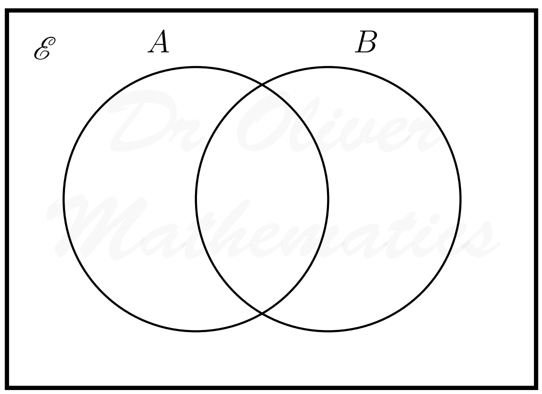
Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2011 November Paper 2 Variant 2: Calculator
2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

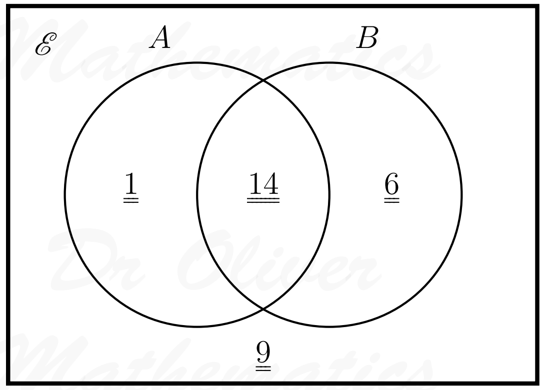
1. The universal set and the sets A and B shown in the Venn diagram below are such that

   - $\text{n}(A) = 15$,
   - $\text{n}(B) = 20$,
   - $\text{n}(A' \cap B) = 6$, and
   - $\text{n}(\mathscr{E}) = 30$.

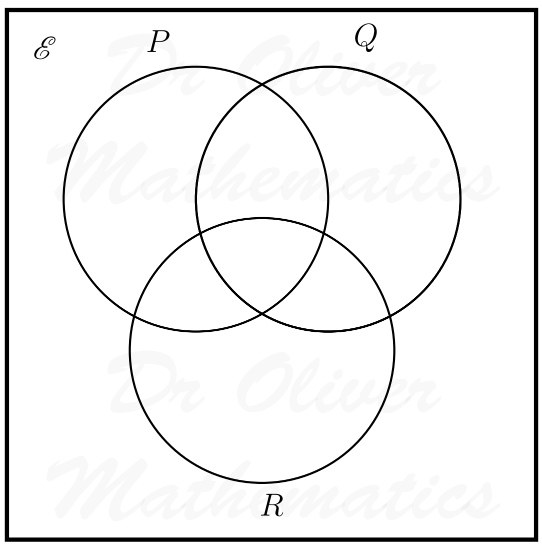
   (a) In the Venn diagram below insert the number of elements in the set represented by each of the four regions. (4)



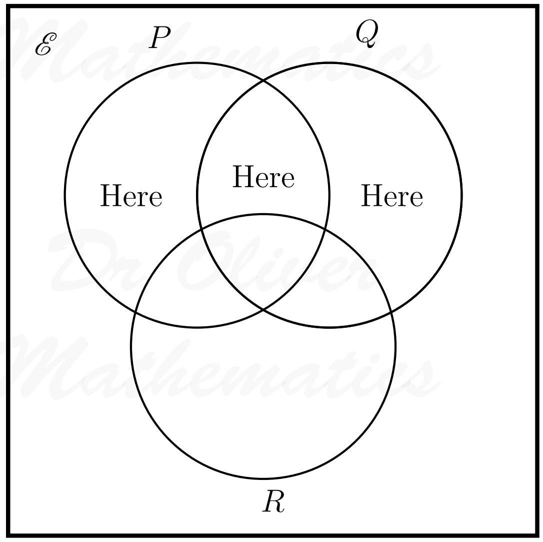
Solution



(b) In the Venn diagram below shade the region that represents $(P \cup Q) \cap R'$. (1)

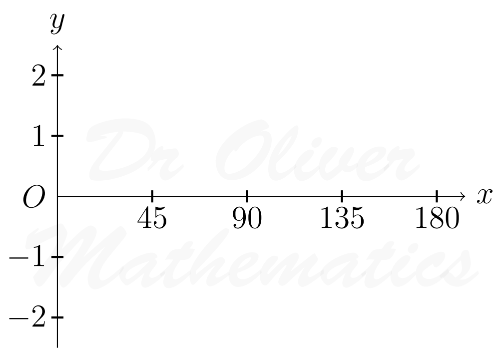


Solution

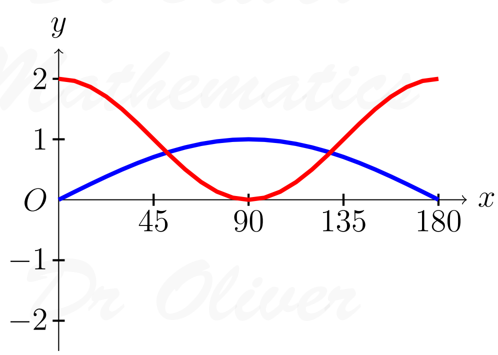


2. (a) On the grid below, draw on the same axes, for $0^\circ \leqslant x \leqslant 180^\circ$, the graphs of (3)

$$y = \sin x \text{ and } y = 1 + \cos 2x.$$



Solution



(b) State the number of roots of the equation (1)

$$\sin x = 1 + \cos 2x \text{ for } 0^\circ \leqslant x \leqslant 180^\circ.$$

Solution

$\underline{\underline{2}}$.

(c) Without extending your graphs, state the number of roots of the equation (1)

$$\sin x = 1 + \cos 2x \text{ for } 0^\circ \leqslant x \leqslant 360^\circ.$$

Solution

$\underline{\underline{2}}$.

3. It is given that $(2x - 1)$ is a factor of the expression (6)

$$4x^3 + ax^2 - 11x + b$$

and that the remainder when the expression is divided by $(x + 2)$ is 25.

Find the remainder when the expression is divided by $(x + 1)$.

Solution

We use synthetic division twice:

$\frac{1}{2}$	4	a	-11	b
	$\downarrow$	2	$\frac{1}{2}a + 1$	$\frac{1}{4}a - 5$
	4	$a + 2$	$\frac{1}{2}a - 10$	$\frac{1}{4}a + b - 5$

and

-2	4	a	-11	b
	$\downarrow$	-8	$-2a + 16$	$4a - 10$
	4	$a - 8$	$-2a + 5$	$4a + b - 10$

So

$$\tfrac{1}{4}a + b - 5 = 0 \Rightarrow b = -\tfrac{1}{4}a + 5 \quad (1)$$
$$4a + b - 10 = 25 \Rightarrow b = -4a + 35 \quad (2).$$

Do (1) = (2):

$$\begin{aligned} -\tfrac{1}{4}a + 5 = -4a + 35 &\Rightarrow \tfrac{15}{4}a = 30 \\ &\Rightarrow a = 8 \\ &\Rightarrow b = 3. \end{aligned}$$

[Check: $-4 \times 8 + 35 = 3$ ✓]

Finally,

-1	4	8	-11	3
	↓	-4	-4	15
	4	4	-15	18

Hence, the remainder is $\underline{\underline{18}}$.

4. It is given that

$$\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 1 & -5 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 1 & -4 & 2 \\ -3 & 5 & 0 \end{pmatrix}, \text{ and } \mathbf{C} = \begin{pmatrix} 4 \\ 2 \\ -7 \end{pmatrix}.$$

(a) Calculate **AB**. (2)

Solution

$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} 3 & -2 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} 1 & -4 & 2 \\ -3 & 5 & 0 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 9 & -22 & 6 \\ 16 & -29 & 2 \end{pmatrix}}}. \end{aligned}$$

(b) Calculate **BC**. (2)

Solution

$$\mathbf{BC} = \begin{pmatrix} 1 & -4 & 2 \\ -3 & 5 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ -7 \end{pmatrix}$$
$$= \underline{\underline{\begin{pmatrix} -18 \\ -2 \end{pmatrix}}}.$$

(c) Find the inverse matrix, $\mathbf{A}^{-1}$. (2)

Solution

Well,

$$\det \mathbf{A} = -15 + 2 = -13$$

and

$$\mathbf{A}^{-1} = \underline{\underline{-\tfrac{1}{13} \begin{pmatrix} -5 & 2 \\ -1 & 3 \end{pmatrix}}}.$$

5. Four boys and three girls are to be seated in a row.

Calculate the number of different ways that this can be done if

(a) the boys and girls sit alternately, (2)

Solution

$$4! \times 3! = \underline{\underline{144}}.$$

(b) the boys sit together and the girls sit together, (2)

Solution

$$2 \times 4! \times 3! = \underline{\underline{288}}.$$

(c) a boy sits at each end of the row. (2)

Solution

$$4 \times 3 \times 5! = \underline{\underline{1\,440}}.$$

6. The length of a rectangular garden is x m and the width of the garden is 10 m less than the length.

(a) Given that the perimeter of the garden is greater than 140 m, write down a linear inequality in x. (1)

Solution

Well,

$$\begin{aligned} x + (x-10) + x + (x-10) > 140 &\Rightarrow 4x - 20 > 140 \\ &\Rightarrow 4x > 160 \\ &\Rightarrow \underline{\underline{x > 40}}. \end{aligned}$$

(b) Given that the area of the garden is less than $3\,000$ m^2, write down a quadratic inequality in x. (1)

Solution

$$\underline{\underline{x(x-10) < 3\,000}}.$$

(c) By solving these two inequalities, find the set of possible values of x. (4)

Solution

Now,

$$x(x-10) < 3\,000 \Rightarrow x^2 - 10x - 3\,000 < 0$$

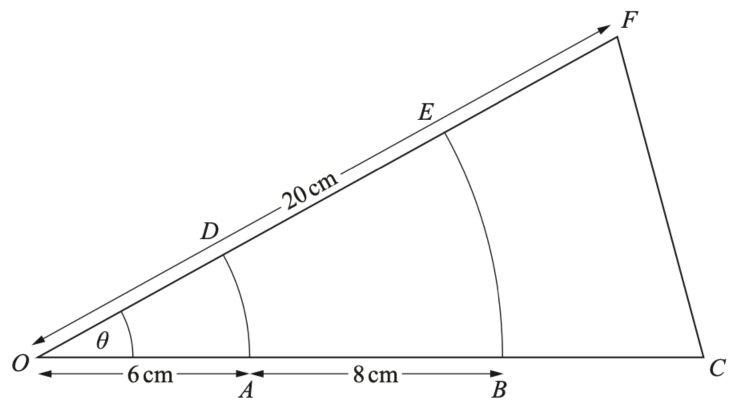
$$\left.\begin{array}{ll} \text{add to:} & -10 \\ \text{multiply to:} & 3\,000 \end{array}\right\} -60,\ +50$$

$$\begin{aligned} &\Rightarrow (x-60)(x+50) < 0 \\ &\Rightarrow -50 < x < 60. \end{aligned}$$

But $x > 40$ which means

$$\underline{\underline{40 < x < 60}}.$$

7. In the diagram, AD and BE are arcs of concentric circles centre O, where $OA = 6$ cm and $AB = 8$ cm.



- The area of the region $ABED$ is 32 cm^2.
- The triangle OCF is isosceles with $OC = OF = 20$ cm.

(a) Find the angle θ in radians. (3)

Solution
Well,

$$\begin{aligned} & \left(\tfrac{1}{2} \times 14^2 \times \theta\right) - \left(\tfrac{1}{2} \times 6^2 \times \theta\right) = 32 \\ \Rightarrow \quad & 98\theta - 18\theta = 32 \\ \Rightarrow \quad & 80\theta = 32 \\ \Rightarrow \quad & \underline{\underline{\theta = 0.4 \text{ radians}}}. \end{aligned}$$

(b) Find the perimeter of the region $BCFE$. (5)

Solution
Now,

$$BC = EF = 6 \text{ cm}$$

Next,

$$\begin{aligned} & FC^2 = OF^2 + OC^2 - 2 \times OF \times OC \times \cos FOC \\ \Rightarrow \quad & FC^2 = 20^2 + 20^2 - 2 \times 20 \times 20 \times \cos 0.4 \\ \Rightarrow \quad & FC^2 = 63.151\,204\,8 \text{ (FCD)} \\ \Rightarrow \quad & FC = 7.946\,773\,232 \text{ (FCD)}. \end{aligned}$$

Finally,

$$\begin{aligned} \text{perimeter} &= BE + EF + FC + BC \\ &= (14 \times 0.4) + 6 + 7.946\ldots + 6 \\ &= 25.546\,773\,232 \text{ (FCD)} \\ &= \underline{\underline{25.5 \text{ cm (3 sf)}}}. \end{aligned}$$

8. A particle travels in a straight line so that, t s after passing through a fixed point O, its velocity, v ms^{-1}, is given by

$$v = 12\cos(\tfrac{1}{3}t).$$

(a) Find the value of t when the velocity of the particle first equals 2 ms^{-1}. (2)

Solution

Well,

$$\begin{aligned} 12\cos(\tfrac{1}{3}t) = 2 &\Rightarrow \cos(\tfrac{1}{3}t) = \tfrac{1}{6} \\ &\Rightarrow \tfrac{1}{3}t = 1.403\,348\,248 \text{ (FCD)} \\ &\Rightarrow t = 4.210\,044\,743 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{t = 4.21 \text{ s (3 sf)}}}. \end{aligned}$$

(b) Find the acceleration of the particle when $t = 3$. (3)

Solution

Now,

$$v = 12\cos(\tfrac{1}{3}t) \Rightarrow a = -4\sin(\tfrac{1}{3}t)$$

and

$$\begin{aligned} t = 3 &\Rightarrow a = -3.365\,883\,939 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{a = -3.37 \text{ ms}^{-2} \text{ (FCD)}}}. \end{aligned}$$

(c) Find the distance of the particle from O when it first comes to instantaneous rest. (4)

Solution

Well,

$$v = 12\cos(\tfrac{1}{3}t) \Rightarrow s = 36\sin(\tfrac{1}{3}t) + c,$$

for some constant c. Now,

$$\begin{aligned} t = 0, s = 0 &\Rightarrow 0 = 0 + c \\ &\Rightarrow c = 0 \end{aligned}$$

and

$$s = 36\sin(\tfrac{1}{3}t).$$

Next,

$$\begin{aligned} v = 0 &\Rightarrow 12\cos(\tfrac{1}{3}t) = 0 \\ &\Rightarrow \cos(\tfrac{1}{3}t) = 0 \\ &\Rightarrow \tfrac{1}{3}t = \tfrac{1}{2}\pi \\ &\Rightarrow t = \tfrac{3}{2}\pi \end{aligned}$$

and, finally,

$$\begin{aligned} t = \tfrac{3}{2}\pi &\Rightarrow s = 36\sin(\tfrac{1}{2}\pi) \\ &\Rightarrow \underline{\underline{s = 36 \text{ m}}}. \end{aligned}$$

9. It is given that

$$\mathrm{f}(x) = 2x^2 - 12x + 10.$$

(a) Find the value of a, of b, and of c for which (3)

$$\mathrm{f}(x) = a(x + b)^2 + c.$$

Solution

Well,

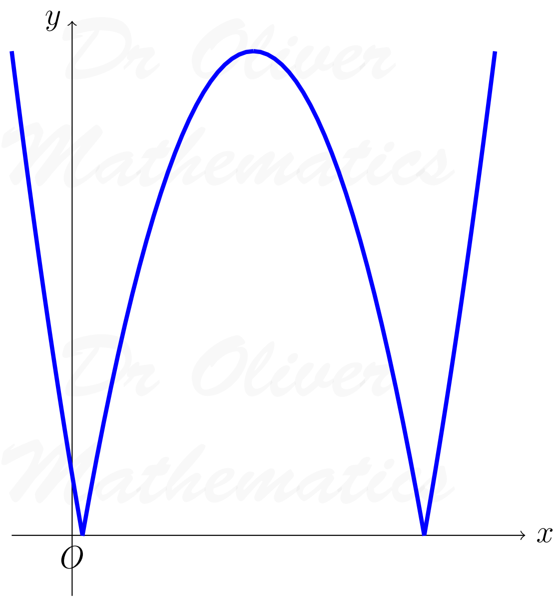
$$\begin{aligned} \mathrm{f}(x) &= 2x^2 - 12x + 10 \\ &= 2[x^2 - 6x] + 10 \\ &= 2[(x^2 - 6x + 9) - 9] + 10 \\ &= 2(x - 3)^2 - 18 + 10 \\ &= \underline{\underline{2(x - 3)^2 - 8}}; \end{aligned}$$

hence, $\underline{\underline{a = 2}}$, $\underline{\underline{b = -3}}$, and $\underline{\underline{c = -8}}$.

(b) Sketch the graph of (4)

$$y = |\,\mathrm{f}(x)| \text{ for } -1 < x < 7.$$

Solution



(c) Find the set of values of k for which the equation (2)

$$|\,\mathrm{f}(x)| = k$$

has 4 distinct roots.

Solution

Well,

$$x = 3 \Rightarrow |\,\mathrm{f}(3)| = 8$$

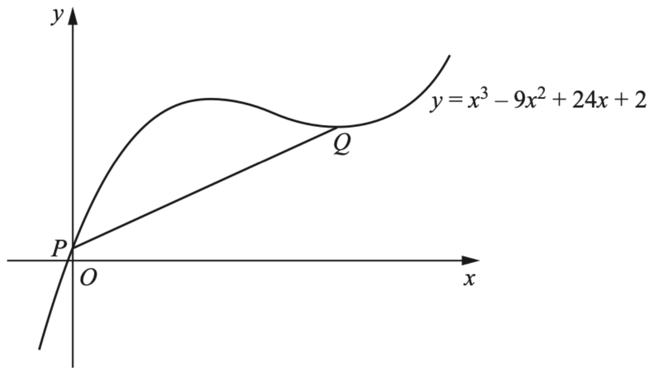
and the set of values is

$$\underline{\underline{0 < k < 8}}.$$

10. The diagram shows part of the curve

$$y = x^3 - 9x^2 + 24x + 2$$

cutting the y-axis at the point P.



The curve has a minimum point at Q.

(a) Find the coordinates of the point Q. (4)

Solution

Well,

$$y = x^3 - 9x^2 + 24x + 2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 18x + 24$$

and

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 3x^2 - 18x + 24 = 0 \\ &\Rightarrow 3(x^2 - 6x + 8) = 0\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & -6 \\ \text{multiply to:} & +8\end{array}\right\} -4, -2$$

$$\begin{aligned}&\Rightarrow 3(x-4)(x-2) = 0 \\ &\Rightarrow x = 2 \text{ or } x = 4.\end{aligned}$$

Now,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x - 18$$

and

$$\begin{aligned}x = 2 &\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -6 < 0 \text{ (maximum)} \\ x = 4 &\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6 > 0 \text{ (minimum).}\end{aligned}$$

Next,

$$x = 4 \Rightarrow y = 18$$

so the coordinates are $\underline{\underline{Q(4, 18)}}$.

(b) Find the area of the region enclosed by the curve and the line PQ. (6)

Solution
Now,

$$x = 0 \Rightarrow y = 2$$

so $P(0, 2)$. Next,

$$\begin{aligned} m_{PQ} &= \frac{18-2}{4-0} \\ &= \frac{16}{4} \\ &= 4 \end{aligned}$$

and the equation of PQ is

$$y - 2 = 4(x - 0) \Rightarrow y = 4x + 2.$$

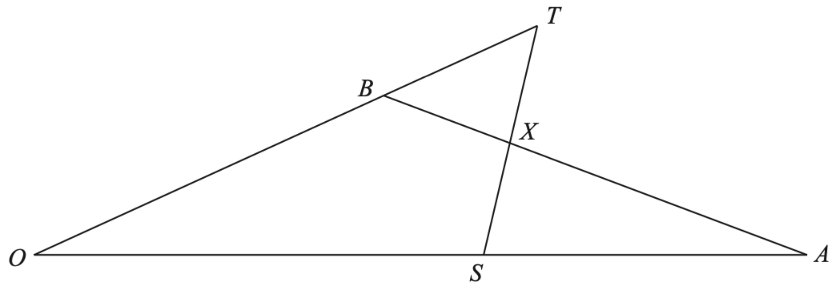
Finally,

$$\begin{aligned} \text{area} &= \int_0^4 \big[(x^3 - 9x^2 + 24x + 2) - (4x + 2)\big] \, \mathrm{d}x \\ &= \int_0^4 (x^3 - 9x^2 + 20x) \, \mathrm{d}x \\ &= \left[\tfrac{1}{4}x^4 - 3x^3 + 10x^2\right]_{x=0}^4 \\ &= (64 - 192 + 160) - (0 - 0 + 0) \\ &= \underline{\underline{32}}. \end{aligned}$$

EITHER

11. In the diagram above,

- $\overrightarrow{OA} = \mathbf{a}$,
- $\overrightarrow{OB} = \mathbf{b}$,
- $\overrightarrow{OS} = \frac{3}{5}\overrightarrow{OA}$, and
- $\overrightarrow{OT} = \frac{7}{5}\overrightarrow{OB}$.



(a) Given that $\overrightarrow{AX} = \mu\overrightarrow{AB}$, where μ is a constant, express $\overrightarrow{OX}$ in terms of μ, $\mathbf{a}$, and $\mathbf{b}$. (2)

Solution

Well,

$$\begin{aligned}\overrightarrow{OX} &= \overrightarrow{OA} + \overrightarrow{AX} \\ &= \overrightarrow{OA} + \mu\overrightarrow{AB} \\ &= \overrightarrow{OA} + \mu(\overrightarrow{AO} + \overrightarrow{OB}) \\ &= \overrightarrow{OA} + \mu(-\overrightarrow{OA} + \overrightarrow{OB}) \\ &= \mathbf{a} + \mu(-\mathbf{a} + \mathbf{b}) \\ &= \underline{\underline{(1-\mu)\mathbf{a} + \mu\mathbf{b}}}.\end{aligned}$$

(b) Given that $\overrightarrow{SX} = \lambda\overrightarrow{ST}$, where λ is a constant, express $\overrightarrow{OX}$ in terms of λ, $\mathbf{a}$, and $\mathbf{b}$. (4)

Solution

Now,

$$\begin{aligned}\overrightarrow{OX} &= \overrightarrow{OS} + \overrightarrow{SX} \\ &= \tfrac{3}{5}\overrightarrow{OA} + \lambda\overrightarrow{ST} \\ &= \tfrac{3}{5}\overrightarrow{OA} + \lambda(\overrightarrow{SO} + \overrightarrow{OT}) \\ &= \tfrac{3}{5}\overrightarrow{OA} + \lambda(-\overrightarrow{OS} + \overrightarrow{OT}) \\ &= \tfrac{3}{5}\overrightarrow{OA} + \lambda(-\tfrac{3}{5}\overrightarrow{OA} + \tfrac{7}{5}\overrightarrow{OB}) \\ &= \underline{\underline{(\tfrac{3}{5} - \tfrac{3}{5}\lambda)\mathbf{a} + \tfrac{7}{5}\lambda\mathbf{b}}}.\end{aligned}$$

(c) Hence evaluate μ and λ. (4)

Solution
Well,

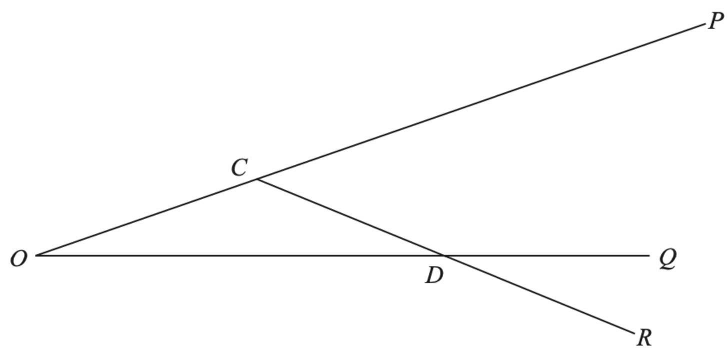
$$1-\mu = \tfrac{3}{5} - \tfrac{3}{5}\lambda \quad (1)$$
$$\mu = \tfrac{7}{5}\lambda \quad (2).$$

Insert (2) into (1):

$$1-\tfrac{7}{5}\lambda = \tfrac{3}{5} - \tfrac{3}{5}\lambda \Rightarrow \tfrac{2}{5} = \tfrac{4}{5}\lambda$$
$$\Rightarrow \underline{\underline{\lambda = \tfrac{1}{2}}}$$
$$\Rightarrow \underline{\underline{\mu = \tfrac{7}{10}}}.$$

OR

12. In the diagram above, $\overrightarrow{OC} = \mathbf{c}$ and $\overrightarrow{OD} = \mathbf{d}$.



- The points P and Q lie on OC and OD produced respectively, so that

$$OC : CP = 1 : 2$$

and

$$OD : DQ = 2 : 1.$$

- The line CD is extended to R so that $CD = DR$.

(a) Find, in terms of $\mathbf{c}$ and/or $\mathbf{d}$, the vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$, and $\overrightarrow{OR}$. (5)

Solution

Well,

$$\begin{aligned}\overrightarrow{OP} &= 3\overrightarrow{OC} \\ &= \underline{\underline{3\mathbf{c}}}\end{aligned}$$

and

$$\begin{aligned}\overrightarrow{OQ} &= \tfrac{3}{2}\overrightarrow{OD} \\ &= \underline{\underline{\tfrac{3}{2}\mathbf{d}}}.\end{aligned}$$

Now,

$$\begin{aligned}\overrightarrow{OR} &= \overrightarrow{OC} + \overrightarrow{CR} \\ &= \overrightarrow{OC} + 2\overrightarrow{CD} \\ &= \overrightarrow{OC} + 2(\overrightarrow{CO} + \overrightarrow{OD}) \\ &= \overrightarrow{OC} + 2(-\overrightarrow{OC} + \overrightarrow{OD}) \\ &= \mathbf{c} + 2(-\mathbf{c} + \mathbf{d}) \\ &= \underline{\underline{-\mathbf{c} + 2\mathbf{d}}}.\end{aligned}$$

(b) Show that the points P, Q, and R are collinear and find the ratio $PQ : QR$. (5)

Solution

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{PO} + \overrightarrow{OQ} \\ &= -\overrightarrow{OP} + \overrightarrow{OQ} \\ &= -3\mathbf{c} + \tfrac{3}{2}\mathbf{d}\end{aligned}$$

and

$$\begin{aligned}\overrightarrow{QR} &= \overrightarrow{QO} + \overrightarrow{OR} \\ &= -\overrightarrow{OQ} + \overrightarrow{OR} \\ &= -\tfrac{3}{2}\mathbf{d} + (-\mathbf{c} + 2\mathbf{d}) \\ &= -\mathbf{c} + \tfrac{1}{2}\mathbf{d} \\ &= \tfrac{1}{3}(-3\mathbf{c} + \tfrac{3}{2}\mathbf{d}) \\ &= \tfrac{1}{3}\overrightarrow{PQ}.\end{aligned}$$

Hence, the points P, Q, and R are $\underline{\underline{\text{collinear}}}$ (since they have the same starting point) and

$$\begin{aligned} PQ : QR &= 1 : \tfrac{1}{3} \\ &= \underline{\underline{3 : 1}}. \end{aligned}$$