Dr Oliver Mathematics
Further Mathematics: Further Pure Mathematics 2 (Paper 4A)
November 2021: Calculator
1 hour 30 minutes

The total number of marks available is 75.
You must write down all the stages in your working.
Inexact answers should be given to three significant figures unless otherwise stated.

1. Without performing any division, explain why (4)

$$n = 20\,210\,520$$

is divisible by 66.

2. A binary operation $\star$ on the set of non-negative integers, $\mathbb{Z}_0^+$, is defined by

$$m \star n = |m - n|,\ m\,, n \in \mathbb{Z}_0^+.$$

   (a) Explain why $\mathbb{Z}_0^+$ is closed under the operation $\star$. (1)

   (b) Show that 0 is an identity for $(\mathbb{Z}_0^+, \star)$. (2)

   (c) Show that all elements of $\mathbb{Z}_0^+$ have an inverse under $\star$. (2)

   (d) Determine if $\mathbb{Z}_0^+$ forms a group under $\star$, giving clear justification for your answer. (3)

3. (a) Use the Euclidean Algorithm to find integers a and b such that (5)

$$125a + 87b = 1.$$

   (b) Hence write down a multiplicative inverse of 87 modulo 125. (1)

   (c) Solve the linear congruence (2)

$$87x \equiv 16 \pmod{125}.$$

4. Let G be a group of order $46^{46} + 47^{47}$. (7)

   Using Fermat's Little Theorem and explaining your reasoning, determine which of the following are possible orders for a subgroup of G

   (a) 11,

   (b) 21.

5. The point P in the complex plane represents a complex number z such that

$$|z+9| = 4|z-12\mathrm{i}|.$$

Given that, as z varies, the locus of P is a circle,

(a) determine the centre and radius of this circle. (6)

(b) Shade on an Argand diagram the region defined by the set (4)

$$\{z \in \mathbb{C} : |z+9| < 4|z-12\mathrm{i}|\} \cap \left\{z \in \mathbb{C} : -\tfrac{1}{4}\pi < \arg(z - \tfrac{1}{5}(3+44\mathrm{i})) < \tfrac{1}{4}\pi\right\}.$$

6. A recurrence system is defined by (6)

$$\begin{aligned} u_{n+2} &= 9(n+1)^2 u_n - 3u_{n+1},\ n \geqslant 1, \\ u_1 &= -3, \text{ and} \\ u_2 &= 18. \end{aligned}$$

Prove by induction that, for $n \in \mathbb{N}$,

$$u_n = (-3)^n\, n!.$$

7.

$$I_n = \int t^n \sqrt{4+5t^2}\,\mathrm{d}t.$$

(a) Show that, for $n > 1$, (5)

$$I_n = \frac{t^{n-1}}{5(n+2)}(4+5t^2)^{\frac{3}{2}} - \frac{4(n-1)}{5(n+2)} I_{n-2}.$$

The curve shown in Figure 1 is defined by the parametric equations

$$x = \tfrac{1}{\sqrt{5}}t^5,\ y = \tfrac{1}{2}t^4,\ 0 \leqslant t \leqslant 1.$$

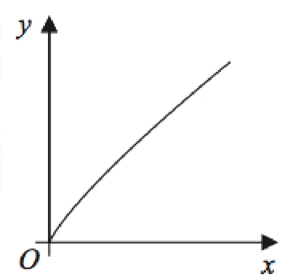

Figure 1: $x = \frac{1}{\sqrt{5}}t^5$, $y = \frac{1}{2}t^4$

This curve is rotated through 2π radians about the x-axis to form a hollow open shell.

(b) Show that the external surface area of the shell is given by (5)

$$\pi \int_0^1 t^7 \sqrt{4 + 5t^2}\, \mathrm{d}t.$$

(c) determine the value of the external surface area of the shell, giving your answer to 3 significant figures. (5)

8.

$$\mathbf{A} = \begin{pmatrix} 5 & -2 & 5 \\ 0 & 3 & p \\ -6 & 6 & -4 \end{pmatrix}.$$

Given that

$$\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

is an eigenvector for $\mathbf{A}$,

(a) (i) determine the eigenvalue corresponding to this eigenvector, (1)

(ii) hence show that $p = 2$, (2)

(iii) determine the remaining eigenvalues and corresponding eigenvectors of $\mathbf{A}$. (7)

(b) Write down a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that (1)

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}.$$

(c) (i) Solve the differential equation (2)

$$\dot{u} = ku,$$

where k is a constant.

With respect to a fixed origin O, the velocity of a particle moving through space is modelled by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

By considering

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \mathbf{P}^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

so that

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \mathbf{P}^{-1} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix},$$

(ii) determine a general solution for the displacement of the particle. (4)