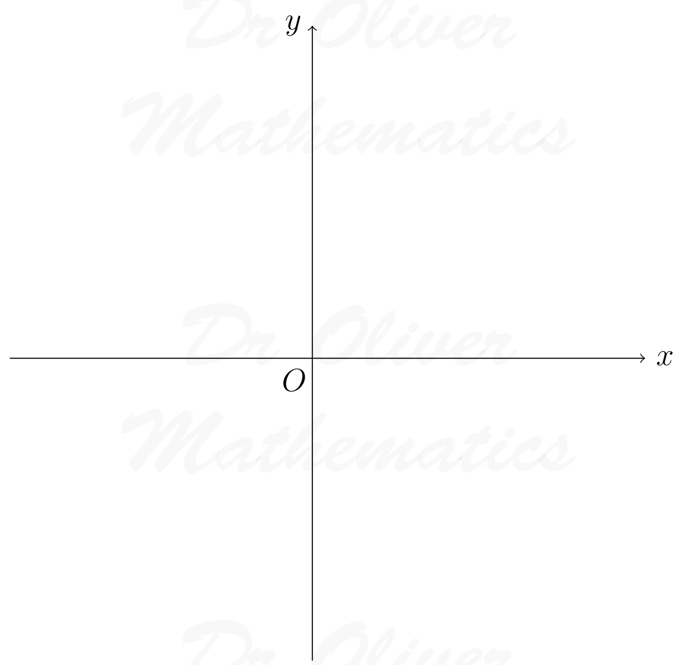
Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2012 June Paper 1 Variant 1: Calculator
2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

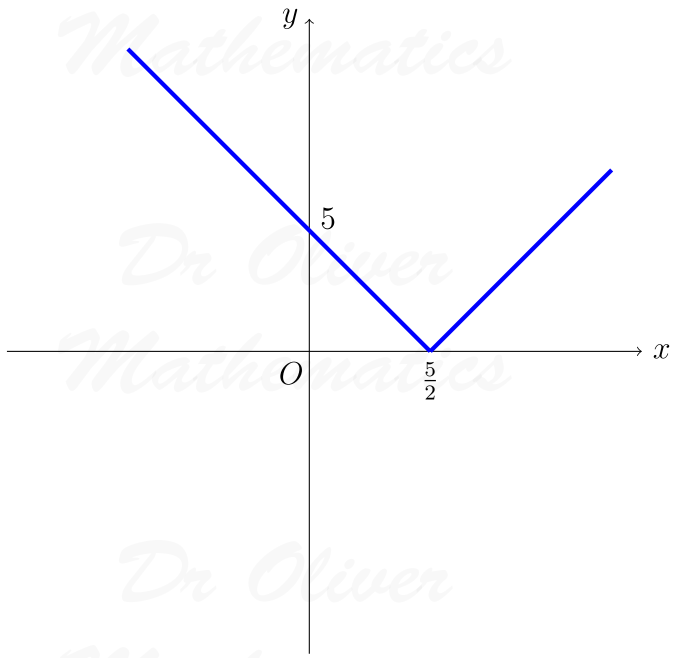
1. (a) Sketch the graph of (2)

$$y = |2x - 5|,$$

showing the coordinates of the points where the graph meets the coordinate axes.



Solution



(b) Solve (2)

$$|2x - 5| = 3.$$

Solution

$\underline{2x - 5 = 3}$:

$$\begin{aligned} 2x - 5 = 3 &\Rightarrow 2x = 8 \\ &\Rightarrow x = 4. \end{aligned}$$

$\underline{-(2x - 5) = 3}$:

$$\begin{aligned} -(2x - 5) = 3 &\Rightarrow -2x + 5 = 3 \\ &\Rightarrow -2x = -2 \\ &\Rightarrow x = 1. \end{aligned}$$

Hence,

$$\underline{\underline{x = 1 \text{ or } x = 4}}.$$

2. The expression (5)

$$2x^3 + ax^2 + bx - 30$$

is divisible by $(x+2)$ and leaves a remainder of -35 when divided by $(2x-1)$.

Find the values of the constants a and b.

Solution

We use synthetic division twice:

$$\begin{array}{r|cccc}
-2 & 2 & a & b & -30 \\
 & \downarrow & -4 & -2a+8 & 4a-2b-16 \\
\hline
 & 2 & a-4 & -2a+b+8 & 4a-2b-46
\end{array}$$

and

$$\begin{array}{r|cccc}
\frac{1}{2} & 2 & a & b & -30 \\
 & \downarrow & 1 & \frac{1}{2}a+\frac{1}{2} & \frac{1}{4}a+\frac{1}{2}b+\frac{1}{4} \\
\hline
 & 2 & a+1 & \frac{1}{2}a+b+\frac{1}{2} & \frac{1}{4}a+\frac{1}{2}b-\frac{119}{4}
\end{array}$$

So,

$$4a-2b-46=0 \Rightarrow 4a-2b=46 \quad (1)$$

and

$$\begin{aligned}
\tfrac{1}{4}a+\tfrac{1}{2}b-\tfrac{119}{4}=-35 &\Rightarrow a+2b-119=-140 \\
&\Rightarrow a+2b=-21 \quad (2).
\end{aligned}$$

Add (1) + (2):

$$5a=25 \Rightarrow \underline{\underline{a=5}}$$

insert in to (1):

$$\begin{aligned}
&\Rightarrow 4(5)-2b=46 \\
&\Rightarrow 20-2b=46 \\
&\Rightarrow 2b=-26 \\
&\Rightarrow \underline{\underline{b=-13}}.
\end{aligned}$$

[Check in (2): $5+2\times(-13)=-21$ ✓]

3. Find the set of values of k for which the line (6)

$$y = 2x + k$$

cuts the curve

$$y = x^2 + kx + 5$$

at two distinct points.

Solution

Well,

$$x^2 + kx + 5 = 2x + k \Rightarrow x^2 + (k-2)x + (5-k) = 0$$

and we want $a = 1$, $b = k - 2$, and $c = 5 - k$.

As the question says, "at two distinct points" so

$$b^2 - 4ac > 0 \Rightarrow (k-2)^2 - 4(1)(5-k) > 0$$

$\times$	k	-2
k	k^2	$-2k$
-2	$-2k$	$+4$

$$\Rightarrow (k^2 - 4k + 4) - (20 - 4k) > 0$$
$$\Rightarrow k^2 - 16 > 0$$

$$\left.\begin{array}{ll}\text{add to:} & 0 \\ \text{multiply to:} & -16\end{array}\right\} -4, +4$$

e.g.,

$$\Rightarrow (k-4)(k+4) > 0.$$

We need a 'table of signs':

	$k < -4$	$k = -4$	$-4 < k < 4$	$k = 4$	$k > 4$
$k+4$	$-$	0	$+$	$+$	$+$
$k-4$	$-$	$-$	$-$	0	$+$
$(k+4)(k-4)$	$+$	0	$-$	0	$+$

Hence,

$$\underline{\underline{k < -4 \text{ or } k > 4}}.$$

4. Arrangements containing 5 different letters from the word AMPLITUDE are to be made.

Find

(a) (i) the number of 5-letter arrangements if there are no restrictions, (1)

Solution

$$9 \times 8 \times 7 \times 6 \times 5 = \underline{\underline{15\,120}}.$$

(ii) the number of 5-letter arrangements which start with the letter A and end with the letter E. (1)

Solution

$$7 \times 6 \times 5 = \underline{\underline{210}}.$$

(b) Tickets for a concert are given out randomly to a class containing 20 students.

- No student is given more than one ticket.
- There are 15 tickets.

(i) Find the number of ways in which this can be done. (1)

Solution

$$\binom{20}{15} = \underline{\underline{15\,504}}.$$

There are 12 boys and 8 girls in the class.

Find the number of different ways in which

(ii) 10 boys and 5 girls get tickets, (3)

Solution

$$\begin{aligned}\binom{12}{10} \times \binom{8}{5} &= 66 \times 56 \\ &= \underline{\underline{3\,696}}.\end{aligned}$$

(iii) all the boys get tickets. (1)

Solution

Well, 3 girls must get a ticket:

$$\binom{8}{3} = \underline{\underline{56}}.$$

5. (a) Find the equation of the tangent to the curve (4)

$$y = x^3 + 2x^2 - 3x + 4$$

at the point where the curve crosses the y-axis.

Solution

Well,

$$y = x^3 + 2x^2 - 3x + 4 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 4x - 3$$

and

$$x = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -3.$$

Now,

$$x = 0 \Rightarrow y = 4$$

and so the equation of the tangent is

$$y - 4 = -3(x - 0) \Rightarrow \underline{\underline{y = -3x + 4}}.$$

(b) Find the coordinates of the point where this tangent meets the curve again. (3)

Solution

Now,

$$\begin{aligned} x^3 + 2x^2 - 3x + 4 = -3x + 4 &\Rightarrow x^3 + 2x^2 = 0 \\ &\Rightarrow x^2(x + 2) = 0 \\ &\Rightarrow x = 0 \text{ or } x = -2. \end{aligned}$$

Next,

$$x = -2 \Rightarrow y = 10$$

and the coordinates are $\underline{\underline{(-2, 10)}}$.

6. (a) Given that (4)

$$15\cos^2\theta + 2\sin^2\theta = 7,$$

show that

$$\tan^2\theta = \tfrac{8}{5}.$$

Solution

Well,

$$\begin{aligned}15\cos^2\theta + 2\sin^2\theta = 7 &\Rightarrow (2\cos^2\theta + 2\sin^2\theta) + 13\cos^2\theta = 7\\ &\Rightarrow 2(\cos^2\theta + \sin^2\theta) + 13\cos^2\theta = 7\\ &\Rightarrow 2 + 13\cos^2\theta = 7\\ &\Rightarrow 13\cos^2\theta = 5\\ &\Rightarrow \cos^2\theta = \tfrac{5}{13}\\ &\Rightarrow 1 - \cos^2\theta = 1 - \tfrac{5}{13}\\ &\Rightarrow \sin^2\theta = \tfrac{8}{13}\\ &\Rightarrow \underline{\underline{\tan^2\theta = \tfrac{8}{5}}},\end{aligned}$$

as required.

(b) Solve (3)

$$15\cos^2\theta + 2\sin^2\theta = 7 \text{ for } 0 \leqslant \theta \leqslant \pi \text{ radians.}$$

Solution

Well,

$$\begin{aligned}15\cos^2\theta + 2\sin^2\theta = 7 &\Rightarrow \tan^2\theta = \tfrac{8}{5}\\ &\Rightarrow \tan\theta = \pm\tfrac{2\sqrt{10}}{5}.\end{aligned}$$

$\underline{\tan\theta = \tfrac{2\sqrt{10}}{5}}$:

$$\begin{aligned}\tan\theta = \tfrac{2\sqrt{10}}{5} &\Rightarrow \theta = 0.901\,832\,252\,5 \text{ (FCD)}\\ &\Rightarrow \theta = 0.902 \text{ (3 sf)}.\end{aligned}$$

$\underline{\tan\theta = -\tfrac{2\sqrt{10}}{5}}$:

$$\begin{aligned}\tan\theta = -\tfrac{2\sqrt{10}}{5} &\Rightarrow \theta = 2.239\,760\,401 \text{ (FCD)}\\ &\Rightarrow \theta = 2.30 \text{ (3 sf)}.\end{aligned}$$

Hence,

$$\underline{\underline{\theta = 0.902,\ 2.30 \text{ (3 sf)}}}.$$

7. The table shows values of variables x and y.

x	1	3	6	10	14
y	2.5	4.5	0	-20	-56

(a) By plotting a suitable straight line graph, show that y and x are related by the equation (4)

$$y = Ax + Bx^2,$$

where A and B are constants.

Solution

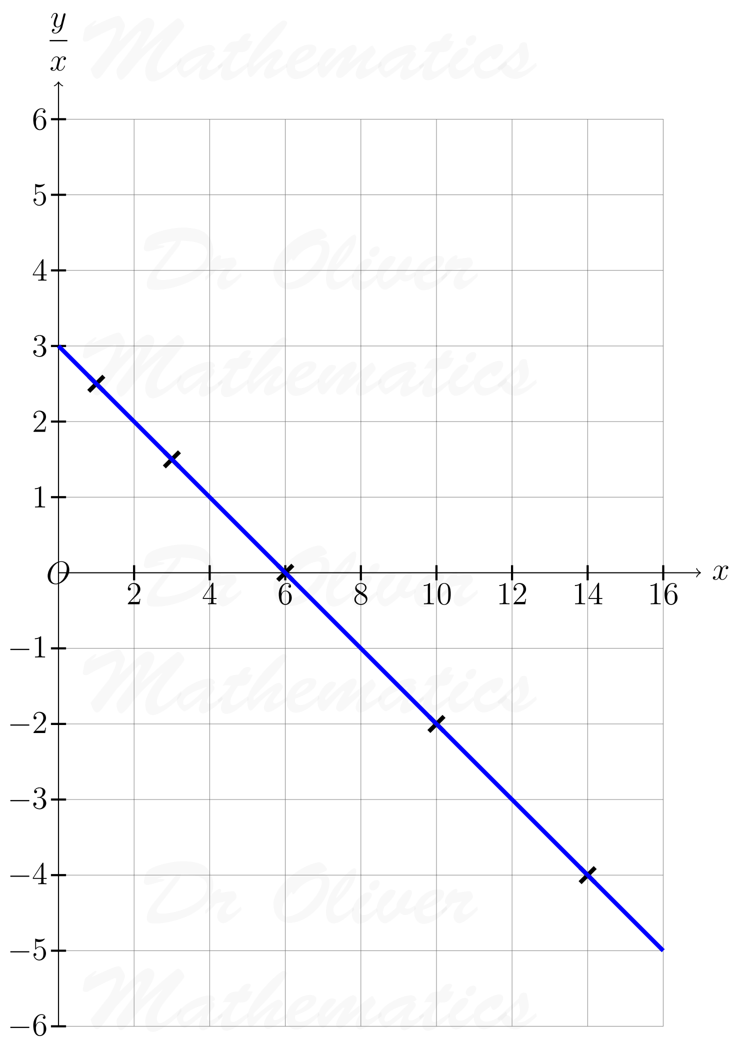
Well,

$$\begin{aligned} y = Ax + Bx^2 &\Rightarrow y = x(A + Bx) \\ &\Rightarrow \frac{y}{x} = A + Bx \end{aligned}$$

so, we want a straight line, then we plot $\frac{y}{x}$ against x:

x	1	3	6	10	14
$\frac{y}{x}$	2.5	1.5	0	-2	-4

We plot a graph:



So, there is a straight line between $\frac{y}{x}$ against x and, hence, there is a linear relationship between the two variables.

(b) Use your graph to find the value of A and of B. (4)

Solution

Well, the straight line goes through $(0, 3)$ and $(6, 0)$ and

$$\begin{aligned} m &= \frac{3-0}{0-6} \\ &= -\tfrac{1}{2} \end{aligned}$$

and, clearly, the $\frac{y}{x}$-intercept is 3. Now,

$$\begin{aligned}\frac{y}{x} - 3 = -\tfrac{1}{2}(x-0) &\Rightarrow \frac{y}{x} = -\tfrac{1}{2}x + 3\\ &\Rightarrow \underline{\underline{y = 3x - \tfrac{1}{2}x^2}};\end{aligned}$$

hence, $\underline{\underline{A = 3}}$ and $\underline{\underline{B = -\frac{1}{2}}}$.

8. (a) Find the value of x for which (5)

$$2\log_{10} x - \log_{10}(5x+60) = 1.$$

Solution

Well,

$$\begin{aligned}2\log_{10} x - \log_{10}(5x+60) = 1 &\Rightarrow \log_{10} x^2 - \log_{10}(5x+60) = 1\\ &\Rightarrow \log_{10}\left(\frac{x^2}{5x+60}\right) = 1\\ &\Rightarrow \frac{x^2}{5x+60} = 10\\ &\Rightarrow x^2 = 10(5x+60)\\ &\Rightarrow x^2 = 50x + 600\\ &\Rightarrow x^2 - 50x - 600 = 0\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & -50\\ \text{multiply to:} & -600\end{array}\right\} -60,\ +10$$

$$\begin{aligned}&\Rightarrow (x-60)(x+10) = 0\\ &\Rightarrow x = 60 \text{ or } x = -10.\end{aligned}$$

But $x \neq -10$ (why?) so $\underline{\underline{x = 60}}$.

(b) Solve (4)

$$\log_5 y = 4\log_y 5.$$

Solution

Now,

$$\begin{aligned}\log_5 y = 4\log_y 5 \Rightarrow \log_5 y &= \frac{4}{\log_5 y}\\ \Rightarrow (\log_5 y)^2 &= 4\\ \Rightarrow \log_5 y &= \pm 2.\end{aligned}$$

$\underline{\log_5 y = 2}$:

$$\log_5 y = 2 \Rightarrow y = 25.$$

$\underline{\log_5 y = -2}$:

$$\log_5 y = -2 \Rightarrow y = \tfrac{1}{25}.$$

Hence,

$$\underline{\underline{y = 25 \text{ or } y = \tfrac{1}{25}}}.$$

9. Find the values of the positive constants p and q such that, in the binomial expansion of (8)

$$(p+qx)^{10},$$

the coefficient of x^5 is 252 and the coefficient of x^3 is 6 times the coefficient of x^2.

Solution

Well,

$$\begin{aligned}\binom{10}{5}p^5q^5 = 252 \Rightarrow 252p^5q^5 &= 252\\ \Rightarrow p^5q^5 &= 1\\ \Rightarrow (pq)^5 &= 1\\ \Rightarrow pq &= 1 \quad (1)\end{aligned}$$

and

$$\begin{aligned}\binom{10}{3}p^7q^3 = 6\times\binom{10}{2}p^8q^2 &\Rightarrow 120p^7q^3 = 270p^8q^2\\ &\Rightarrow 120p^7q^3 - 270p^8q^2 = 0\\ &\Rightarrow 30p^7q^2(4q-9p) = 0\\ &\Rightarrow 4q - 9p = 0\\ &\Rightarrow 4q = 9p\\ &\Rightarrow q = \tfrac{9}{4}p \quad (2).\end{aligned}$$

Insert (2) into (1):

$$\begin{aligned}\tfrac{9}{4}p^2 = 1 &\Rightarrow p^2 = \tfrac{4}{9}\\ &\Rightarrow \underline{\underline{p = \tfrac{2}{3}}}\\ &\Rightarrow \underline{\underline{q = \tfrac{3}{2}}},\end{aligned}$$

as both p and q are positive constants.

10. Variables x and y are such that

$$y = \mathrm{e}^{2x} + \mathrm{e}^{-2x}.$$

(a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. (2)

Solution

$$y = \mathrm{e}^{2x} + \mathrm{e}^{-2x} \Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = 2\mathrm{e}^{2x} - 2\mathrm{e}^{-2x}}}.$$

(b) By using the substitution $u = \mathrm{e}^{2x}$, find the value of y when $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3$. (4)

Solution

Well,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 3 &\Rightarrow 2\mathrm{e}^{2x} - 2\mathrm{e}^{-2x} = 3\\ &\Rightarrow 2u - 2u^{-1} = 3\end{aligned}$$

multiply by u:

$$\begin{aligned}&\Rightarrow 2u^2 - 2 = 3u\\ &\Rightarrow 2u^2 - 3u - 2 = 0\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & -3 \\ \text{multiply to:} & (+2)\times(-2) = -4\end{array}\right\} -4,\ +1$$

e.g.,

$$\begin{aligned}
&\Rightarrow 2u^2 - 4u + u - 2 = 0 \\
&\Rightarrow 2u(u-2) + 1(u-2) = 0 \\
&\Rightarrow (2u+1)(u-2) = 0 \\
&\Rightarrow u = -\tfrac{1}{2} \text{ or } u = 2 \\
&\Rightarrow \mathrm{e}^{2x} = -\tfrac{1}{2} \text{ (cannot happen) or } \mathrm{e}^{2x} = 2 \\
&\Rightarrow 2x = \ln 2 \\
&\Rightarrow \underline{\underline{x = \tfrac{1}{2}\ln 2}}.
\end{aligned}$$

(c) Given that x is decreasing at the rate of 0.5 units s^{-1}, find the corresponding rate of change of y when $x = 1$. (3)

Solution

Well,

$$x = 1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2\mathrm{e}^2 - 2\mathrm{e}^{-2}$$

and

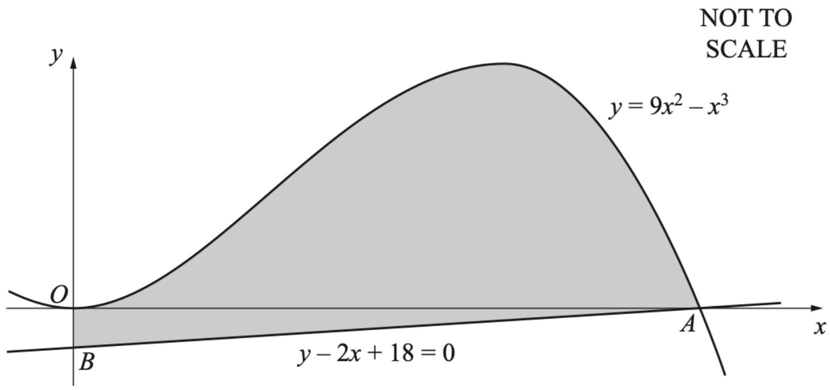
$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\mathrm{d}y}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t} \\
&= (2\mathrm{e}^2 - 2\mathrm{e}^{-2}) \times (-0.5) \\
&= \underline{\underline{\mathrm{e}^{-2} - \mathrm{e}^2}}.
\end{aligned}$$

EITHER

11. The diagram shows part of the curve

$$y = 9x^2 - x^3,$$

which meets the x-axis at the origin O and at the point A.



The line

$$y - 2x + 18 = 0$$

passes through A and meets the y-axis at the point B.

(a) Show that, for $x \geqslant 0$, (4)

$$9x^2 - x^3 \leqslant 108.$$

Solution

$$\begin{aligned} 9x^2 - x^3 = 0 &\Rightarrow x^2(9 - x) = 0 \\ &\Rightarrow x = 0 \text{ or } x = 9, \end{aligned}$$

so $A(9, 0)$.

Well,

$$y = 9x^2 - x^3 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 18x - 3x^2$$

and

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 18x - 3x^2 = 0 \\ &\Rightarrow 3x(6 - x) = 0 \\ &\Rightarrow x = 0 \text{ or } x = 6. \end{aligned}$$

Now,

$$x = 0 \Rightarrow y = 0$$

and

$$x = 6 \Rightarrow y = 108;$$

so the minimum is $(0, 0)$ and the maximum is $(6, 108)$. Hence,

$$0 \leqslant x \leqslant 9 \Rightarrow \underline{\underline{9x^2 - x^3 \leqslant 108}},$$

as required.

(b) Find the area of the shaded region bounded by the curve, the line AB, and the y-axis. (6)

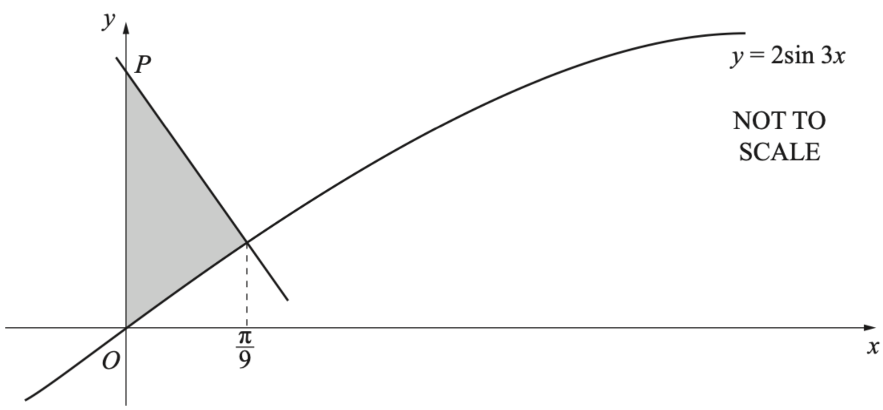
Solution

$$\begin{aligned}
\text{shaded region} &= \int_0^9 \left[(9x^2 - x^3) - (2x - 18)\right] \mathrm{d}x \\
&= \int_0^9 (18 - 2x + 9x^2 - x^3)\, \mathrm{d}x \\
&= \left[18x - x^2 + 3x^3 - \tfrac{1}{4}x^4\right]_{x=0}^{9} \\
&= \left(162 - 81 + 2\,187 - 1\,640\tfrac{1}{4}\right) - (0 - 0 + 0 - 0) \\
&= \underline{\underline{627\tfrac{3}{4}}}.
\end{aligned}$$

OR

12. The diagram shows part of the curve

$$y = 2 \sin 3x.$$



The normal to the curve

$$y = 2 \sin 3x$$

at the point where $x = \frac{1}{9}\pi$ meets the y-axis at the point P.

(a) Find the coordinates of P. (5)

Solution
Well,

$$x = \tfrac{1}{9}\pi \Rightarrow y = \sqrt{3}$$

so the point is $(\frac{1}{9}\pi, \sqrt{3})$. Now,

$$y = 2\sin 3x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 6\cos 3x.$$

Next,

$$x = \tfrac{1}{9}\pi \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3$$

and

$$m_{\text{normal}} = -\tfrac{1}{3}.$$

The normal to the point has the equation

$$y - \sqrt{3} = -\tfrac{1}{3}(x - \tfrac{1}{9}\pi)$$

and

$$\begin{aligned} x = 0 \Rightarrow y - \sqrt{3} &= -\tfrac{1}{3}(0 - \tfrac{1}{9}\pi) \\ \Rightarrow y - \sqrt{3} &= \tfrac{1}{27}\pi \\ \Rightarrow y &= \sqrt{3} + \tfrac{1}{27}\pi; \end{aligned}$$

hence, $\underline{\underline{P(0, \sqrt{3} + \frac{1}{27}\pi)}}$.

(b) Find the area of the shaded region bounded by the curve, the normal, and the y-axis. (5)

Solution
Well,

$$\begin{aligned} \int_0^{\frac{1}{9}\pi} 2\sin 3x \,\mathrm{d}x &= \left[-\tfrac{2}{3}\cos 3x\right]_{x=0}^{\frac{1}{9}\pi} \\ &= -\tfrac{1}{3} - (-\tfrac{2}{3}) \\ &= \tfrac{1}{3}. \end{aligned}$$

Hence,

$$\begin{aligned}\text{shaded region} &= \text{trapezium} - \text{integral}\\ &= \left[\tfrac{1}{2} \times (\sqrt{3} + \tfrac{1}{27}\pi + \sqrt{3}) \times \tfrac{1}{9}\pi\right] - \tfrac{1}{3}\\ &= \underline{\underline{\tfrac{1}{18}\pi(2\sqrt{3} + \tfrac{1}{27}\pi) - \tfrac{1}{3}}}.\end{aligned}$$