Dr Oliver Mathematics
OCR FMSQ Additional Mathematics
2017 Paper
2 hours

The total number of marks available is 100.
You must write down all the stages in your working.
You are permitted to use a scientific or graphical calculator in this paper.
Final answers should be given correct to three significant figures where appropriate.

Section A

1. Solve the inequality (3)

$$-2 < 3x + 1 < 7.$$

Solution

$$\begin{aligned} -2 < 3x + 1 < 7 &\Rightarrow -3 < 3x < 6 \\ &\Rightarrow \underline{\underline{-1 < x < 2}}. \end{aligned}$$

2. Find the equation of the line which is perpendicular to the line with equation $2x+3y = 4$ and which passes through the point $(3, -1)$. (4)

Solution

$$\begin{aligned} 2x + 3y = 4 &\Rightarrow 3y = -2x + 4 \\ &\Rightarrow y = -\tfrac{2}{3}x + \tfrac{4}{3}. \end{aligned}$$

So, the gradient of this line is $-\frac{2}{3}$ and the gradient of the normal is

$$\frac{-1}{-\frac{2}{3}} = \tfrac{3}{2}.$$

Hence, the equation is

$$\begin{aligned} y + 1 = \tfrac{3}{2}(x - 3) &\Rightarrow y + 1 = \tfrac{3}{2}x - \tfrac{9}{2} \\ &\Rightarrow \underline{\underline{y = \tfrac{3}{2}x - \tfrac{11}{2}}}. \end{aligned}$$

3. Find the equation of the tangent to the curve (4)

$$y = x^2 - 3x$$

at the point $(4, 4)$.

Solution

$$y = x^2 - 3x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 3$$

and

$$x = 4 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 5.$$

Hence, the equation of the tangent is

$$\begin{aligned} y - 4 = 5(x - 4) &\Rightarrow y - 4 = 5x - 20 \\ &\Rightarrow \underline{\underline{y = 5x - 16}}. \end{aligned}$$

4. The coordinates of A and B are $(1, 5)$ and $(-3, 7)$ respectively.

   (a) Calculate the **exact** length of AB. (2)

   Solution

$$\begin{aligned} AB &= \sqrt{[(-3-1)^2 + (7-5)^2} \\ &= \sqrt{(-4)^2 + 2^2} \\ &= \sqrt{16 + 4} \\ &= \sqrt{20} \\ &= \sqrt{4 \times 5} \\ &= \sqrt{4} \times \sqrt{5} \\ &= \underline{\underline{2\sqrt{5}}}. \end{aligned}$$

   (b) Find the coordinates of the midpoint of AB. (1)

   Solution

$$\left(\frac{1 + (-3)}{2}, \frac{5 + 7}{2} \right) = \underline{\underline{(-1, 6)}}.$$

5. (a) Find the equation of the circle which has its centre at the origin and passes through the point $(1, 7)$. (2)

Solution

$$x^2 + y^2 = 1^2 + 7^2 \Rightarrow \underline{\underline{x^2 + y^2 = 50}}.$$

(b) Find the coordinates of the two points where the line $2x + y = 15$ cuts this circle. (4)

Solution

$$2x + y = 15 \Rightarrow y = -2x + 15$$

and substitute:

$$\begin{aligned} x^2 + y^2 = 50 &\Rightarrow x^2 + (-2x + 15)^2 = 50 \\ &\Rightarrow x^2 + (4x^2 - 60x + 225) = 50 \\ &\Rightarrow 5x^2 - 60x + 175 = 0 \\ &\Rightarrow 5(x^2 - 12x + 35) = 0 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & -12 \\ \text{multiply to:} & +35 \end{array}\right\} -7, -5$$

$$\begin{aligned} &\Rightarrow 5(x - 5)(x - 7) = 0 \\ &\Rightarrow x = 5 \text{ or } x = 7 \\ &\Rightarrow y = 5 \text{ or } y = 1; \end{aligned}$$

hence, the points are $\underline{\underline{(5, 5)}}$ and $\underline{\underline{(7, 1)}}$.

6. You are given that the equation

$$x^3 - x^2 - 10x + 6 = 0$$

has two non-integer positive roots and one negative integer root.

(a) Using the factor theorem, find the negative root. (2)

Solution

Let

$$\mathrm{f}(x) = x^3 - x^2 - 10x + 6.$$

Then

$$\begin{aligned}
\mathrm{f}(-1) &= 14 \\
f(-2) &= 14 \\
f(-3) &= 0
\end{aligned}$$

and so $x = -3$ is a root.

(b) Hence solve the equation. (4)

Solution

We use synthetic division:

-3	1	-1	-10	6
	$\downarrow$	-3	12	-6
	1	-4	2	0

Hence,

$$x^3 - x^2 - 10x + 6 = (x+3)(x^2 - 4x + 2).$$

Now,

$$\begin{aligned}
x^2 - 4x + 2 = 0 &\Rightarrow x^2 - 4x + 4 = 2 \\
&\Rightarrow (x-2)^2 = 2 \\
&\Rightarrow x - 2 = \pm\sqrt{2} \\
&\Rightarrow x = 2 \pm \sqrt{2}.
\end{aligned}$$

Hence, the solutions are

$$\underline{\underline{-3,\ 2 \pm \sqrt{2}}}.$$

7. (a) Find (4)

$$\int_3^5 (x^2 - 7)\,\mathrm{d}x.$$

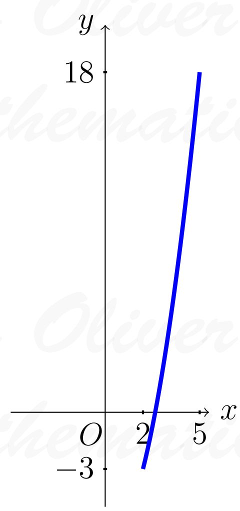
Solution

$$\int_3^5 (x^2 - 7)\,\mathrm{d}x = \left[\tfrac{1}{3}x^3 - 7x\right]_{x=3}^{5}$$
$$= \left(41\tfrac{2}{3} - 35\right) - (9 - 21)$$
$$= \underline{\underline{18\tfrac{2}{3}}}.$$

(b) Explain by means of a sketch why the area between the curve $y = x^2 - 7$ and the lines $x = 2$ and $x = 5$ is not (2)

$$\int_3^5 (x^2 - 7)\,\mathrm{d}x.$$

Solution



Because, in the graph, the portion of $2 \leqslant x < \sqrt{7}$ is <u>below the x-axis</u> and the portion of $\sqrt{7} < x \leqslant 5$ is <u>above the x-axis</u>. In fact,

$$\text{area} = \underline{\underline{-\int_2^{\sqrt{7}} (x^2 - 7)\,\mathrm{d}x + \int_{\sqrt{7}}^5 (x^2 - 7)\,\mathrm{d}x}}.$$

8. Four ordinary six-sided dice are rolled. Find the probability that at least 2 sixes are obtained. (6)

Solution

$$\begin{aligned} \text{P(at least 2 sixes)} &= \text{P(2 sixes)} + \text{P(3 sixes)} + \text{P(4)} \\ &= \binom{4}{2}(\tfrac{1}{6})^2(\tfrac{5}{6})^2 + \binom{4}{3}(\tfrac{1}{6})^3(\tfrac{5}{6}) + (\tfrac{1}{6})^4 \\ &= \tfrac{25}{216} + \tfrac{5}{324} + \tfrac{1}{1\,296} \\ &= \underline{\underline{\tfrac{19}{144} \text{ or } 0.132 \text{ (3 sf)}}}. \end{aligned}$$

9. A car moves from rest away from traffic lights such that after t seconds its velocity, v ms^{-1}, is given by

$$v = \tfrac{7}{128}(12t^2 - t^3).$$

(a) Show that the acceleration is 0 when $t = 0$ and $t = 8$. (3)

Solution

$$\begin{aligned} v = \tfrac{7}{128}(12t^2 - t^3) &\Rightarrow v = \tfrac{21}{32}t^2 - \tfrac{7}{128}t^3 \\ &\Rightarrow a = \tfrac{21}{16}t - \tfrac{21}{128}t^2 \\ &\Rightarrow a = \tfrac{21}{128}t(8 - t) \end{aligned}$$

Now,

$$t = 0 \Rightarrow a = \underline{\underline{0 \text{ ms}^{-2}}}$$

and

$$t = 8 \Rightarrow a = \underline{\underline{0 \text{ ms}^{-2}}}.$$

(b) Find the distance travelled in the first 8 seconds. (4)

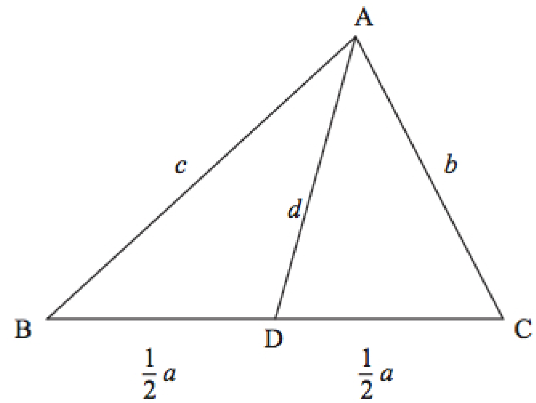
Solution

$$v = \tfrac{21}{32}t^2 - \tfrac{7}{128}t^3 \Rightarrow s = \tfrac{7}{32}t^3 - \tfrac{7}{512}t^4 + c,$$

for some constant c. Hence,

$$\begin{aligned} \text{distance} &= s(8) - s(0) \\ &= (112 - 56 + c) - (0 + 0 + c) \\ &= \underline{\underline{56 \text{ m}}}. \end{aligned}$$

10. The triangle shown in the diagram below is such that $AB = c$, $BC = a$, and $CA = b$.



D is the midpoint of the line BC so that $BD = DC = \frac{1}{2}a$ and $AD = d$.

(a) Write down a formula for $\cos ADC$ in terms of d, b, and a. (1)

Solution

$$\cos ADC = \frac{(\frac{1}{2}a)^2 + d^2 - b^2}{2(\frac{1}{2}a)d} = \frac{\frac{1}{4}a^2 + d^2 - b^2}{ad}.$$

(b) Write down a formula for $\cos ADB$ in terms of d, c, and a. (1)

Solution

$$\cos ADB = \frac{(\frac{1}{2}a)^2 + d^2 - c^2}{2(\frac{1}{2}a)d} = \frac{\frac{1}{4}a^2 + d^2 - c^2}{ad}.$$

(c) Using the property that connects angles ADB and ADC, show that (3)

$$d^2 = \frac{2b^2 + 2c^2 - a^2}{4}.$$

Solution

$$\begin{aligned}\cos ADC = -\cos ADB &\Rightarrow \frac{\frac{1}{4}a^2 + d^2 - b^2}{ad} = -\left(\frac{\frac{1}{4}a^2 + d^2 - c^2}{ad}\right)\\ &\Rightarrow \tfrac{1}{4}a^2 + d^2 - b^2 = -\tfrac{1}{4}a^2 - d^2 + c^2\\ &\Rightarrow 2d^2 = b^2 + c^2 - \tfrac{1}{2}a^2\\ &\Rightarrow 4d^2 = 2b^2 + 2c^2 - a^2\\ &\Rightarrow \underline{\underline{d^2 = \frac{2b^2 + 2c^2 - a^2}{4}}},\end{aligned}$$

as required.

(d) In the triangle ABC where $AB = 9$, $AC = 7$, and $BC = 10$, find the exact length of the line from A to the midpoint of BC. (2)

Solution

$c = 9$, $b = 7$, and $a = 10$:

$$\begin{aligned}d^2 = \frac{2b^2 + 2c^2 - a^2}{4} &\Rightarrow d^2 = \frac{98 + 162 - 100}{4}\\ &\Rightarrow d^2 = 40\\ &\Rightarrow \underline{\underline{d = 2\sqrt{10}}}.\end{aligned}$$

Section B

11. A farmer conducts a trial on plots of land to decide what amounts of fertiliser will yield the greatest crop. He knows that if he uses no fertiliser then the average yield is 24 tonnes per plot. He finds from his trial that if he uses 2 kg per plot then the average yield is 34 tonnes and if he uses 4 kg per plot then the average yield is 32 tonnes. All plots in the trial have the same area. He decides to use the equation

$$y = -x^3 + ax^2 + bx + c,$$

where the amount of fertiliser, x, in kg produces a yield, y, in tonnes of crop.

(a) Show that $c = 24$. (1)

Solution

$$\begin{aligned} x = 0,\ y = 24 &\Rightarrow 24 = 0 + 0 + 0 + c \\ &\Rightarrow \underline{\underline{c = 24}}, \end{aligned}$$

as required.

(b) Using the data above, show that (5)

$$y = -x^3 + \tfrac{9}{2}x^2 + 24.$$

Solution

$$\begin{aligned} x = 2,\ y = 34 &\Rightarrow 34 = -8 + 4a + 2b + 24 \\ &\Rightarrow 4a + 2b = 18 \quad (1) \end{aligned}$$

and

$$\begin{aligned} x = 4,\ y = 32 &\Rightarrow 32 = -64 + 16a + 4b + 24 \\ &\Rightarrow 16a + 4b = 72 \quad (2). \end{aligned}$$

Now, $2 \times (1)$ gives

$$8a + 4b = 36 \quad (3)$$

and $(2) - (3)$ gives

$$8a = 36 \Rightarrow \underline{\underline{a = \tfrac{9}{2}}}$$

$$\begin{aligned} &\Rightarrow 4(\tfrac{9}{2}) + 2b = 18 \\ &\Rightarrow 18 + 2b = 18 \\ &\Rightarrow 2b = 0 \\ &\Rightarrow \underline{\underline{b = 0}}, \end{aligned}$$

as required.

(c) Using calculus, find the amount of fertiliser that should be used to maximise the yield and find the yield for this amount of fertiliser. (6)

Solution

$$y = -x^3 + \tfrac{9}{2}x^2 + 24 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -3x^2 + 9x$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow -3x^2 + 9x = 0$$
$$\Rightarrow -3x(x-3) = 0$$
$$\Rightarrow x = 0 \text{ or } x = 3.$$

Now,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -3x^2 + 9x \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -6x + 9$$

and

$$x = 0 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 9 > 0$$
$$x = 3 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -9 < 0;$$

hence, $\underline{\underline{x = 3 \text{ kg per plot}}}$ and this corresponds to

$$y = -27 + 40\tfrac{1}{2} + 24 = \underline{\underline{37\tfrac{1}{2} \text{ tonnes}}}.$$

12. A school wishes to transport students and teachers totalling 300 people to a concert. It uses a coach firm that can provide minibuses which can seat 10 or coaches that can seat 30. The coach firm has 15 minibuses and 8 coaches that it can hire out.

Let x be the number of minibuses that the school hires and y be the number of coaches the school hires.

(a) Write down an inequality in x and y that must be met in order to transport the students and teachers. (1)

Solution
$10x + 30y \geqslant 300 \Rightarrow \underline{\underline{x + 3y \geqslant 30}}$.

(b) State two more inequalities regarding the maximum number of each vehicle that can be hired. (1)

Solution
$\underline{\underline{x \leqslant 15}}$ and $\underline{\underline{y \leqslant 8}}$.

It costs £100 to hire a minibus and £150 to hire a coach. The school allocates a maximum of £2 400 for the hire of the vehicles.

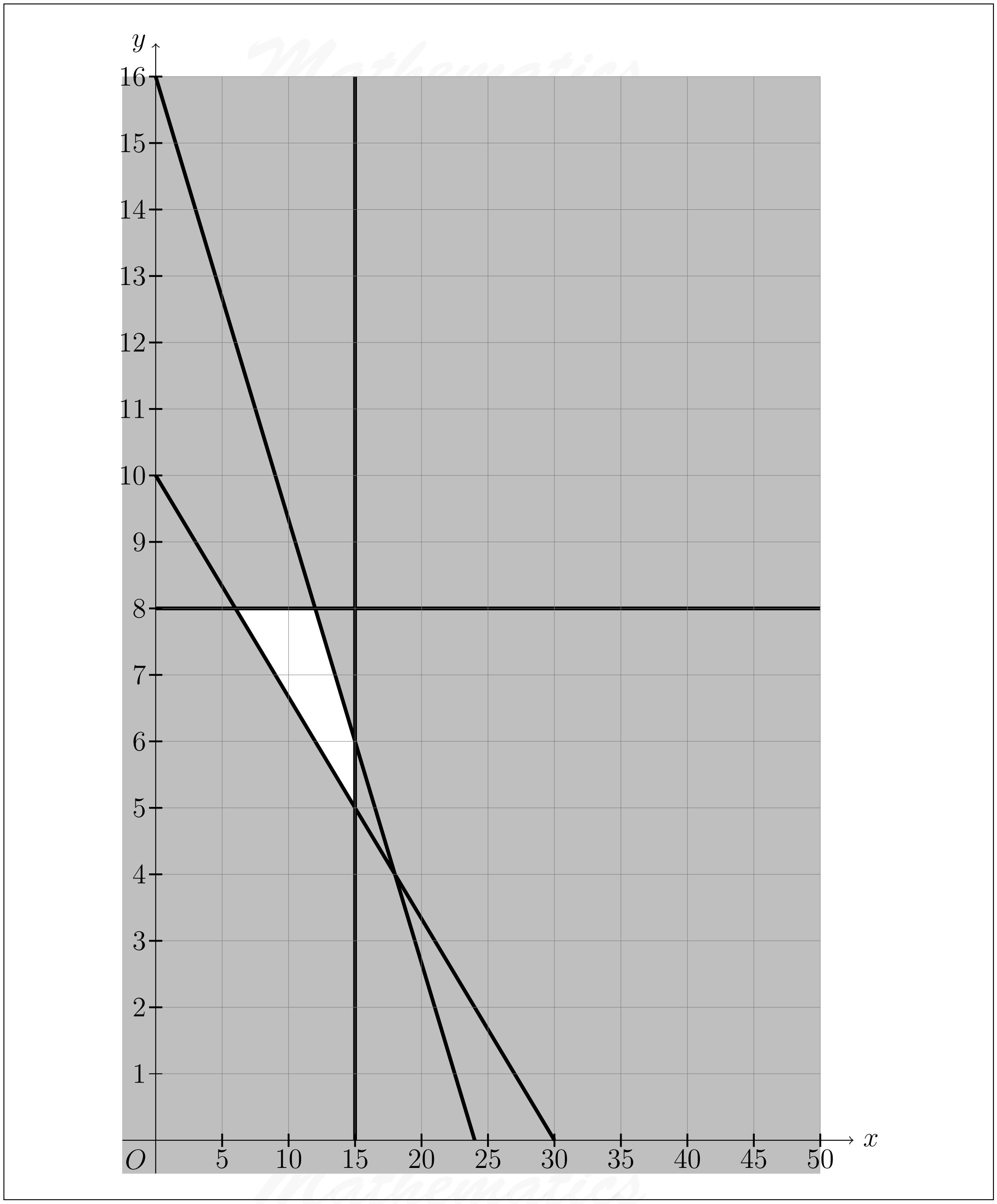
(c) Write down another inequality to represent this cost requirement. (1)

Solution

$100x + 150y \leqslant 2\,400 \Rightarrow \underline{\underline{2x + 3y \leqslant 48}}$.

(d) Plot the 4 inequalities on the grid provided. You should shade the region that does **not** satisfy the inequalities. (5)

Solution



One teacher suggests that the best arrangement is to hire as many minibuses as possible.

(e) From your graph find the combination of minibuses and coaches that achieves this for as small a cost as possible and the number of vehicles used. (2)

Solution

15 minibuses and 5 coaches $\Rightarrow$ 20 vehicles.

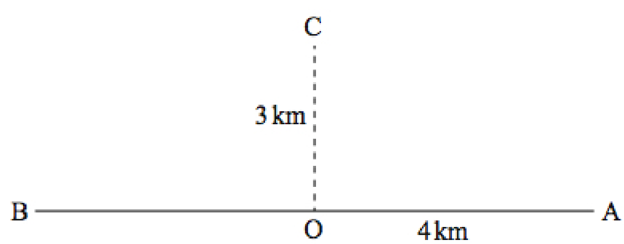
The organiser decides, however, to use the combination of coaches and minibuses that minimises the cost.

(f) From your graph find this combination and the minimum cost. (2)

Solution

$$\begin{aligned}\text{6 minibuses and 8 coaches} \Rightarrow \text{cost} &= (6 \times 100) + (8 \times 150)\\ \Rightarrow \text{cost} &= 600 + 1\,200\\ \Rightarrow \text{cost} &= \underline{\underline{\pounds 1\,800}}.\end{aligned}$$

13. A path AB crosses a section of moorland in an east-west direction. John wishes to walk from A to a point C which is due north of a point O on the path AB as shown in the figure below. A is 4 km due east of O and C is 3 km due north of O.



On the path John can walk at 5 km/hr and on the moorland he can only walk at 2 km/hr.

(a) Find the time he takes to walk from A to C

(i) along the path to O and then up to C across the moor, (2)

Solution

$$\begin{aligned}\text{Time} &= \frac{4}{5} + \frac{3}{2}\\ &= 0.8 + 1.5\\ &= \underline{\underline{\text{2 hours 18 minutes}}}.\end{aligned}$$

(ii) direct from A to C across the moor. (3)

Solution

$$AC = \sqrt{3^2 + 4^2} = 5$$

and so

$$\begin{aligned} \text{time} &= \frac{5}{2} \\ &= \underline{\underline{2 \text{ hours } 30 \text{ minutes}}}. \end{aligned}$$

John finds that he can minimise the time taken to walk from A to C if he sets off towards O on the path but at X, a distance of x km from A, he turns to walk directly to C across the moor.

(b) (i) Find an expression for the time, t hours, that he takes to complete this walk. (4)

Solution

$$\text{Time} = \underline{\underline{\frac{x}{5} + \frac{\sqrt{3^2 + (4-x)^2}}{2}}}.$$

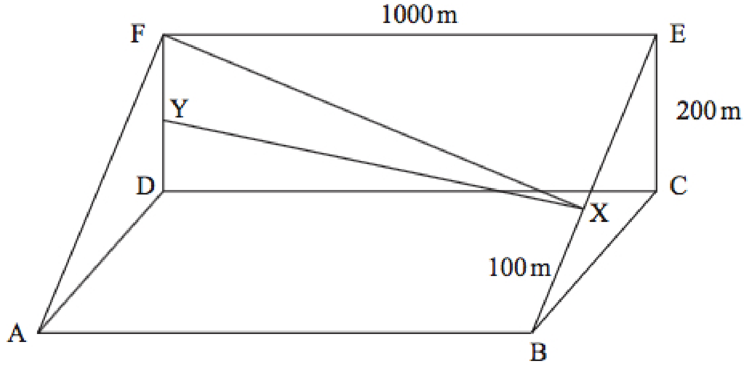
(ii) Using this expression and by substituting values for x, $x = 2.6$, 2.7, and 2.8, show that there is justification for $x = 2.7$ being the distance for which the time taken to walk from A to C is a minimum. (3)

Solution

x	t
2.6	2.175 294 ...
2.7	2.174 778 ...
2.8	2.175 549 ...

t is decreasing from $x = 0$ to $x = 2.7$ and then seems to increase from $x = 2.7$ to $x = 4$.

14. A hillside can be modelled by a prism $ABCDEF$, as shown in the figure below. $ABCD$ is a horizontal rectangle and $DCEF$ is a rectangle in the vertical plane. BCE and ADF are right-angled triangles in the vertical plane. The angle of slope $EBC = FAD = 28°$. $AB = DC = FE = 1000$ m and $EC = FD = 200$ m.



John sets off from B walking up the line BE to a point X where $BX = 100$ m. He then walks across the slope directly to F, as shown in the diagram. Y is on FD such that XY is horizontal.

(a) Find the height of X above the base line BC. (2)

Solution

$$\begin{aligned} \sin = \frac{\text{opp}}{\text{hyp}} &\Rightarrow \sin 28^\circ = \frac{h}{100} \\ &\Rightarrow h = 100 \sin 28^\circ \\ &\Rightarrow h = 46.947\,156\,28 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{h = 47.0 \text{ m (3 sf)}}}. \end{aligned}$$

(b) Find the length FX. (5)

Solution

$$\begin{aligned} \sin = \frac{\text{opp}}{\text{hyp}} &\Rightarrow \sin 28^\circ = \frac{200}{BE} \\ &\Rightarrow BE = \frac{200}{\sin 28^\circ} \\ &\Rightarrow XE = \frac{200}{\sin 28^\circ} - 100 \\ &\Rightarrow FX = \sqrt{1\,000^2 + \left(\frac{200}{\sin 28^\circ} - 100\right)^2} \\ &\Rightarrow FX = 1\,051.799\,935 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{FX = 1\,050 \text{ m (3 sf)}}}. \end{aligned}$$

(c) Hence calculate the angle of slope of the line FX. (5)

Solution

$$FY = 200 - h$$

and

$$\begin{aligned}\sin = \frac{\text{opp}}{\text{hyp}} &\Rightarrow \sin \angle FXY = \frac{FY}{FX} \\ &\Rightarrow \angle FXY = \sin^{-1}\left(\frac{153.052\ldots}{1\,051.799\ldots}\right) \\ &\Rightarrow \angle FXY = 8.367\,112\,531 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{\angle FXY = 8.37^\circ \text{ (3 sf)}}}.\end{aligned}$$