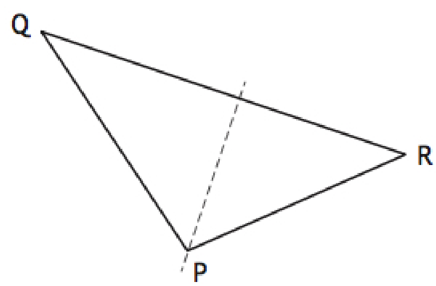
Dr Oliver Mathematics
Mathematics: Higher
2023 Paper 2: Calculator
1 hour 30 minutes

The total number of marks available is 65.
You must write down all the stages in your working.

1. Triangle PQR has vertices $P(5, -1)$, $Q(-2, 8)$, and $R(13, 3)$.



(a) Find the equation of the altitude from P. (3)

Solution
Well,

$$m_{QR} = \frac{3 - 8}{13 - (-2)} = \frac{-5}{15} = -\tfrac{1}{3}$$

and

$$m_{\text{normal}} = -\frac{1}{-\frac{1}{3}} = 3.$$

Finally, the equation of the altitude from P is

$$y - (-1) = 3(x - 5) \Rightarrow y + 1 = 3x - 15 \Rightarrow \underline{\underline{y = 3x - 16}}.$$

(b) Calculate the angle that the side PR makes with the positive direction of the x-axis. (2)

Solution

$$\overrightarrow{PR} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$

and

$$\begin{aligned} \text{angle} &= \tan^{-1}(\tfrac{4}{8}) \\ &= 26.565\,051\,18 \text{ (FCD)} \\ &= \underline{\underline{26.6^\circ \text{ (3 sf)}}}. \end{aligned}$$

2. Find the equation of the tangent to the curve with equation (4)

$$y = 2x^5 - 3x$$

at the point where $x = 1$.

Solution

$$y = 2x^5 - 3x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 10x^4 - 3$$

and

$$\begin{aligned} x = 1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= 10(1^4) - 3 \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= 7. \end{aligned}$$

Now,

$$x = 1 \Rightarrow y = 2(1^5) - 3(1) = -1$$

and, finally, the equation of the tangent is

$$\begin{aligned} y - (-1) = 7(x - 1) &\Rightarrow y + 1 = 7x - 7 \\ &\Rightarrow \underline{\underline{y = 7x - 8}}. \end{aligned}$$

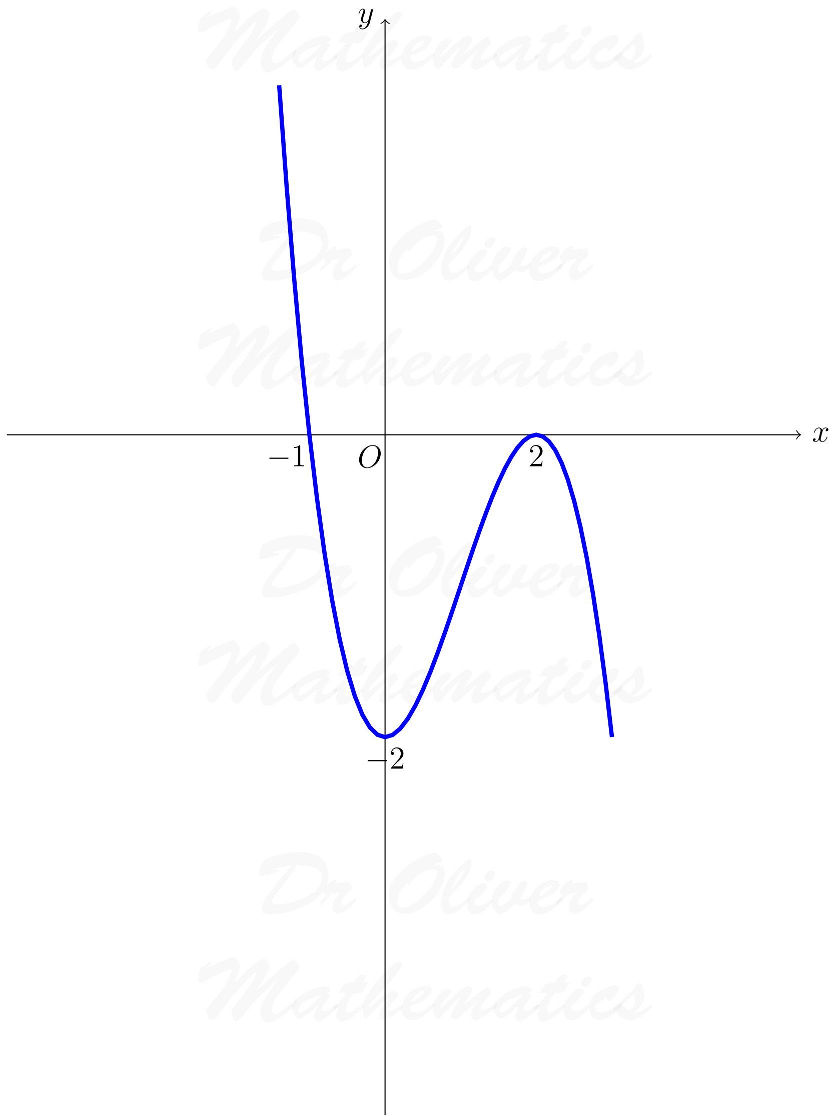
3. Find (2)

$$\int 7\cos(4x + \tfrac{1}{3}\pi)\,\mathrm{d}x.$$

Solution

$$\int 7\cos(4x+\tfrac{1}{3}\pi)\,\mathrm{d}x = \underline{\underline{\tfrac{7}{4}\sin(4x+\tfrac{1}{3}\pi)+c}}.$$

4. The diagram shows the cubic graph of $y = \mathrm{f}(x)$, with stationary points at $(2, 0)$ and $(0, -2)$. (2)



Sketch the graph of $y = 2\,\mathrm{f}(-x)$.

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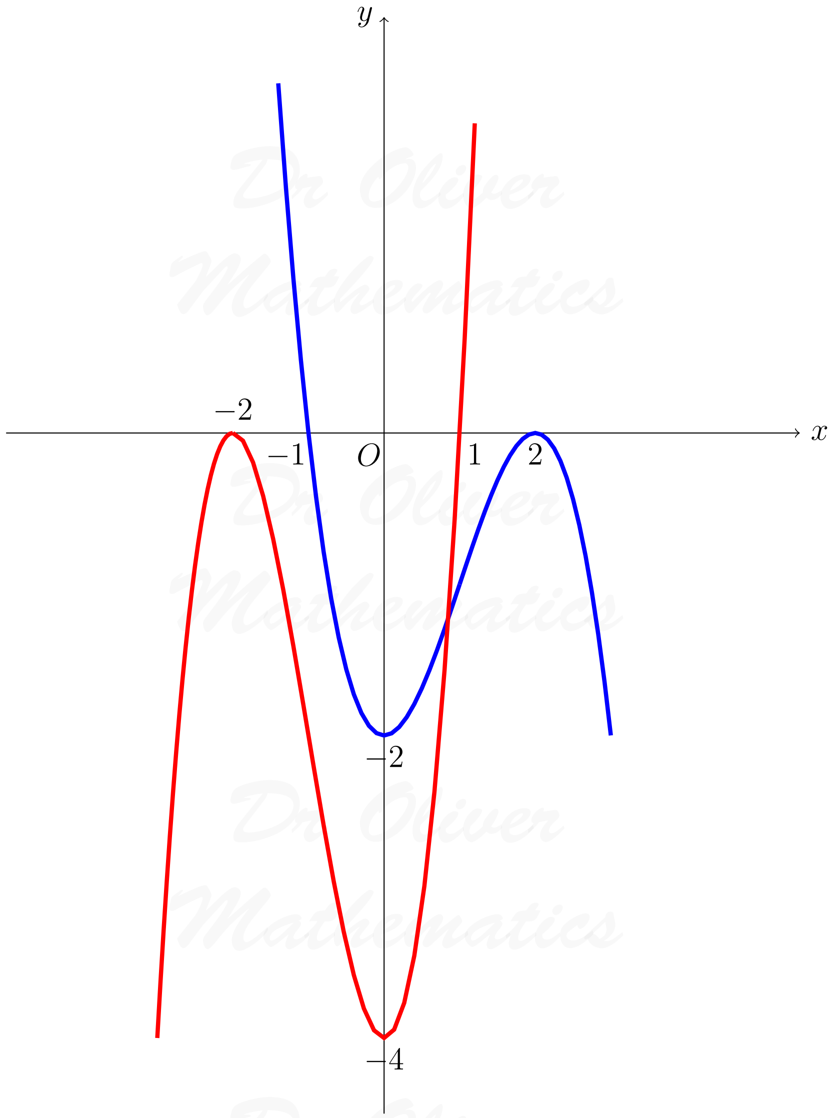
Solution



5. A function, f, is defined by (3)

$$f(x) = (3 - 2x)^4, \text{ where } x \in \mathbb{R}.$$

Calculate the rate of change of f when $x = 4$.

Solution

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$$\begin{aligned} f(x) = (3-2x)^4 &\Rightarrow f'(x) = 4(3-2x)^3 \times (-2) \\ &\Rightarrow f'(x) = -8(3-2x)^3 \end{aligned}$$

and

$$\begin{aligned} f'(4) &= -8[3-2(4)]^3 \\ &= \underline{\underline{1\,000}}. \end{aligned}$$

6. A function $f(x)$ is defined by (3)

$$f(x) = \frac{2}{x} + 3, \; x > 0.$$

Find the inverse function, $f^{-1}(x)$.

Solution

$$\begin{aligned} y = \frac{2}{x} + 3 &\Rightarrow y - 3 = \frac{2}{x} \\ &\Rightarrow x = \frac{2}{y-3} \end{aligned}$$

and so the inverse function is

$$f^{-1}(x) = \underline{\underline{\frac{2}{x-3}}}.$$

7. Solve the equation (5)

$$\sin x^\circ + 2 = 3\cos 2x^\circ$$

for $0 \leqslant x < 360$.

Solution

$$\begin{aligned} \sin x^\circ + 2 = 3\cos 2x^\circ &\Rightarrow \sin x^\circ + 2 = 3(1 - 2\sin^2 x^\circ) \\ &\Rightarrow \sin x^\circ + 2 = 3 - 6\sin^2 x^\circ \\ &\Rightarrow 6\sin^2 x^\circ + \sin x^\circ - 1 = 0 \end{aligned}$$

$$\begin{array}{ll} \text{add to:} & +1 \\ \text{multiply to:} & (+6)\times(-1) = -6 \end{array} \Bigg\} + 3, \ -2$$

$$\begin{aligned} &\Rightarrow 6\sin^2 x^\circ + 3\sin x^\circ - 2\sin x^\circ - 1 = 0 \\ &\Rightarrow 3\sin x^\circ(2\sin x^\circ + 1) - 1(2\sin x^\circ + 1) = 0 \\ &\Rightarrow (3\sin x^\circ - 1)(2\sin x^\circ + 1) = 0 \\ &\Rightarrow 3\sin x^\circ - 1 = 0 \text{ or } 2\sin x^\circ + 1 = 0 \\ &\Rightarrow \sin x^\circ = \tfrac{1}{3} \text{ or } \sin x^\circ = -\tfrac{1}{2}. \end{aligned}$$

$\underline{\sin x^\circ = \frac{1}{3}}$:

$$\begin{aligned} \sin x^\circ = \tfrac{1}{3} &\Rightarrow x = 19.471\,220\,63,\ 160.528\,779\,4 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 19.5,\ 161 \text{ (3 sf)}}}. \end{aligned}$$

$\underline{\sin x^\circ = -\frac{1}{2}}$:

$$\sin x^\circ = -\tfrac{1}{2} \Rightarrow \underline{\underline{x = 210,\ 330}}.$$

8. The diagram shows part of the curve with equation (5)

$$y = x^3 - 2x^2 - 4x + 1$$

and the line with equation

$$y = x - 5.$$

The curve and the line intersect at the points where $x = -2$ and $x = 1$.

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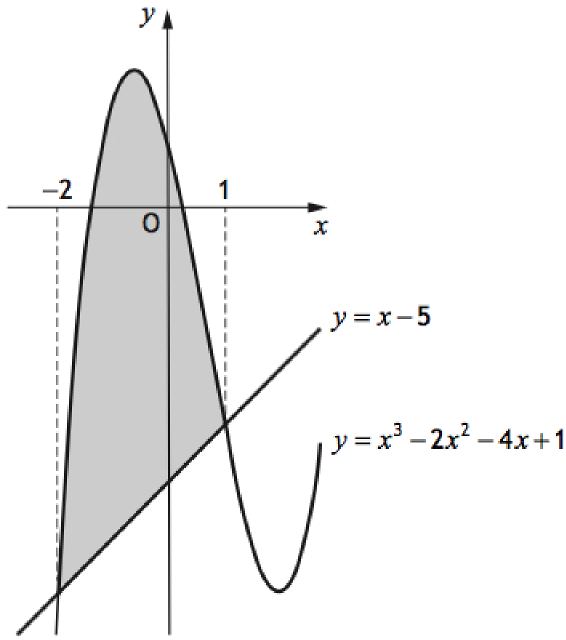


Calculate the shaded area.

Solution

$$
\begin{aligned}
\text{Shaded area} &= \int_{-2}^{1} \left[(x^3 - 2x^2 - 4x + 1) - (x - 5)\right] \, \mathrm{d}x \\
&= \int_{-2}^{1} (x^3 - 2x^2 - 5x + 6) \, \mathrm{d}x \\
&= \left[\tfrac{1}{4}x^4 - \tfrac{2}{3}x^3 - \tfrac{5}{2}x^2 + 6x\right]_{x=-1}^{2} \\
&= \left(\tfrac{1}{4} - \tfrac{2}{3} - \tfrac{5}{2} + 12\right) - \left(4 + \tfrac{16}{3} - 10 - 6\right) \\
&= 9\tfrac{1}{12} - (-6\tfrac{2}{3}) \\
&= \underline{\underline{15\tfrac{3}{4}}}.
\end{aligned}
$$

9. (a) Express (4)

$$7 \cos x^\circ - 3 \sin x^\circ$$

in the form

$$k \sin(x + a)^\circ,$$

where $k > 0$ and $0 < a < 360$.

Solution
Well,

$$\begin{aligned} 7\cos x^\circ - 3\sin x^\circ &= k\sin(x+a)^\circ \\ &= k\sin x^\circ \cos a^\circ + k\cos x^\circ \sin a^\circ \end{aligned}$$

and

$$k\sin a^\circ = 7 \text{ and } k\cos a^\circ = -3.$$

Now,

$$\begin{aligned} k &= \sqrt{7^2 + (-3)^2} \\ &= \sqrt{49+9} \\ &= \sqrt{58}. \end{aligned}$$

Next,

$$\begin{aligned} \tan a^\circ &= \frac{k\sin a^\circ}{k\cos a^\circ} \\ &= -\tfrac{7}{3} \end{aligned}$$

and

$$\begin{aligned} a^\circ &= \tan^{-1}(-\tfrac{7}{3}) \\ &= -66.801\,409\,49^\circ \text{ (not in range)}, 113.195\,590\,5 \text{ (FCD)}. \end{aligned}$$

Finally,

$$7\cos x^\circ - 3\sin x^\circ = \underline{\underline{\sqrt{58}\sin(x+113)^\circ \text{ (3 sf)}}}.$$

(b) Hence, or otherwise, find:

(i) the maximum value of $14\cos x^\circ - 6\sin x^\circ$, (1)

Solution
Well,

$$\begin{aligned} 14\cos x^\circ - 6\sin x^\circ &= 2(7\cos x^\circ - 3\sin x^\circ) \\ &= 2\sqrt{58}\sin(x-66.8)^\circ; \end{aligned}$$

so, the maximum value is

$$\underline{\underline{2\sqrt{58}}}.$$

(ii) the value of x for which it occurs where $0 \leqslant x < 360$. (2)

Solution
The value for x when it occurs is

$$\begin{aligned} x + 113.195\ldots = 90 &\Rightarrow x = -23.198\,590\,51 \text{ (not in range)} \\ &\Rightarrow x = 336.801\,409\,5 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 337 \text{ (3 sf)}}}. \end{aligned}$$

10. Determine the range of values of x for which the function (4)

$$\mathrm{f}(x) = 2x^3 + 9x^2 - 24x + 6$$

is strictly decreasing.

Solution

$$\mathrm{f}(x) = 2x^3 + 9x^2 - 24x + 6 \Rightarrow \mathrm{f}'(x) = 6x^2 + 18x - 24$$

and

$$\begin{aligned} \mathrm{f}'(x) = 0 &\Rightarrow 6x^2 + 18x - 24 = 0 \\ &\Rightarrow 6(x^2 + 3x - 4) = 0 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & +3 \\ \text{multiply to:} & = -4 \end{array}\right\} + 4, -1$$

$$\begin{aligned} &\Rightarrow 6[x^2 + 4x - x - 4] = 0 \\ &\Rightarrow 6[x(x + 4) - 1(x + 4)] = 0 \\ &\Rightarrow 6(x - 1)(x + 4) = 0 \\ &\Rightarrow x - 1 = 0 \text{ or } x + 4 = 0 \\ &\Rightarrow x = 1 \text{ or } x = -4. \end{aligned}$$

We need a 'table of signs':

	$x < -4$	$x = -4$	$-4 < x < 1$	$x = 1$	$x > 1$
$x + 4$	$-$	0	$+$	$+$	$+$
$x - 1$	$-$	$-$	$-$	0	$+$
$\mathrm{f}'(x)$	$+$	0	$-$	0	$+$

Finally, the range of values of x for which the function $\mathrm{f}(x)$ is strictly decreasing is

$$\underline{\underline{-4 < x < 1}}.$$

11. Circle C_1 has equation

$$(x-4)^2 + (y+2)^2 = 37.$$

Circle C_2 has equation

$$x^2 + y^2 + 2x - 6y - 7 = 0.$$

(a) Calculate the distance between the centres of C_1 and C_2. (3)

Solution

Circle C_2:

$$\begin{aligned} x^2 + y^2 + 2x - 6y - 7 = 0 &\Rightarrow x^2 + 2x + y^2 - 6y = 7 \\ &\Rightarrow (x^2 + 2x + 1) + (y^2 - 6y + 9) = 7 + 1 + 9 \\ &\Rightarrow (x+1)^2 + (y-3)^2 = 17. \quad (1) \end{aligned}$$

Hence, C_1 has centre $(4, -2)$ and C_2 has centre $(-1, 3)$.

So, the distance between the centres of C_1 and C_2 will be

$$\begin{aligned} \text{distance} &= \sqrt{[4-(-1)]^2 + [-2-3]^2} \\ &= \sqrt{5^2 + [-5]^2} \\ &= \sqrt{50} \\ &= \underline{\underline{5\sqrt{2}}}. \end{aligned}$$

(b) Hence, show that C_1 and C_2 intersect at two distinct points. (3)

Solution

Well,

$$\begin{aligned} \sqrt{37} + \sqrt{17} &= 10.20\ldots \\ \text{distance} &= 7.07\ldots \\ \sqrt{37} - \sqrt{17} &= 1.95\ldots; \end{aligned}$$

hence, C_1 and C_2 intersect at two distinct points.

12. A curve, for which (4)

$$\frac{dy}{dx} = 8x^3 + 3,$$

passes through the point $(-1, 3)$.

Express y in terms of x.

Solution

$$\frac{dy}{dx} = 8x^3 + 3 \Rightarrow y = 2x^4 + 3x + c,$$

for some constant c. Now,

$$\begin{aligned} x = -1, y = 3 &\Rightarrow 3 = 2[(-1)^4] + 3(-1) + c \\ &\Rightarrow 3 = 2 - 3 + c \\ &\Rightarrow c = 4. \end{aligned}$$

Hence,

$$\underline{\underline{y = 2x^4 + 3x + 4}}.$$

13. A patient is given a dose of medicine.

The concentration of the medicine in the patient's blood is modelled by where

$$C_t = 11e^{-0.0053t},$$

where

- t is the time, in minutes, since the dose of medicine was given aand
- C_t is the concentration of the medicine, in mg/l, at time t.

(a) Calculate the concentration of the medicine 30 minutes after the dose was given (1)

Solution

$$\begin{aligned} t = 30 &\Rightarrow C_{30} = 11e^{-0.0053 \times 30} \\ &\Rightarrow C_{30} = 11e^{-0.159} \\ &\Rightarrow C_{30} = 9.382\,959\,949 \text{ (FCD)} \\ &\Rightarrow C_{30} = \underline{\underline{9.38 \text{ mg/l (3 sf)}}}. \end{aligned}$$

The dose of medicine becomes ineffective when its concentration falls to 0.66 mg/l.

(b) Calculate the time taken for this dose of the medicine to become ineffective. (3)

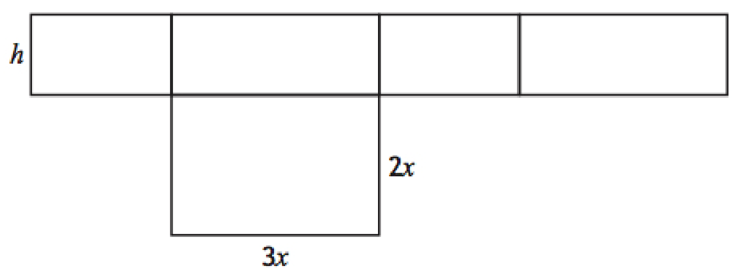
Solution

$$\begin{aligned} 11\mathrm{e}^{-0.0053t} = 0.66 &\Rightarrow \mathrm{e}^{-0.0053t} = 0.06 \\ &\Rightarrow -0.0053t = \ln 0.06 \\ &\Rightarrow t = -\frac{\ln 0.06}{0.0053} \\ &\Rightarrow t = 530.83221007 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{t = 531 \text{ mins (3 sf)}}}. \end{aligned}$$

14. A net of an open box is shown.

The box is a cuboid with height h centimetres.

The base is a rectangle measuring $3x$ centimetres by $2x$ centimetres.



(a) (i) Express the area of the net, A cm^2, in terms of h and x. (1)

Solution

Well,

$$\begin{aligned} A &= (2x \times 3x) + 2(2x \times h) + 2(3x \times h) \\ &= 6x^2 + 4hx + 6hx \\ &= \underline{\underline{(6x^2 + 10hx) \text{ cm}^2}}. \end{aligned}$$

(ii) Given that $A = 7\,200$ cm^2, show that the volume of the box, V cm^3, is given by (2)

$$V = 4\,320x - \tfrac{18}{5}x^3.$$

Solution

$$6x^2 + 10hx = 7\,200 \Rightarrow 10hx = 7\,200 - 6x^2$$
$$\Rightarrow h = \frac{720}{x} - \tfrac{3}{5}x$$

and

$$\begin{aligned} V &= h(2x)(3x) \\ &= 6x^2\left[\frac{720}{x} - \tfrac{3}{5}x\right] \\ &= \underline{\underline{4\,320x - \tfrac{18}{5}x^3}}, \end{aligned}$$

as required.

(b) Determine the value of x that maximises the volume of the box. (4)

Solution

$$V = 4\,320x - \tfrac{18}{5}x^3 \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}x} = 4\,320 - \tfrac{54}{5}x^2$$

and

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}x} = 0 &\Rightarrow 4\,320 - \tfrac{54}{5}x^2 = 0 \\ &\Rightarrow \tfrac{54}{5}x^2 = 4\,320 \\ &\Rightarrow x^2 = 400 \\ &\Rightarrow \underline{\underline{x = 20}}. \end{aligned}$$

We need a 'table of signs':

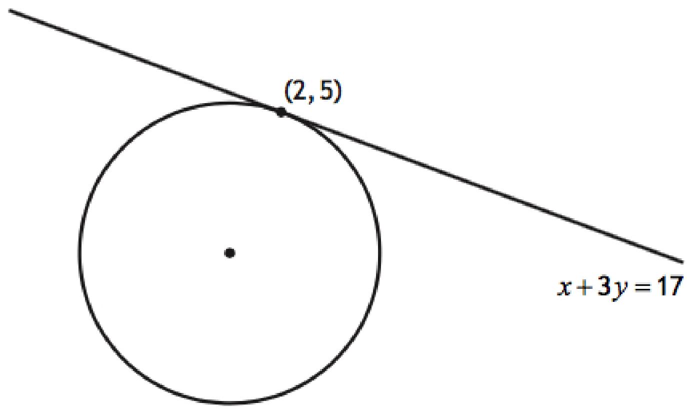
	$x < 20$	$x = 20$	$x > 20$
$V'(x)$	$+$	0	$-$

So the stationary point is a maximum.

15. The line (4)

$$x + 3y = 17$$

is a tangent to a circle at the point $(2, 5)$.



The centre of the circle lies on the y-axis.

Find the coordinates of the centre of the circle.

Solution

Let the centre of the circle be $(0, a)$ (why do we know the x-coordinate is 0?).

Find another point on the line $x + 3y = 17$: $(5, 4)$ will do.

Now,

$$\begin{aligned} m_{\text{tangent}} &= \frac{4-5}{5-2} \\ &= -\tfrac{1}{3} \end{aligned}$$

and

$$\begin{aligned} m_{\text{radius}} &= \frac{5-a}{2-0} \\ &= \tfrac{5-a}{2}. \end{aligned}$$

Now, the product of the two gradients is -1:

$$\begin{aligned} -1 = -\tfrac{1}{3} \times \tfrac{5-a}{2} &\Rightarrow 3 = \tfrac{5-a}{2} \\ &\Rightarrow 6 = 5 - a \\ &\Rightarrow a = -1. \end{aligned}$$

Hence, the coordinates of the centre of the circle are

$$\underline{\underline{(0, -1)}}.$$