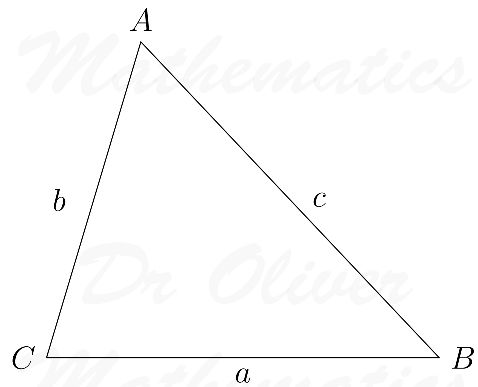
Dr Oliver Mathematics
Full Sine Rule

In this note, we will investigate the *full* sine rule.

What's that when it's at home?

First, we recall the sine rule:



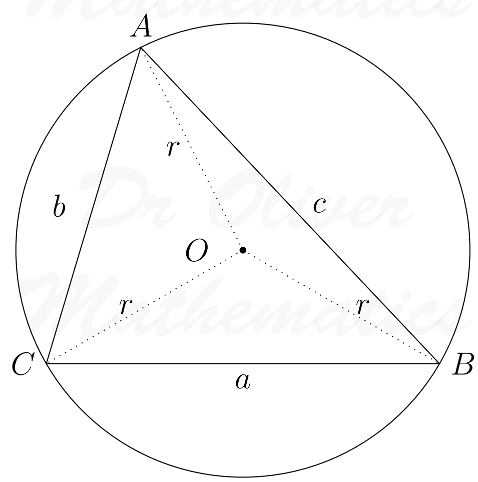
Then

$$\frac{a}{\sin A^\circ} = \frac{b}{\sin B^\circ} = \frac{c}{\sin C^\circ}$$

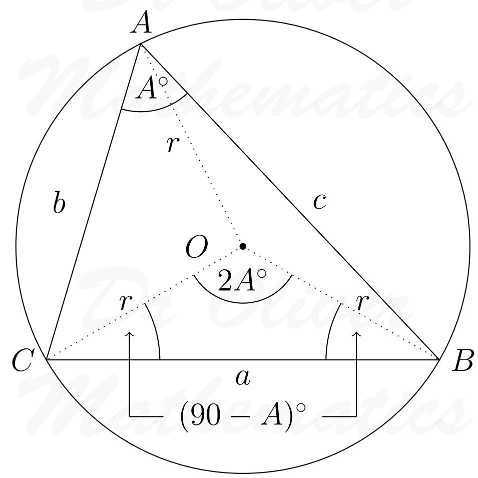
or

$$\frac{\sin A^\circ}{a} = \frac{\sin B^\circ}{b} = \frac{\sin C^\circ}{c}.$$

Now, draw the circumcircle and mark in the the radii:



Next, we consider $\angle BOC$: the angle at the centre is twice the angle at the circumference ($\angle BAC$) and, since the triangle is isosceles, we know what the base angles are.



Now, we apply the sine rule to $\triangle OBC$:

$$\frac{r}{\sin(90 - A)^\circ} = \frac{a}{\sin 2A^\circ}.$$

But $\sin(90 - A)^\circ = \cos A^\circ$ and we can write this as

$$\frac{r}{\cos A^\circ} = \frac{a}{\sin 2A^\circ}.$$

Next, we apply a wee bit of Pure Mathematics to the problem:

$$\sin 2A^\circ \equiv 2 \sin A^\circ \cos A^\circ$$

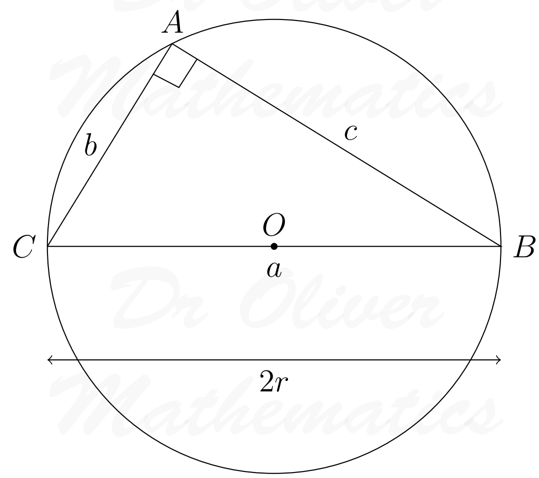
and this leads us to

$$\begin{aligned} \frac{r}{\cos A^\circ} = \frac{a}{\sin 2A^\circ} &\Rightarrow \frac{r}{\cos A^\circ} = \frac{a}{2 \sin A^\circ \cos A^\circ} \\ &\Rightarrow r = \frac{a}{2 \sin A^\circ} \\ &\Rightarrow 2r = \frac{a}{\sin A^\circ}. \end{aligned}$$

And that is the full sine rule:

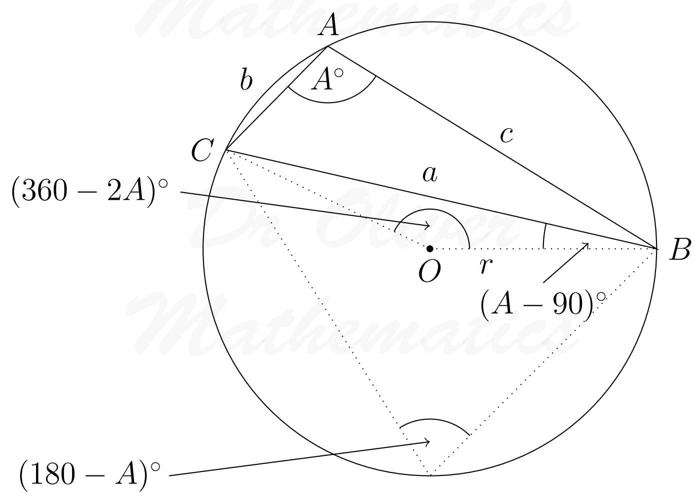
$$\boxed{\frac{a}{\sin A^\circ} = \frac{b}{\sin B^\circ} = \frac{c}{\sin C^\circ} = 2r.}$$

What about right-angled triangles?



$$2r = a = \frac{a}{\sin 90^\circ}.$$

Finally, what about the obtuse case?



Now,

$$\begin{aligned}\sin(A-90)^\circ &= \sin A^\circ \cos 90^\circ - \sin 90^\circ \cos A^\circ \\ &= -\cos A^\circ\end{aligned}$$

and

$$\begin{aligned}\sin(360-2A)^\circ &= \sin 360^\circ \cos 2A^\circ - \sin 2A^\circ \cos 360^\circ \\ &= -\sin 2A^\circ.\end{aligned}$$

Next,

$$\begin{aligned}\frac{r}{\sin(A-90)^\circ} = \frac{a}{\sin(360-2A)^\circ} &\Rightarrow \frac{r}{-\cos A^\circ} = \frac{a}{-\sin 2A^\circ} \\ &\Rightarrow \frac{r}{\cos A^\circ} = \frac{a}{\sin 2A^\circ},\end{aligned}$$

and we proceed as outlined in the case above.