Dr Oliver Mathematics
Mathematics: Higher
2009 Paper 2: Calculator
1 hour 10 minutes

The total number of marks available is 60.
You must write down all the stages in your working.

1. Find the coordinates of the turning points of the curve with equation (8)

$$y = x^3 - 3x^2 - 9x + 12$$

and determine their nature.

Solution

$$\begin{aligned} y = x^3 - 3x^2 - 9x + 12 &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 6x - 9 \\ &\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x - 6. \end{aligned}$$

Now,

$$\begin{aligned} 3x^2 - 6x - 9 = 0 &\Rightarrow 3(x^2 - 2x - 3) = 0 \\ &\Rightarrow 3(x-3)(x+1) = 0 \end{aligned}$$

and so the candidates of the stationary points are $x = -1$ and $x = 3$. Next,

$$x = -1 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -12 < 0$$

which means that $(-1, 17)$ is a local maximum turning point and

$$x = 3 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12 > 0$$

which means that $(3, -15)$ is a local minimum turning point.

2. Functions f and g are given by

$$\mathrm{f}(x) = 3x + 1 \text{ and } \mathrm{g}(x) = x^2 - 2.$$

(a) (i) Find $\mathrm{p}(x)$ where $\mathrm{p}(x) = \mathrm{f}(\mathrm{g}(x))$. (3)

Solution

$$\begin{aligned}\mathrm{p}(x) &= \mathrm{f}(x^2 - 2)\\ &= 3(x^2 - 2) + 1\\ &= \underline{\underline{3x^2 - 5}}.\end{aligned}$$

(ii) Find $\mathrm{q}(x)$ where $\mathrm{q}(x) = \mathrm{g}(\mathrm{f}(x))$.

Solution

$$\begin{aligned}\mathrm{q}(x) &= \mathrm{g}(3x + 1)\\ &= (3x + 1)^2 - 2\\ &= (9x^2 + 6x + 1) - 2\\ &= \underline{\underline{9x^2 + 6x - 1}}.\end{aligned}$$

(b) Solve (3)

$$\mathrm{p}'(x) = \mathrm{q}'(x).$$

Solution

$$\begin{aligned}\mathrm{p}'(x) = \mathrm{q}'(x) &\Rightarrow 6x = 18x + 6\\ &\Rightarrow 12x = -6\\ &\Rightarrow \underline{\underline{x = -\tfrac{1}{2}}}.\end{aligned}$$

3. (a) (i) Show that $x = 1$ is a root of (4)

$$x^3 + 8x^2 + 11x - 20 = 0.$$

Solution
Let

$$\mathrm{f}(x) = x^3 + 8x^2 + 11x - 20.$$

Then

$$f(1) = 1 + 9 + 11 - 20 = 0$$

and $\underline{\underline{x = 1}}$ is a root

(ii) Hence factorise

$$x^3 + 8x^2 + 11x - 20$$

fully.

Solution

$$\begin{aligned} x^3 + 8x^2 + 11x - 20 &= (x-1)(x^2 + 9x + 20) \\ &= \underline{\underline{(x-1)(x+4)(x+5)}}. \end{aligned}$$

(b) Solve (5)

$$\log_2(x+3) + \log_2(x^2 + 5x - 4) = 3.$$

Solution

$$\begin{aligned} \log_2(x+3) + \log_2(x^2 + 5x - 4) = 3 &\Rightarrow \log_2[(x+3)(x^2 + 5x - 4)] = 3 \\ &\Rightarrow (x+3)(x^2 + 5x - 4) = 2^3 \end{aligned}$$

$\times$	x^2	$+5x$	-4
x	x^3	$+5x^2$	$-4x$
$+3$	$+3x^2$	$+15x$	-12

$$\begin{aligned} &\Rightarrow x^3 + 8x^2 + 11x - 12 = 8 \\ &\Rightarrow x^3 + 8x^2 + 11x - 20 = 0 \\ &\Rightarrow (x-1)(x+4)(x+5) = 0 \\ &\Rightarrow x = -5, x = -4, \text{ or } x = 1; \end{aligned}$$

but $x \neq -5$ or $x \neq -4$ (why?) and we are left with $\underline{\underline{x = 1}}$.

4. (a) Show that the point $P(5, 10)$ lies on circle C_1 with equation (1)

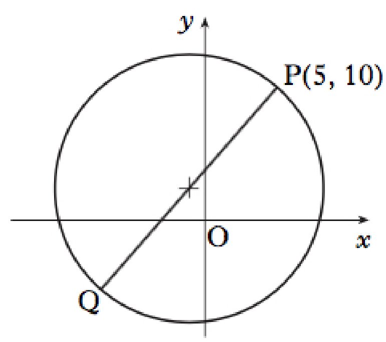
$$(x+1)^2 + (y-2)^2 = 100.$$

Solution

$$\begin{aligned}(5+1)^2 + (10-2)^2 &= 6^2 + 8^2 \\ &= 100\end{aligned}$$

and so $\underline{\underline{P(5, 10) \text{ lies on circle } C_1}}$.

PQ is a diameter of this circle as shown in the diagram.



(b) Find the equation of the tangent at Q. (5)

Solution

The circle's centre is $(-1, 2)$ and so $Q(-7, -6)$. Now,

$$\begin{aligned}m_{PQ} &= \frac{10-(-8)}{5-(-7)} \\ &= \frac{18}{12} \\ &= \tfrac{4}{3}\end{aligned}$$

and so the tangent at Q is $-\frac{3}{4}$. Hence, the equation of the tangent at Q is

$$\begin{aligned}y + 6 = -\frac{3}{4}(x+7) &\Rightarrow 4y + 24 = -3x - 21 \\ &\Rightarrow \underline{\underline{3x + 4y + 45 = 0}}.\end{aligned}$$

Two circles, C_2 and C_3, touch circle C_1 at Q.
The radius of each of these circles is twice the radius of circle C_1.

(c) Find the equations of circles C_2 and C_3. (4)

Solution
The centres for C_2 and C_3 are $(5, 10)$ and $(-19, -22)$ and the new radius is

$$2 \times 10 = 20.$$

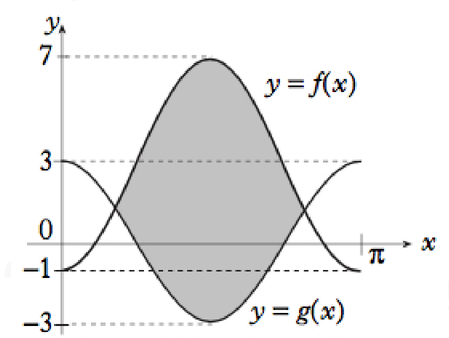
Hence, the equations of circles C_2 and C_3 are

$$\underline{\underline{(x-5)^2 + (y-10)^2 = 400}}$$

and

$$\underline{\underline{(x+19)^2 + (y+22)^2 = 400}}.$$

5. The graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$ are shown in the diagram.



$\mathrm{f}(x) = -4\cos(2x) + 3$ and $\mathrm{g}(x)$ is of the form $\mathrm{g}(x) = m\cos(nx)$.

(a) Write down the values of m and n. (1)

Solution
$\underline{\underline{m = 3}}$ and $\underline{\underline{n = 2}}$.

(b) Find, correct to one decimal place, the coordinates of the points of intersection of the two graphs in the interval $0 \leqslant x \leqslant \pi$. (5)

Solution

$$
\begin{aligned}
\mathrm{f}(x) = \mathrm{g}(x) &\Rightarrow -4\cos(2x) + 3 = 3\cos(2x) \\
&\Rightarrow 7\cos(2x) = 3 \\
&\Rightarrow \cos(2x) = \tfrac{3}{7} \\
&\Rightarrow 2x = 1.127\,885\,283, 5.155\,300\,024 \text{ (FCD)} \\
&\Rightarrow x = 0.563\,942\,641\,4, 2.577\,650\,012 \text{ (FCD)} \\
&\Rightarrow x = 0.6, 2.6 \text{ (1 dp)}
\end{aligned}
$$

Now,

$$x = 0.563\ldots \Rightarrow y = 1\tfrac{2}{7} = 1.285\ldots$$

and

$$x = 2.577\ldots \Rightarrow y = 1\tfrac{2}{7}.$$

Hence, the coordinates are $\underline{\underline{(0.6, 1.3) \text{ (1 dp)}}}$ and $\underline{\underline{(2.6, 1.3) \text{ (1 dp)}}}$

(c) Calculate the shaded area. (6)

Solution

$$
\begin{aligned}
\text{Area} &= \int_{0.563\ldots}^{2.577\ldots} [(-4\cos(2x) + 3) - 3\cos(2x)]\,\mathrm{d}x \\
&= \int_{0.563\ldots}^{2.577\ldots} [-7\cos(2x) + 3]\,\mathrm{d}x \\
&= \left[-\tfrac{7}{2}\sin(2x) + 3x\right]_{x=0.563\ldots}^{2.577\ldots} \\
&= [3.162\ldots + 7.732\ldots] - [-3.162\ldots + 1.691\ldots] \\
&= 12.365\,677\,43 \text{ (FCD)} \\
&= \underline{\underline{12.4 \text{ (1 dp)}}}.
\end{aligned}
$$

6. The size of the human population, N, can be modelled using the equation

$$N = N_0 \mathrm{e}^{rt},$$

where N_0 is the population in 2006, t is the time in years since 2006, and r is the annual rate of increase in the population.

(a) In 2006 the population of the United Kingdom was approximately 61 million, with an annual rate of increase of 1.6%. Assuming this growth rate remains constant, what would be the population in 2020? (2)

Solution

$$\begin{aligned}\text{Population} &= 61 \text{ million} e^{0.016\times 14}\\ &= 76.315\,332\,19 \text{ million (FCD)}\\ &= \underline{\underline{76.3 \text{ million (3 sf)}}}.\end{aligned}$$

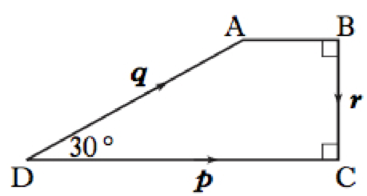
In 2006 the population of Scotland was approximately 5.1 million, with an annual rate of increase of 0.43%.

(b) Assuming this growth rate remains constant, how long would it take for Scotland's population to double in size? (3)

Solution

$$\begin{aligned}5.1e^{0.004\,3t} = 10.2 &\Rightarrow e^{0.004\,3t} = 2\\ &\Rightarrow 0.004\,3t = \ln 2\\ &\Rightarrow t = \frac{1}{0.004\,3}\ln 2\\ &\Rightarrow t = 161.197\,018\,7 \text{ (FCD)}\\ &\Rightarrow \underline{\underline{t = 161 \text{ years (3 sf)}}}.\end{aligned}$$

7. Vectors **p**, **q**, and **r** are represented on the diagram shown where angle $ADC = 30°$.



It is also given that $|\mathbf{p}| = 4$ and $|\mathbf{q}| = 3$.

(a) Evaluate (6)

$$\mathbf{p}.(\mathbf{q}+\mathbf{r}) \text{ and } \mathbf{r}.(\mathbf{p}-\mathbf{q}).$$

Solution

Well,

$$\mathbf{p} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \mathbf{q} = \begin{pmatrix} 3\cos 30^\circ \\ 3\sin 30^\circ \end{pmatrix}, \text{ and } \mathbf{r} = \begin{pmatrix} 0 \\ -3\sin 30^\circ \end{pmatrix}.$$

$$\begin{aligned}
\mathbf{p}.(\mathbf{q}+\mathbf{r}) &= \begin{pmatrix} 4 \\ 0 \end{pmatrix} . \left[\begin{pmatrix} 3\cos 30^\circ \\ 3\sin 30^\circ \end{pmatrix} + \begin{pmatrix} 0 \\ -3\sin 30^\circ \end{pmatrix} \right] \\
&= \begin{pmatrix} 4 \\ 0 \end{pmatrix} . \begin{pmatrix} 3\cos 30^\circ \\ 0 \end{pmatrix} \\
&= 12\cos 30^\circ \\
&= \underline{\underline{6\sqrt{3}}}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{r}.(\mathbf{p}-\mathbf{q}) &= \begin{pmatrix} 0 \\ -3\sin 30^\circ \end{pmatrix} . \left[\begin{pmatrix} 4 \\ 0 \end{pmatrix} - \begin{pmatrix} 3\cos 30^\circ \\ 3\sin 30^\circ \end{pmatrix} \right] \\
&= \begin{pmatrix} 0 \\ -3\sin 30^\circ \end{pmatrix} . \begin{pmatrix} 4 - 3\cos 30^\circ \\ -3\sin 30^\circ \end{pmatrix} \\
&= 9\sin^2 30^\circ \\
&= \underline{\underline{2\tfrac{1}{4}}}.
\end{aligned}$$

(b) Find (4)

$$|\mathbf{q}+\mathbf{r}| \text{ and } |\mathbf{p}-\mathbf{q}|.$$

Solution

$$\begin{aligned}
|\mathbf{q}+\mathbf{r}| &= \left| \begin{pmatrix} 3\cos 30^\circ \\ 3\sin 30^\circ \end{pmatrix} + \begin{pmatrix} 0 \\ -3\sin 30^\circ \end{pmatrix} \right| \\
&= \left| \begin{pmatrix} 3\cos 30^\circ \\ 0 \end{pmatrix} \right| \\
&= \underline{\underline{\tfrac{3}{2}\sqrt{3}}}
\end{aligned}$$

and

$$
\begin{aligned}
|\mathbf{p} - \mathbf{q}| &= \left| \begin{pmatrix} 4 \\ 0 \end{pmatrix} - \begin{pmatrix} 3\cos 30^\circ \\ 3\sin 30^\circ \end{pmatrix} \right| \\
&= \left| \begin{pmatrix} 4 - 3\cos 30^\circ \\ -3\sin 30^\circ \end{pmatrix} \right| \\
&= \sqrt{(4 - 3\cos 30^\circ)^2 + (-3\sin 30^\circ)^2} \\
&= \sqrt{(16 - 12\cos 30^\circ + 9\cos^2 30^\circ)^2 + 9\sin^2 30^\circ} \\
&= \sqrt{25 - 12\cos 30^\circ} \\
&= \underline{\underline{\sqrt{25 - 6\sqrt{3}}}}.
\end{aligned}
$$