinh x.$$

22. The curve C, with equation $y = \cosh 3x - 4x$, has a minimum point, as shown in Figure 5.

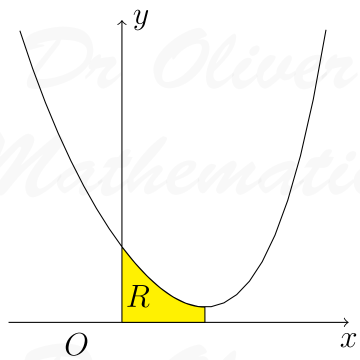

Figure 5: $y = \cosh 3x - 4x$

(a) Use calculus to find the x-coordinate of A. Give your answer in terms of natural logarithm. (5)

The region R, as shown shaded in the figure, is bounded by the curve, the x-axis, the y-axis, and the line through A parallel to the y-axis.

(b) Show that the area of R is $\frac{2}{9}[2 - (\ln 3)^2]$. (6)

23. (a) Using the substitution $x = \dfrac{a}{u}$, or otherwise, find (6)

$$\int \frac{1}{x\sqrt{a^2 - x^2}}\,\mathrm{d}x.$$

(b) Hence find (5)

$$\int_3^4 \frac{1}{x\sqrt{25 - x^2}}\,\mathrm{d}x,$$

giving your answer in the form $a \ln b$, where a and b are rational numbers.

24. Solve the equation (5)

$$7\,\mathrm{sech}\, x - \tanh x = 5.$$

Give your answers in the form $\ln a$ where a is a rational number.

25. Given that $y = \mathrm{arsinh}(\sqrt{x})$, $x > 0$,

(a) find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, giving your answer as a simplified fraction. (3)

(b) Hence, or otherwise, find (6)

$$\int_{\frac{1}{4}}^{4} \frac{1}{\sqrt{x(x+1)}}\,\mathrm{d}x,$$

giving your answer in the form $\ln\left(\frac{a+b\sqrt{5}}{2}\right)$, where a and b are integers.

26.

$$R = 10\pi \int_0^1 \sqrt{16\,\mathrm{c}^2\,\theta + 9}\,\mathrm{d}c, \quad \text{where } c = \cos\theta.$$

Using the substitution $\cos\theta = \frac{3}{4}\sinh u$, or otherwise, evaluate the integral, showing all of your working and giving the final answer to 3 significant figures.

27. (a) Starting from the definitions of $\cosh x$ and $\sinh x$ in terms of exponentials, prove that (3)

$$\cosh 2x \equiv 1 + 2\sinh^2 x.$$

(b) Solve the equation (5)

$$\cosh 2x - 3\sinh x = 15,$$

giving your answers as exact logarithms.

28. Given that $y = (\operatorname{arcosh} 3x)^2$, where $3x > 1$, show that

(a) $(9x^2 - 1)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 36y,$ (5)

(b) $(9x^2 - 1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 9x\frac{\mathrm{d}y}{\mathrm{d}x} = 18.$ (4)

29. Given that $y = \arctan(3\,\mathrm{e}^{2x})$, show that (5)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{5\cosh 2x + 4\sinh 2x}.$$

30. Show that (4)

$$\int_5^8 \frac{1}{\sqrt{x^2 - 10x + 34}}\,\mathrm{d}x = \ln(A + \sqrt{n}),$$

giving the values of the integers A and n.

31. The curve C_1 has equation $y = 3\sinh 2x$ and the curve C_2 has equation $y = 13 - 3\,\mathrm{e}^{2x}$.

(a) Sketch the graphs of the curves C_1 and C_2 on one set of axes, giving the equation of any asymptote and the coordinates of points where the curves cross the axes. (4)

(b) Solve the equation (5)

$$3\sinh 2x = 13 - 3\,\mathrm{e}^{2x},$$

giving your answer in the form $\frac{1}{2}\ln k$, where k is an integer.

32. The hyperbola H has equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

(a) Use calculus to show that the equation of the tangent to H at the point $(a\cosh\theta, b\sinh\theta)$ may be written in the form (4)

$$xb\cosh\theta - ya\sinh\theta = ab.$$

The line l_1 is the tangent to H at the point $(a\cosh\theta, b\sinh\theta)$, $\theta \neq 0$. Given that $l_1$meets the x-axis at the point P,

(b) find, in terms of a and θ, the coordinates of P. (2)

The line l_2 is the tangent to H at the point $(a, 0)$. Given that l_1 and l_2 meets the at the point Q,

(c) find, in terms of a, b, and θ, the coordinates of Q. (2)

(d) Show that, as θ varies, the locus of the midpoint of PQ has equation (6)

$$x(4y^2 + b^2) = ab^2.$$

33. The curve C, as shown in Figure 6, has equation (6)

$$y = \tfrac{1}{3}\cosh 3x,\ 0 \leqslant x \leqslant \ln a,$$

where a is a constant and $a > 1$.

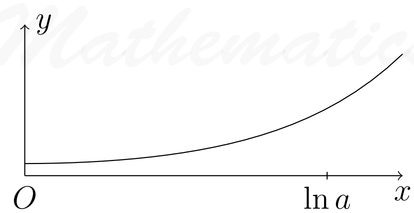

Figure 6: $y = \frac{1}{3}\cosh 3x$

Using calculus, show that the length of curve C is

$$k\left(a^3 - \frac{1}{a^3}\right)$$

and state the value of the constant k.

34. (a) Differentiate $x\,\text{arsinh}\,2x$ with respect to x. (3)

(b) Hence, or otherwise, find the exact value of (7)

$$\int_0^{\sqrt{2}} x \operatorname{arsinh} 2x \, \mathrm{d}x,$$

giving your answer in the form $A \ln B + C$, where A, B, C are real numbers.

35.

$$\mathrm{f}(x) = 5\cosh x - 4\sinh x.$$

(a) Show that $\mathrm{f}(x) = \frac{1}{2}(\mathrm{e}^x + 9\,\mathrm{e}^{-x})$. (2)

Hence

(b) solve $\mathrm{f}(x) = 5$, (4)

(c) show that (5)

$$\int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{1}{5\cosh x - 4\sinh x} \, \mathrm{d}x = \tfrac{\pi}{18}.$$

36. (a) Find (2)

$$\int \frac{1}{\sqrt{4x^2 + 9}} \, \mathrm{d}x.$$

(b) Use your answer to part (a) to find the exact value of (3)

$$\int_{-3}^{3} \frac{1}{\sqrt{4x^2 + 9}} \, \mathrm{d}x,$$

giving your answer in the form $k \ln(a + b\sqrt{5}$, where a and b are integers and k is a constant.

37. The curve with parametric equations (7)

$$x = \cosh 2\theta, y = 4\sinh\theta, 0 \leqslant \theta \leqslant 1,$$

is rotated through 2π radians about the x-axis. Show that the area of of the surface generated is $\lambda(\cosh^3 1 - 1)$, where λ is a constant to be found.

38. Figure 7 shows a sketch of the curve with equation (7)

$$y = 40 \operatorname{arcosh} x - 9x, \; x \geqslant 1.$$

Figure 7: $y = 40 \operatorname{arcosh} x - 9x$

Use calculus to find the exact coordinates of the turning point of the curve, giving your answer in the form $\left(\frac{p}{q}, r \ln 3 + s\right)$, where p, q, r, and s are integers.

39. Figure 8 shows a sketch of the curve with equations

$$y = 6 \cosh x \text{ and } y = 9 - 2 \sinh x.$$

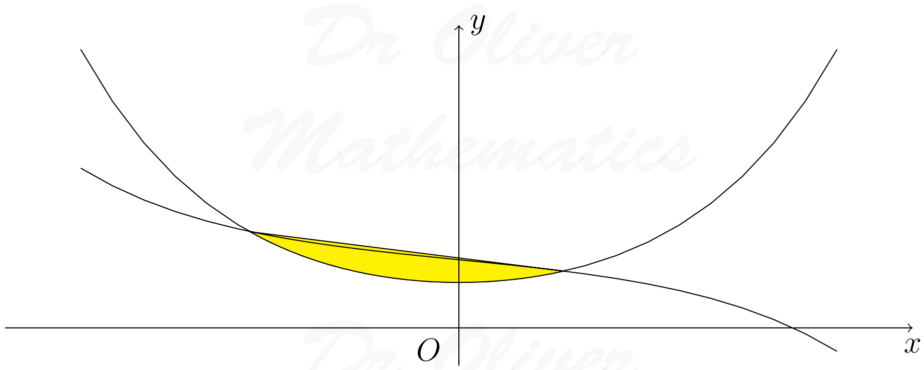

Figure 8: $y = 6 \cosh x$ and $y = 9 - 2 \sinh x$

(a) Using the definitions of $\sinh x$ and $\cosh x$ in terms of e^x, find exact values for the x-coordinates of the two points where the curves intersect. (6)

The finite region between the two curves is shown shaded in the figure.

(b) Using calculus, find the area of the shaded region, giving your answer in the form $a \ln b + c$, where a, b, and c are integers. (6)

40. The curve C, shown in Figure 9, has equation

$$y = 2x^{\frac{1}{2}},\ 1 \leqslant x \leqslant 8.$$

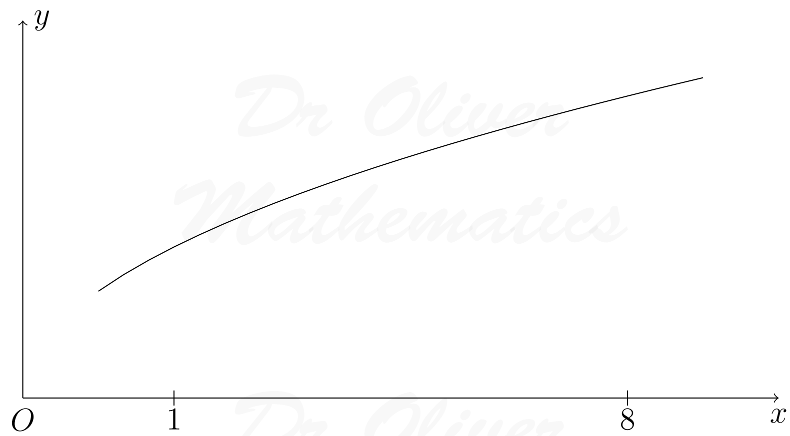

Figure 9: $y = 2x^{\frac{1}{2}}$

(a) Show that the length s of the curve C is given by the equation (2)

$$s = \int_1^8 \sqrt{1 + \frac{1}{x}}\,\mathrm{d}x.$$

(b) Using the substitution $x = \sinh^2 u$, or otherwise, find an exact value for s. Give your answer in the form $a\sqrt{2} + \ln(b + c\sqrt{2})$, where a, b, and c are integers. (9)

41. Using calculus, find the exact value of

(a) $\displaystyle\int_1^2 \frac{1}{\sqrt{x^2 - 2x + 3}}\,\mathrm{d}x$, (4)

(b) $\displaystyle\int_0^1 \mathrm{e}^{2x} \sinh x\,\mathrm{d}x$. (4)

42. Using the definitions of hyperbolic functions in terms of exponentials,

(a) show that (3)

$$\operatorname{sech}^2 x \equiv 1 - \tanh^2 x,$$

(b) solve the equation (4)

$$4 \sinh x - 3 \cosh x = 3.$$

43. Given that $y = \operatorname{artanh} \dfrac{x}{\sqrt{1 + x^2}}$, show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{\sqrt{1 + x^2}}$.

44. Solve the equation

$$5\tanh x + 7 = 5\,\text{sech}\,x.$$

Give each answer in the form $\ln k$ where k is a rational number.

45.

$$9x^2 + 6x + 5 \equiv a(x+b)^2 + c.$$

(a) Find the values of the constants a, b, and c. (3)

Hence, or otherwise, find

(b) $\displaystyle\int \frac{1}{9x^2+6x+5}\,\mathrm{d}x$, (2)

(c) $\displaystyle\int \frac{1}{\sqrt{9x^2+6x+5}}\,\mathrm{d}x$. (2)

46. The curve C has equation $y = \frac{1}{2}\ln(\coth x)$, $x > 0$.

(a) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\,\text{cosech}\,2x$. (3)

The points A and B lie on C. The x-coordinates of A and B are $\ln 2$ and $\ln 3$ respectively.

(b) Find the length of the arc AB, giving your answer in the form $p\ln q$, where p and q are rational numbers. (6)

47. The curve C has equation

$$y = \mathrm{e}^{-x},\ x \in \mathbb{R}.$$

The part of the curve C between $x = 0$ and $x = \ln 3$ is rotated through 2π radians about the x-axis.

(a) Show that the area S of the curved surface generated is given by (3)

$$S = 2\pi \int_0^{\ln 3} \mathrm{e}^{-x}\sqrt{1+\mathrm{e}^{-2x}}\,\mathrm{d}x.$$

(b) Use the substitution $\mathrm{e}^{-x} = \sinh u$ to show that (5)

$$S = 2\pi \int_{\text{arsinh}\,\alpha}^{\text{arsinh}\,\beta} \cosh^2 u\,\mathrm{d}u,$$

where α and β are constants to be determined.

(c) Show that (2)

$$2\int \cosh^2 u\,\mathrm{d}u = \tfrac{1}{2}\sinh 2u + u + k,$$

where k is an arbitrary constant.

(d) Hence find the value of S, giving your answer to 3 decimal places. (2)

48. Solve the equation (6)

$$2\cosh^2 x - 3\sinh x = 1,$$

giving your answers in terms of natural logarithms.

49. A curve has equation (5)

$$y = \cosh x,\ 1 \leqslant x \leqslant \ln 5.$$

Find the length of this curve. Give your answer in terms of e.

50. The curve C has equation

$$y = \frac{1}{\sqrt{x^2 + 2x - 3}},\ x > 1.$$

(a) Find $\int y\,\mathrm{d}x$. (3)

The region R is bounded by the curve C, the x-axis, and the lines with equations $x = 2$ and $x = 3$. The region R is rotated through 2π radians about the x-axis.

(b) Find the volume of the solid generated. Give your answer in the form $p\pi \ln q$, where p and q are rational numbers to be found. (4)

51. The hyperbola H is given by the equation $x^2 - y^2 = 1$.

(a) Write down the equations of the two asymptotes of H. (1)

(b) Show that an equation of the tangent to H at the point $P(\cosh t, \sinh t)$ is (3)

$$y \sinh t = x \cosh t - 1.$$

The tangent at P meets the asymptotes of H at the points Q and R.

(c) Show that P is the midpoint of QR. (3)

(d) Show that the area of the triangle OQR, where O is the origin, is independent of t. (3)

52. (a) Prove that (3)

$$\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{arcoth} x) = \frac{1}{1 - x^2}.$$

Given that $y = (\operatorname{arcoth} x)^2$,

(b) show that (5)

$$(1 - x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{k}{1 - x^2},$$

where k is a constant to be determined.

53. (a) Show that (3)

$$5\cosh x - 4\sinh x = \frac{e^{2x} + 9}{2e^x}.$$

(b) Hence, using the substitution $u = e^x$ or otherwise, find (4)

$$\int \frac{1}{5\cosh x - 4\sinh x}\,dx.$$

54. Given that $y = \operatorname{arsinh}(\tanh x)$, show that (5)

$$\frac{dy}{dx} = \frac{\operatorname{sech}^2 x}{\sqrt{1 + \tanh^2 x}}.$$

55. (a) Using the definition for $\cosh x$ in terms of exponentials, show that (3)

$$\cosh 2x \equiv 2\cosh^2 x - 1.$$

(b) Find the exact values of x for which (6)

$$29\cosh x - 3\cosh 2x = 38,$$

giving your answers in terms of natural logarithms.

56.

$$I_n = \int_0^{\ln 2} \cosh^n x\,dx,\ n \geqslant 0.$$

(a) Show that, for $n \geqslant 2$, (6)

$$I_n = \frac{3a^{n-1}}{nb^n} + \frac{n-1}{n} I_{n-2},$$

where a and b are integers to be found.

(b) Hence, or otherwise, find the exact value of (4)

$$\int_0^{\ln 2} \cosh^4 x\,dx.$$

57. The curve C has equation

$$y = \ln\left(\frac{e^x + 1}{e^x - 1}\right),\ \ln 2 \leqslant x \leqslant 3.$$

(a) Show that (4)

$$\frac{dy}{dx} = -\frac{2e^x}{e^{2x} - 1}.$$

(b) Find the length of the curve C, giving your answer in the form $\ln a$, where a is a rational number. (6)