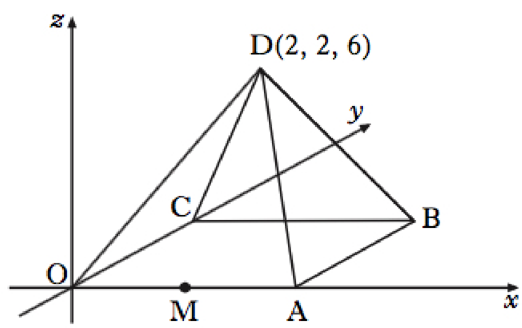
Dr Oliver Mathematics
Mathematics: Higher
2011 Paper 2: Calculator
1 hour 10 minutes

The total number of marks available is 60.
You must write down all the stages in your working.

1. $OABCD$ is a square based pyramid as shown in the diagram below.



O is the origin, D is the point $(2, 2, 6)$, and $OA = 4$ units.
M is the mid-point of OA.

(a) State the coordinates of B. (1)

Solution
$B(4, 4, 0)$.

(b) Express $\overrightarrow{DB}$ and $\overrightarrow{DM}$ in component form. (3)

Solution

$$\begin{aligned}\overrightarrow{DB} &= \overrightarrow{DO} + \overrightarrow{OB} \\ &= \begin{pmatrix} -2 \\ -2 \\ -6 \end{pmatrix} + \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 2 \\ -6 \end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\overrightarrow{DM} &= \overrightarrow{DO} + \overrightarrow{OM} \\ &= \begin{pmatrix} -2 \\ -2 \\ -6 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 0 \\ -2 \\ -6 \end{pmatrix}}}.\end{aligned}$$

(c) Find the size of angle BDM. (5)

Solution

$$\begin{aligned} & \overrightarrow{DB} \cdot \overrightarrow{DM} = DB \cdot DM \cdot \cos BDM \\ \Rightarrow\ & 0 - 4 + 36 = \sqrt{2^2 + 2^2 + 6^2} \cdot \sqrt{0^2 + 2^2 + 6^2} \cdot \cos BDM \\ \Rightarrow\ & 32 = \sqrt{44} \cdot \sqrt{40} \cdot \cos BDM \\ \Rightarrow\ & \cos BDM = \tfrac{4\sqrt{110}}{55} \\ \Rightarrow\ & \angle BDM = 40.290\,986\,39 \text{ (FCD)} \\ \Rightarrow\ & \underline{\underline{\angle BDM = 40.3^\circ \text{ (1 dp)}}}. \end{aligned}$$

2. Functions f, g, and h are defined on the set of real numbers by

$$\begin{aligned} \mathrm{f}(x) &= x^3 - 1 \\ \mathrm{g}(x) &= 3x + 1 \\ \mathrm{h}(x) &= 4x - 5. \end{aligned}$$

(a) Find $\mathrm{g}(\mathrm{f}(x))$. (2)

Solution

$$\begin{aligned} \mathrm{g}(\mathrm{f}(x)) &= \mathrm{g}(x^3 - 1) \\ &= 3(x^3 - 1) + 1 \\ &= \underline{\underline{3x^3 - 2}}. \end{aligned}$$

(b) Show that (1)

$$\mathrm{g}(\mathrm{f}(x)) + x\,\mathrm{h}(x) = 3x^3 + 4x^2 - 5x - 2.$$

Solution

$$\begin{aligned}\mathrm{g}(\mathrm{f}(x)) + x\,\mathrm{h}(x) &= (3x^3 - 2) + x(4x - 5)\\ &= (3x^3 - 2) + (4x^2 - 5x)\\ &= \underline{\underline{3x^3 + 4x^2 - 5x - 2}},\end{aligned}$$

as required.

(c) (i) Show that $(x - 1)$ is a factor of (5)

$$3x^3 + 4x^2 - 5x - 2.$$

Solution

We use synthetic division:

1	3	4	−5	−2
	↓	3	7	2
	3	7	2	0

As the remainder is 0, $(x - 1)$ is a <u>factor</u> of the cubic.

(ii) Factorise

$$3x^3 + 4x^2 - 5x - 2$$

fully.

Solution

$$3x^3 + 4x^2 - 5x - 2 = (x - 1)(3x^2 + 7x + 2)$$

$$\left.\begin{array}{ll}\text{add to:} & +7\\ \text{multiply to:} & (+3) \times (+2) = +6\end{array}\right\} +1,\ +6$$

$$\begin{aligned}&= (x - 1)[3x^2 + 6x + x + 2]\\ &= (x - 1)[3x(x + 2) + (x + 2)]\\ &= \underline{\underline{(x - 1)(3x + 1)(x + 2)}}.\end{aligned}$$

(d) Hence solve (1)

$$\mathrm{g}(\mathrm{f}(x)) + x\,\mathrm{h}(x) = 0.$$

Solution

$$\begin{aligned}\mathrm{g}(\mathrm{f}(x)) + x\,\mathrm{h}(x) = 0 &\Rightarrow (x-1)(3x+1)(x+2) = 0 \\ &\Rightarrow \underline{\underline{x = 1, x = -\tfrac{1}{3}, \text{ or } x = -2}}.\end{aligned}$$

3. A sequence is defined by

$$u_{n+1} = -\tfrac{1}{2}u_n \text{ with } u_0 = -16.$$

(a) Write down the values of u_1 and u_2. (1)

Solution

$u_1 = -\frac{1}{2} \times -16 = \underline{\underline{8}}$.

$u_2 = -\frac{1}{2} \times 8 = \underline{\underline{-4}}$.

A second sequence is given by $4, 5, 7, 11, \ldots$.
It is generated by the recurrence relation

$$v_{n+1} = pv_n + q \text{ with } v_1 = 4.$$

(b) Find the values of p and q. (3)

Solution

We take the first and the second terms:

$$\begin{aligned}4p + q &= 5 \quad (1)\\ 5p + q &= 7 \quad (2).\end{aligned}$$

Now, (2) − (1):

$$\underline{\underline{p = 2}} \Rightarrow 8 + q = 5$$

$$\Rightarrow \underline{\underline{q = -3}}.$$

Either the sequence in (a) or the sequence in (b) has a limit.

(c) (i) Calculate this limit. (3)

Solution
The sequence in (a) has a limit of $\underline{\underline{0}}$.

(ii) Why does the other sequence not have a limit?

Solution
The sequence in (b) is divergent.

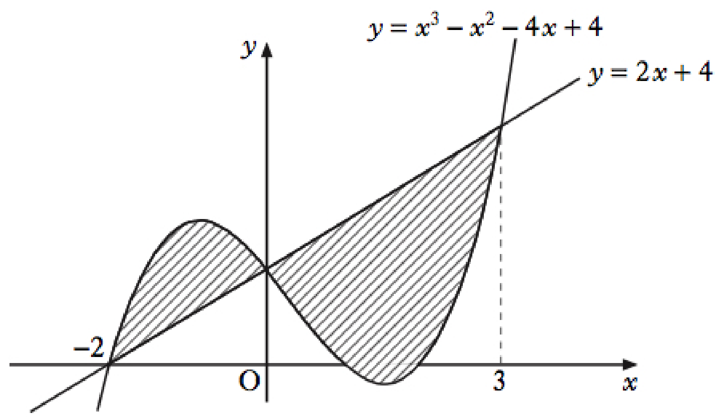
4. The diagram shows the curve with equation (10)

$$y = x^3 - x^2 - 4x + 4$$

and the line with equation

$$y = 2x + 4.$$

The curve and the line intersect at the points $(-2, 0)$, $(0, 4)$, and $(3, 10)$.



Calculate the total shaded area.

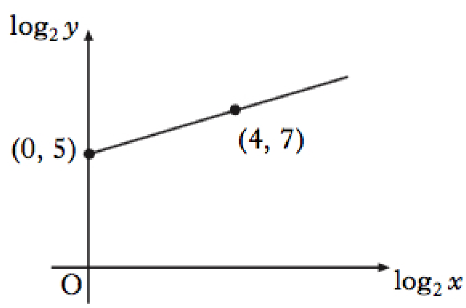
Solution

$$\begin{aligned}
\text{Area} &= \int_{-2}^{0} \left[(x^3 - x^2 - 4x + 4) - (2x + 4)\right] \mathrm{d}x \\
&\qquad + \int_{0}^{3} \left[(2x + 4) - (x^3 - x^2 - 4x + 4)\right] \mathrm{d}x \\
&= \int_{-2}^{0} (x^3 - x^2 - 6x) \,\mathrm{d}x + \int_{0}^{3} (6x + x^2 - x^3) \,\mathrm{d}x \\
&= \left[\tfrac{1}{4}x^4 - \tfrac{1}{3}x^3 - 3x^2\right]_{x=-2}^{0} + \left[3x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\right]_{x=0}^{3} \\
&= \left[(0 - 0 + 0) - (4 + \tfrac{8}{3} - 12)\right] + \left[(27 + 9 - \tfrac{81}{4}) - (0 - 0 + 0)\right] \\
&= 5\tfrac{1}{3} + 15\tfrac{3}{4} \\
&= \underline{\underline{21\tfrac{1}{21}}}.
\end{aligned}$$

5. Variables x and y are related by the equation (5)

$$y = kx^n.$$

The graph of $\log_2 y$ against $\log_2 x$ is a straight line through the points $(0, 5)$ and $(4, 7)$, as shown in the diagram.



Find the values of k and n.

Solution

$$\begin{aligned}
\text{Gradient} &= \frac{7 - 5}{4 - 0} \\
&= \tfrac{1}{2}
\end{aligned}$$

and the equation is

$$\begin{aligned}\log_2 y - 5 = \tfrac{1}{2}(\log_2 x - 0) &\Rightarrow \log_2 y - 5 = \tfrac{1}{2}\log_2 x \\ &\Rightarrow \log_2 y - 5 = \log_2 x^{\frac{1}{2}} \\ &\Rightarrow \log_2 y - \log_2 x^{\frac{1}{2}} = 5 \\ &\Rightarrow \log_2\left(\frac{y}{x^{\frac{1}{2}}}\right) = 5 \\ &\Rightarrow \frac{y}{x^{\frac{1}{2}}} = 2^5 \\ &\Rightarrow y = 32x^{\frac{1}{2}};\end{aligned}$$

hence, $\underline{\underline{k = 32}}$ and $\underline{\underline{n = \frac{1}{2}}}$.

6. (a) The expression (4)

$$3\sin x - 5\cos x$$

can be written in the form

$$R\sin(x + a)$$

where $R > 0$ and $0 \leqslant a < 2\pi$.

Calculate the values of R and a.

Solution

$$R\sin(x + a) \equiv R\sin x\cos a + R\cos x\sin a$$

and hence

$$R\cos a = 3 \text{ and } R\sin a = -5.$$

Now,

$$\begin{aligned}R &= \sqrt{(R\sin a)^2 + (R\cos a)^2} \\ &= \sqrt{(-5)^2 + 3^2} \\ &= \underline{\underline{\sqrt{34}}}\end{aligned}$$

and

$$\tan a = \frac{R\sin a}{R\cos a} \Rightarrow \tan a = -\tfrac{5}{3}$$
$$\Rightarrow a = -1.030\,376\,827 \text{ (no - because the angle is negative)}$$
$$\Rightarrow a = 2.111\,215\,827 \text{ (no - because the sine is positive)}$$
$$\Rightarrow a = \underline{\underline{5.252\,808\,481 \text{ (yes)}}}.$$

(b) Hence find the value of t, where $0 \leqslant t \leqslant 2$, for which (7)

$$\int_0^t (3\cos x + 5\sin x)\,\mathrm{d}x = 3.$$

Solution

$$\int_0^t (3\cos x + 5\sin x)\,\mathrm{d}x = 3 \Rightarrow [3\sin x - 5\cos x]_{x=0}^t = 3$$
$$\Rightarrow (3\sin t - 5\cos t) - (0-5) = 3$$
$$\Rightarrow 3\sin t - 5\cos t + 5 = 3$$
$$\Rightarrow 3\cos t - 5\sin t = -2$$
$$\Rightarrow \sqrt{34}\sin(x + 5.252\ldots) = -2$$
$$\Rightarrow \sin(x + 5.252\ldots) = -\frac{2}{\sqrt{34}}$$
$$\Rightarrow x + 5.252\ldots = -0.350\,105\,777\,8 \text{ (no)}$$
$$\Rightarrow x + 5.252\ldots = 4.171\,969\,48 \text{ (no)}$$
$$\Rightarrow x + 5.252\ldots = 5.933\,079\,29 \text{ (yes)}$$
$$\Rightarrow x = 0.680\,271\,048\,8 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{x = 0.68 \text{ (2 dp)}}}.$$

7. Circle C_1 has equation (9)

$$(x+1)^2 + (y-1)^2 = 121.$$

A circle C_2 with equation

$$x^2 + y^2 - 4x + 6y + p = 0$$

is drawn inside C_1.
The circles have no points of contact.
What is the range of values of p?

Solution

The circle C_1 has centre $A(-1, 1)$ and radius 11. Now,

$$\begin{aligned} x^2 + y^2 - 4x + 6y + p = 0 &\Rightarrow x^2 - 4x + y^2 + 6y = -p \\ &\Rightarrow (x^2 - 4x + 4) + (y^2 + 6y + 9) = 4 + 9 - p \\ &\Rightarrow (x - 2)^2 + (y + 3)^2 = 13 - p \end{aligned}$$

which means that the circle C_2 has centre $B(2, -3)$ and radius $\sqrt{13 - p}$. Now,

$$\sqrt{13 - p} > 0 \Rightarrow 13 - p > 0 \Rightarrow p < 13.$$

Next,

$$\begin{aligned} AB &= \sqrt{3^2 + 4^2} \\ &= 5. \end{aligned}$$

Let C be the point at which AB extended meets C_1. So

$$\begin{aligned} BC < 6 &\Rightarrow \sqrt{13 - p} < 6 \\ &\Rightarrow 13 - p < 36 \\ &\Rightarrow p > -22. \end{aligned}$$

Finally, the range of values of p is

$$\underline{\underline{-22 < p < 13}}.$$