Dr Oliver Mathematics
OCR FMSQ Additional Mathematics
2013 Paper
2 hours

The total number of marks available is 100.
You must write down all the stages in your working.
You are permitted to use a scientific or graphical calculator in this paper.
Final answers should be given correct to three significant figures where appropriate.

Section A

1. (a) Find the gradient of the line, L, whose equation is (2)

$$3x + 2y = 7.$$

Solution

$$\begin{aligned} 3x + 2y = 7 &\Rightarrow 2y = -3x + 7 \\ &\Rightarrow y = -\tfrac{3}{2}x + \tfrac{7}{2}; \end{aligned}$$

hence, the gradient is $\underline{\underline{-\frac{3}{2}}}$.

(b) Find the equation of the line which is perpendicular to L and which passes through the point $(3, 1)$. (3)

Solution

The perpendicular has gradient

$$\frac{-1}{-\frac{3}{2}} = \tfrac{2}{3}$$

and the equation of this line is

$$\begin{aligned} y - 1 = \tfrac{2}{3}(x - 3) &\Rightarrow y - 1 = \tfrac{2}{3}x - 2 \\ &\Rightarrow \underline{\underline{y = \tfrac{2}{3}x - 1}}. \end{aligned}$$

2. Find the integers that satisfy the inequality (4)

$$-7 < 3x + 1 < 12.$$

Solution

$$\begin{aligned} -7 < 3x + 1 < 12 &\Rightarrow -8 < 3x < 11 \\ &\Rightarrow -2\tfrac{2}{3} < 3x < 3\tfrac{2}{3}; \end{aligned}$$

hence, the integers are $\underline{\underline{-2,\ -1,\ 0,\ 1,\ 2,\ \text{and } 3}}$.

3. This year John is 4 times as old as his son Paul. In 5 years' time John will be only 3 times as old as Paul. (4)

Let the age of Paul now be x years.

By forming an equation in x and solving it, find Paul's age now.

Solution

Let y years be the age of John. Then

$$y = 4x \quad (1)$$

and

$$y + 5 = 3(x + 5) \quad (2).$$

Do (1) – (2):

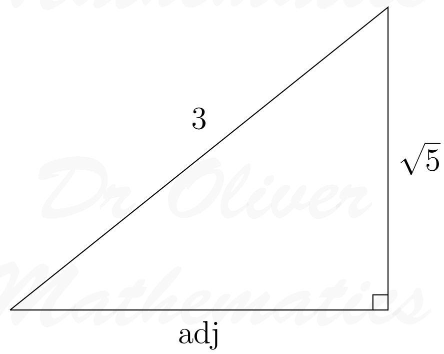
$$\begin{aligned} -5 = 4x - 3(x + 5) &\Rightarrow -5 = 4x - 3x - 15 \\ &\Rightarrow \underline{\underline{x = 10}}. \end{aligned}$$

4. You are given that θ is an acute angle and (3)

$$\sin\theta = \frac{\sqrt{5}}{3}.$$

Find the exact value of $\tan\theta$.

Solution



Now,

$$\begin{aligned}\text{adj}^2 + (\sqrt{5})^2 = 3^2 &\Rightarrow \text{adj}^2 + 5 = 9\\ &\Rightarrow \text{adj}^2 = 4\\ &\Rightarrow \text{adj} = 2.\end{aligned}$$

Hence,

$$\tan\theta = \underline{\underline{\frac{\sqrt{5}}{2}}}.$$

5. (a) Use calculus to find the stationary points on the curve (5)

$$y = x^3 - \tfrac{3}{2}x^2 - 6x + 3.$$

Solution

$$y = x^3 - \tfrac{3}{2}x^2 - 6x + 3 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 3x - 6$$

and

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 3x^2 - 3x - 6 = 0\\ &\Rightarrow 3(x^2 - x - 2) = 0\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & -1 \\ \text{multiply to:} & -2\end{array}\right\} -2,\ +1$$

$$\Rightarrow 3(x-2)(x+1) = 0$$

$$\Rightarrow x - 2 = 0 \text{ or } x + 1 = 0$$

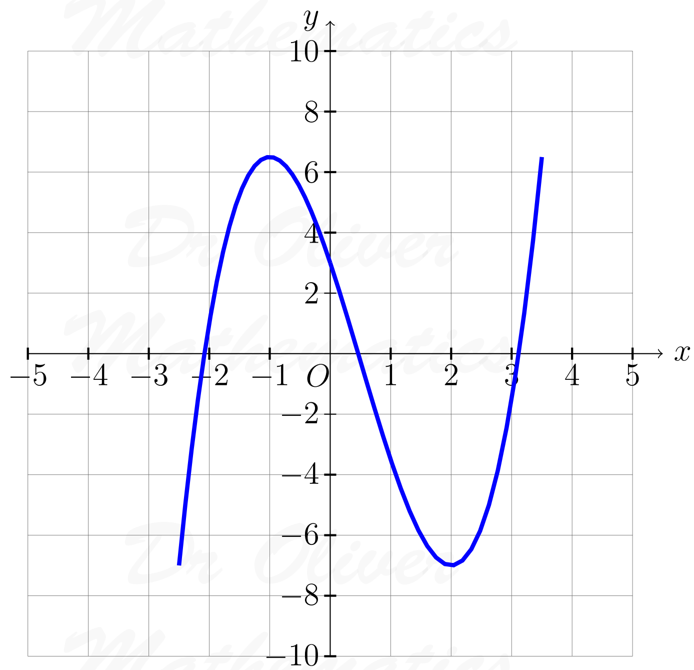
$$\Rightarrow x = 2 \text{ or } x = -1$$

$$\Rightarrow y = -7 \text{ or } y = 6\tfrac{1}{2};$$

hence, $\underline{\underline{(2, -7)}}$ and $\underline{\underline{(-1, 6\frac{1}{2})}}$.

(b) Sketch the curve on the axes provided showing the stationary points and the point where it cuts the y-axis. (2)

Solution



It cuts the y-axis at $\underline{\underline{(0, 3)}}$.

6. Amanda throws 3 fair dice. What is the probability that

(a) exactly 2 sixes are thrown, (3)

Solution

$$\begin{aligned}\text{P(exactly 2 sixes)} &= \binom{3}{2}(\tfrac{1}{6})^2(\tfrac{5}{6}) \\ &= \underline{\underline{\tfrac{5}{72} \text{ or } 0.0694 \text{ (3 sf)}}}.\end{aligned}$$

(b) at least 1 six is thrown? (3)

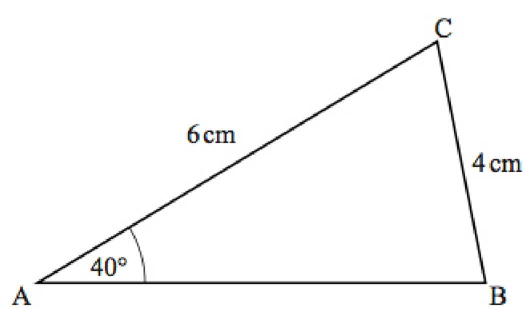
Solution

$$\begin{aligned}\text{P(at least 1 six)} &= 1 - \text{P(no sixes)} \\ &= 1 - (\tfrac{5}{6})^3 \\ &= 1 - \tfrac{125}{216} \\ &= \underline{\underline{\tfrac{91}{216} \text{ or } 0.421 \text{ (3 sf)}}}.\end{aligned}$$

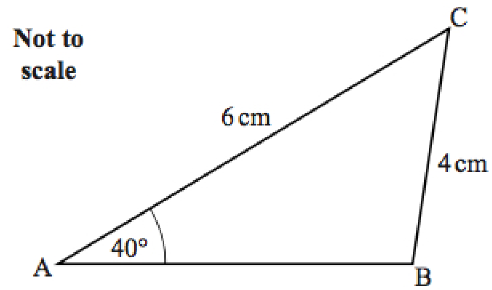
7. John and Jennie are asked to draw a triangle ABC with the following properties: (4)

- $AC = 6$ cm,
- $CB = 4$ cm, and
- the angle $A = 40°$.

John draws the triangle as shown below



and Jennie draws the triangle as shown below.



Calculate the angle B in each case.

Solution

We use the sine rule:

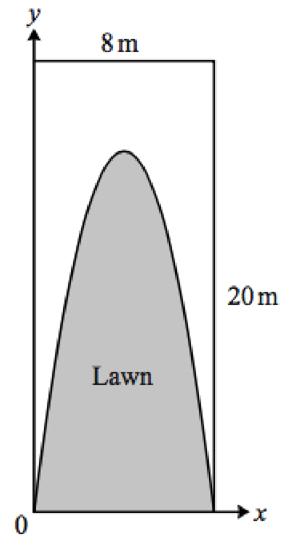
$$\frac{\sin ABC}{6} = \frac{\sin 40^\circ}{4} \Rightarrow \sin ABC = \frac{6 \sin 40^\circ}{4}$$
$$\Rightarrow \angle ABC = 74.618\,568\,31,\ 105.381\,431\,7 \text{ (FCD)}.$$

So, John draws $\underline{\underline{74.6^\circ \text{ (3 sf)}}}$ and Jennie draws $\underline{\underline{105^\circ \text{ (3 sf)}}}$

8. A mathematical gardener has a garden which is rectangular in shape measuring 20 metres by 8 metres. He wishes to arrange the garden so that approximately half of it is lawn and the rest flower bed. (6)

He sets up a coordinate system as shown in the diagram below and maps out the graph of the curve

$$y = 8x - x^2.$$



Show that the area of the lawn is approximately 53% of the total area.

Solution

$$\begin{aligned}\text{Area of the lawn} &= \int_0^8 (8x - x^2)\,\mathrm{d}x \\ &= \left[4x^2 - \tfrac{1}{3}x^3\right]_{x=0}^{8} \\ &= \left(256 - 170\tfrac{2}{3}\right) - (0 - 0) \\ &= 85\tfrac{1}{3}\end{aligned}$$

and

$$\begin{aligned}\text{percentage of the total area} &= \frac{85\frac{1}{3}}{8 \times 20} \times 100\% \\ &= \underline{\underline{53\tfrac{1}{3}\%}}.\end{aligned}$$

9. (a) Find the values of the constants a and b such that, for all values of x, (3)

$$x^2 + 8x + 19 \equiv (x + a)^2 + b.$$

Solution

$$\begin{aligned} x^2 + 8x + 19 &\equiv (x^2 + 8x + 16) + 3 \\ &\equiv \underline{\underline{(x + 4)^2 + 3}}; \end{aligned}$$

hence, $\underline{\underline{a = 4}}$ and $\underline{\underline{b = 3}}$.

(b) Hence state the least value of (2)

$$x^2 + 8x + 19$$

and the value of x at which this occurs.

Solution

$\underline{\underline{x = -4 \Rightarrow y = 3}}$.

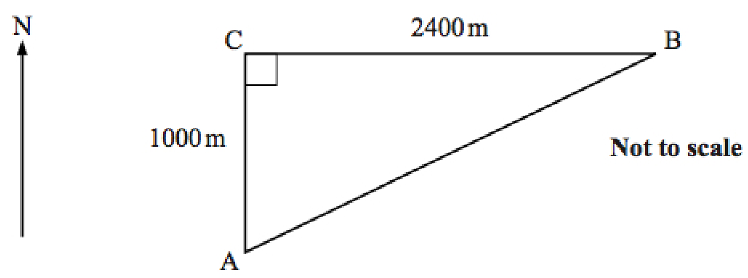
(c) Write down the greatest value of (1)

$$\frac{1}{x^2 + 8x + 19}.$$

Solution

$\underline{\underline{\frac{1}{3}}}$.

10. One leg of a cross-country race is from A to B. The checkpoint B is at the end of a wall that runs due east-west, as shown in the diagram. A is a point 1000 m due south of a point C on the wall. $BC = 2400$ m.



(a) What bearing should a runner take to travel from A to B and what is the distance AB? (4)

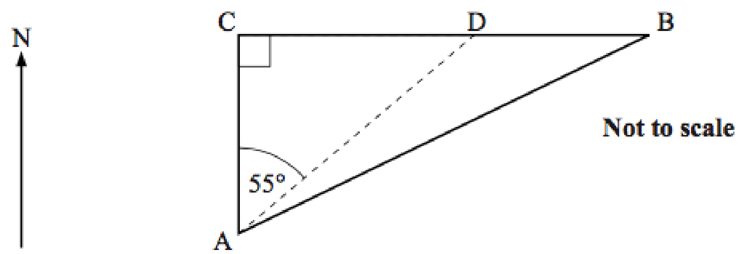
Solution

$$\text{Bearing} = \tan^{-1}\left(\frac{2\,400}{1\,000}\right)$$
$$= 067.380\,135\,05 \text{ (FCD)}$$
$$= \underline{\underline{067.4^\circ \text{ (3 sf)}}}$$

and

$$\text{distance} = \sqrt{2\,400^2 + 1\,000^2}$$
$$= \underline{\underline{2\,600 \text{ m}}}.$$

John sets off from A unable to see the checkpoint, B. He heads out on a bearing of 055° and when he reaches the wall at point D he knows he has to go east along the wall to reach the point B, as shown in the diagram.



(b) How much further than the distance AB does John run? (3)

Solution

$$\frac{1\,000}{AD} = \cos 55^\circ \Rightarrow AD = \frac{1\,000}{\cos 55^\circ}$$

and

$$DE = 2\,400 - 1\,000 \tan 55^\circ.$$

Hence,

$$\text{extra distance} = \left[\frac{1\,000}{\cos 55^\circ} + (2\,400 - 1\,000\tan 55^\circ)\right] - 2\,600$$
$$= 115.298\,788\,9 \text{ (FCD)}$$
$$= \underline{\underline{115 \text{ m (3 sf)}}}.$$

Section B

11. A circle has equation

$$(x-2)^2 + y^2 = 100.$$

(a) Write down the radius and the coordinates of the centre, C, of this circle. (2)

Solution

The centre is $\underline{\underline{(2, 0)}}$ and the radius is $\underline{\underline{10}}$.

The line $y = 2x + 6$ cuts the circle at two points, A and B.

(b) Find

(i) the coordinates of A and B, (5)

Solution

$$(x-2)^2 + y^2 = 100 \Rightarrow (x-2)^2 + (2x+6)^2 = 100$$

$\times$	x	-2
x	x^2	$-2x$
-2	$-2x$	$+4$

$\times$	$2x$	$+6$
$2x$	$4x^2$	$+12x$
$+6$	$+12x$	$+36$

$$\begin{aligned}
&\Rightarrow (x^2 - 4x + 4) + (4x^2 + 24x + 36) = 100 \\
&\Rightarrow 5x^2 + 20x + 40 = 100 \\
&\Rightarrow 5x^2 + 20x - 60 = 0 \\
&\Rightarrow 5(x^2 + 4x - 12) = 0
\end{aligned}$$

add to: $+4$, multiply to: -12 $\Big\} +6, -2$

$$\Rightarrow 5(x+6)(x-2) = 0$$
$$\Rightarrow x = -6 \text{ or } x = 2$$
$$\Rightarrow y = -6 \text{ or } y = 10;$$

hence, the coordinates of A and B are $\underline{\underline{(-6,-6)}}$ and $\underline{\underline{(2,10)}}$ (in whatever order).

(ii) the mid-point, M, of AB, (1)

Solution

$$\left(\frac{-6+2}{2}, \frac{-6+10}{2}\right) = \underline{\underline{M(-2,2)}}.$$

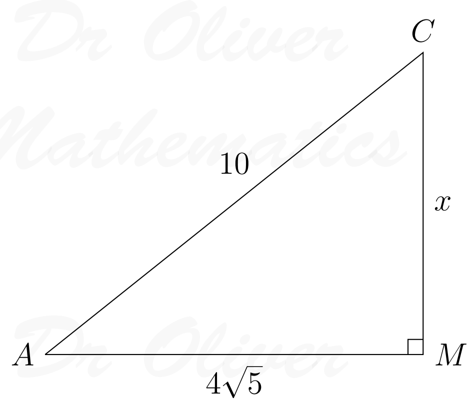
(iii) the length AB. (2)

Solution

$$\begin{aligned} AB &= \sqrt{[2-(-6)]^2 + [10-(-6)]^2} \\ &= \sqrt{8^2 + 16^2} \\ &= \underline{\underline{8\sqrt{5}}}. \end{aligned}$$

(c) Hence find the distance of the centre of the circle from the line AB. (2)

Solution



$$(4\sqrt{5})^2 + x^2 = 10^2 \Rightarrow 80 + x^2 = 100$$
$$\Rightarrow x^2 = 20$$
$$\Rightarrow \underline{\underline{x = 2\sqrt{5} \text{ or } 4.47 \text{ cm (3 sf)}}}.$$

12. An object sinks through a thick liquid such that at time t seconds after being released on the surface the depth, s metres, is given by

$$s = 4t^2 - \tfrac{2}{3}t^3 \text{ for } 0 \leqslant t \leqslant 4.$$

(a) Find the formula for the velocity, v metres per second, t seconds after being released. Hence show that the object stops sinking when $t = 4$. (4)

Solution

$$s = 4t^2 - \tfrac{2}{3}t^3 \Rightarrow \underline{\underline{v = (8t - 2t^2) \text{ ms}^{-1}}}$$
$$\Rightarrow v = 2t(4 - t)$$
$$\Rightarrow v = 0 \text{ or } t = 4;$$

hence, the object stops sinking when $\underline{\underline{t = 4 \text{ s}}}$.

(b) Find

(i) the acceleration of the object when it is released on the surface of the liquid, (4)

Solution

$$v = 8t - 2t^2 \Rightarrow a = (8 - 4t) \text{ ms}^{-2}.$$

When

$$t = 0 \Rightarrow \underline{\underline{a = 8 \text{ ms}^{-2}}}.$$

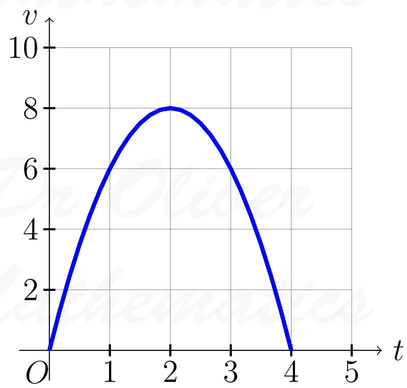
(ii) the greatest depth of the object. (2)

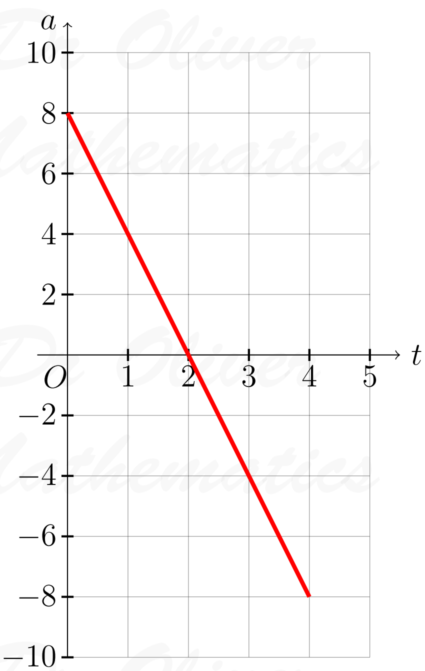
Solution

$$t = 4 \Rightarrow \underline{\underline{s = 21\tfrac{1}{3} \text{ m}}}.$$

(c) Sketch the velocity-time and acceleration-time graphs. (2)

Solution





13. A number of students from a group of 20 boys and 30 girls are to be selected to attend a one-day conference.

The number of girls attending must be at least the same as the number of boys but no more than twice the number of boys.

Let there be x boys and y girls selected.

(a) Given that $x > 0$ and $y > 0$, write down four more inequalities to represent the (3) information.

Solution

$\underline{\underline{x \leqslant 20}}$.

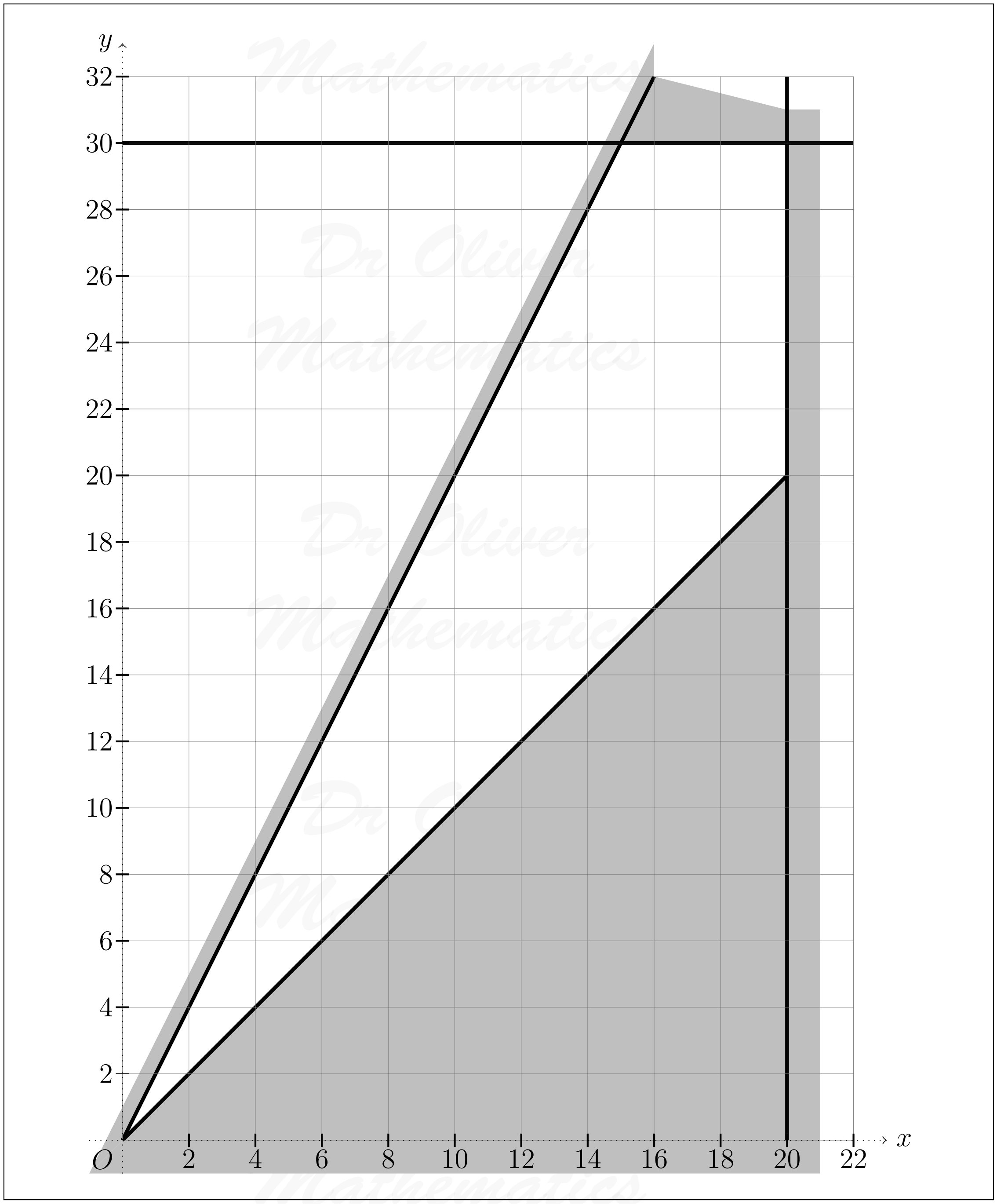
$\underline{\underline{y \leqslant 30}}$.

$\underline{\underline{y \geqslant x}}$.

$\underline{\underline{2x \geqslant y}}$.

(b) Plot these inequalities on the grid provided. Indicate the region for which the inequalities hold. Shade the area that is **not** required. (5)

Solution



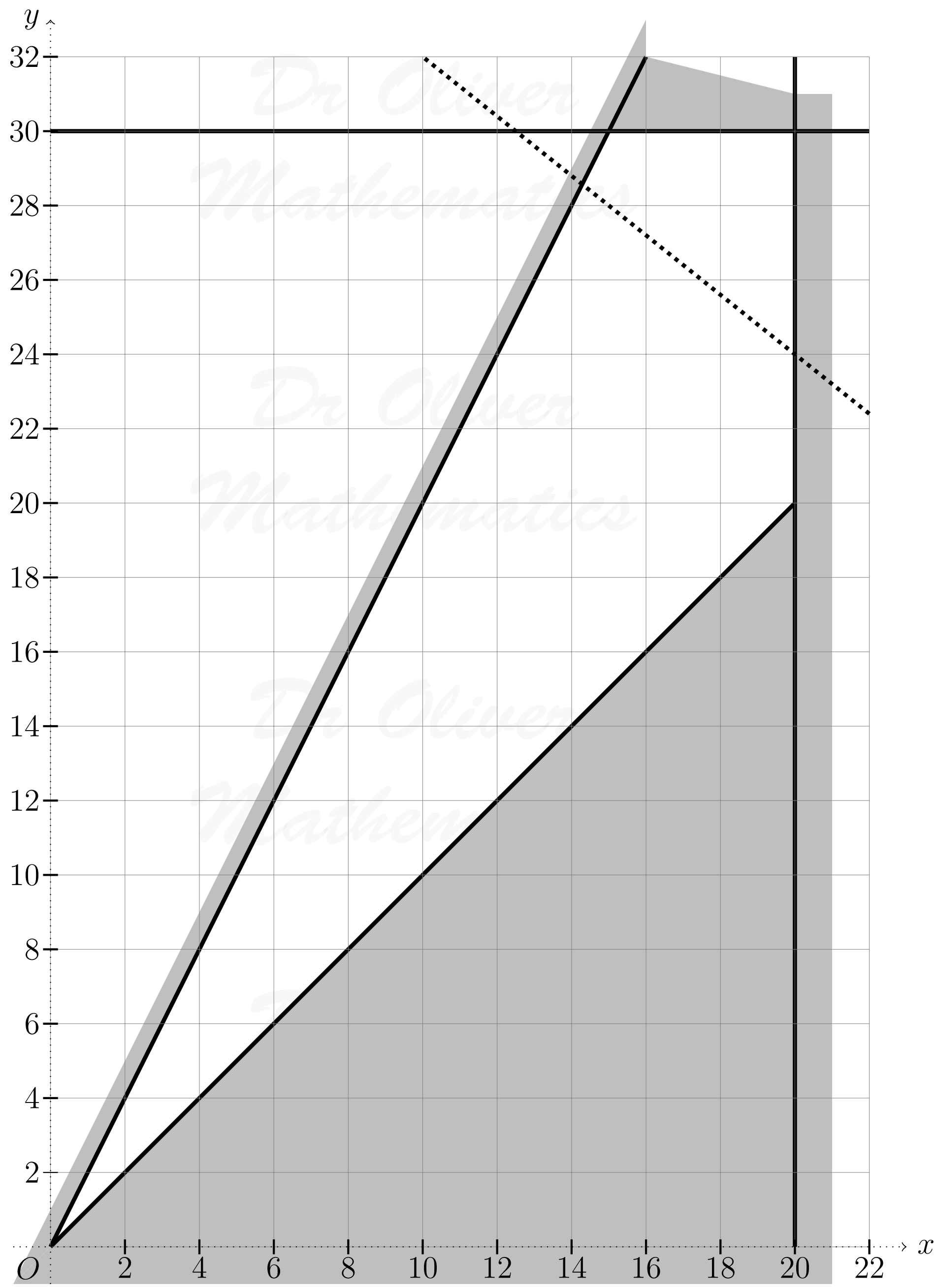
In order to attend the conference the students need to be given a special uniform. The uniform for the boys costs £40 and the uniform for the girls cost £50. The school has £2 000 to spend on the uniforms.

(c) By plotting the appropriate line on your graph, find the maximum number of students that could go to the conference. (4)

Solution

Do

$$40x + 50y = 2\,000$$



We see that it is 20 boys and 24 girls, i.e., 44 students.

14. A curve has equation

$$y = 4x^3 - 5x^2 + 1$$

and passes through the point $A(1, 0)$.

(a) Find the equation of the normal to the curve at A. (5)

Solution

$$y = 4x^3 - 5x^2 + 1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 12x^2 - 10x$$

and

$$x = 1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2 \Rightarrow m_{\text{normal}} = -\tfrac{1}{2}.$$

Hence, the equation of the normal is

$$y - 0 = -\tfrac{1}{2}(x - 1) \Rightarrow \underline{\underline{y = -\tfrac{1}{2}x + \tfrac{1}{2}}}.$$

This normal also cuts the curve in two other points, B and C.

(b) Show that the x-coordinates of the three points where the normal cuts the curve are given by the equation (2)

$$8x^3 - 10x^2 + x + 1 = 0.$$

Solution

$$\begin{aligned} 4x^3 - 5x^2 + 1 = -\tfrac{1}{2}x + \tfrac{1}{2} &\Rightarrow 8x^3 - 10x^2 + 2 = -x + 1 \\ &\Rightarrow \underline{\underline{8x^3 - 10x^2 + x + 1 = 0}}, \end{aligned}$$

as required.

(c) Show that the point $B(\frac{1}{2}, \frac{1}{4})$ satisfies the normal and the curve. (2)

Solution

Curve:

$$x = \tfrac{1}{2} \Rightarrow y = \tfrac{1}{2} - 1\tfrac{1}{4} + 1 = \tfrac{1}{4}$$

and hence satisfies the curve.

Normal:

$$x = \tfrac{1}{2} \Rightarrow y = -\tfrac{1}{4} + \tfrac{1}{2} = \tfrac{1}{4}$$

and hence satisfies the normal.

(d) Find the coordinates of C. (3)

Solution

Let

$$\mathrm{f}(x) = 8x^3 - 10x^2 + x + 1.$$

Now,

$$\mathrm{f}(1) = 8 - 10 + 1 + 1 = 0$$

and $x = 1$ is a root. Now we use synthetic division:

$$\begin{array}{c|cccc} 1 & 8 & -10 & 1 & 1 \\ & \downarrow & 8 & -2 & -1 \\ \hline & 8 & -2 & -1 & 0 \end{array}$$

Now,

$$8x^3 - 10x^2 + x + 1 = (x-1)(8x^2 - 2x - 1)$$

$$\left.\begin{array}{ll} \text{add to:} & -2 \\ \text{multiply to:} & (+8) \times (-1) = -8 \end{array}\right\} -4,\ +2$$

$$\begin{aligned} &= (x-1)[8x^2 - 4x + 2x - 1] \\ &= (x-1)[4x(2x-1) + 1(2x-1)] \\ &= (x-1)(4x+1)(2x-1) \end{aligned}$$

and

$$\mathrm{f}(x) = 0 \Rightarrow x = 1,\ x = -\tfrac{1}{4} \text{ or } x = \tfrac{1}{2}.$$

Finally,

$$x = -\tfrac{1}{4} \Rightarrow y = \tfrac{1}{8} + \tfrac{1}{2} = \tfrac{5}{8}.$$

Hence, $C(-\frac{1}{4}, \frac{5}{8})$.