Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2007 November Paper 2: Calculator
2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

1. The two variables x and y are related by the equation

$$yx^2 = 800.$$

(a) Obtain an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x. (2)

Solution

$$\begin{aligned} yx^2 = 800 &\Rightarrow y = 800x^{-2} \\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = -1\,600x^{-3}}}. \end{aligned}$$

(b) Hence find the approximate change in y as x increases from 10 to $10 + p$, where p is small. (2)

Solution

$$\begin{aligned} \delta y &= \frac{\mathrm{d}y}{\mathrm{d}x} \times \delta x \\ &= \frac{-1\,600}{10^3} p \\ &= \underline{\underline{-1.6p}}. \end{aligned}$$

2. Solve the equation (5)

$$3\sin(\tfrac{1}{2}x - 1) = 1,$$

for $0 < x < 6\pi$ radians.

Solution

Now,

$$\begin{aligned} & 3\sin(\tfrac{1}{2}x-1)=1 \\ \Rightarrow\quad & \sin(\tfrac{1}{2}x-1)=\tfrac{1}{3} \\ \Rightarrow\quad & \tfrac{1}{2}x-1=0.339\,836\,909\,5,\ 2.801\,755\,744 \text{ (FCD)} \\ \Rightarrow\quad & \tfrac{1}{2}x=1.339\,836\,909\,5,\ 1.801\,755\,744 \text{ (FCD)} \\ \Rightarrow\quad & x=2.679\,673\,819,\ 7.603\,511\,488 \text{ (FCD)} \\ \Rightarrow\quad & \underline{\underline{x=2.68,\ 7.60 \text{ (3 sf)}}}. \end{aligned}$$

3. (a) Express (1)

$$9^{x+1}$$

as a power of 3.

Solution

$$\begin{aligned} 9^{x+1} &= (3^2)^{x+1} \\ &= \underline{\underline{3^{2(x+1)}}}. \end{aligned}$$

(b) Express (1)

$$\sqrt[3]{27^{2x}}$$

as a power of 3.

Solution

$$\begin{aligned} \sqrt[3]{27^{2x}} &= ((3^3)^{2x})^{\frac{1}{3}} \\ &= \underline{\underline{3^{2x}}}. \end{aligned}$$

(c) Express (3)

$$\frac{54\times\sqrt[3]{27^{2x}}}{9^{x+1}+216(3^{2x-1}-1)}$$

as a fraction in its simplest form.

Solution

$$\frac{54 \times \sqrt[3]{27^{2x}}}{9^{x+1} + 216(3^{2x-1} - 1)} = \frac{54 \times 3^{2x}}{3^{2x+2} + 216(3^{2x-1})}$$

divide top and bottom by 3^{2x-1}:

$$\begin{aligned} &= \frac{54 \times 3}{3^3 + 216} \\ &= \frac{162}{27 + 216} \\ &= \frac{162}{243} \\ &= \underline{\underline{\tfrac{2}{3}}}. \end{aligned}$$

4. A cycle shop sells three models of racing cycles, A, B, and C.

The table below shows the numbers of each model sold over a four-week period and the cost of each model in £.

	A	B	C
1	8	12	4
2	7	10	2
3	10	12	0
4	6	8	4
Cost, £	300	500	800

In the first two weeks the shop banked 30% of all money received, but in the last two weeks the shop only banked 20% of all money received.

(a) Write down three matrices such that matrix multiplication will give the total amount of money banked over the four-week period. (2)

Solution

E.g.,

$$\underline{\underline{\begin{pmatrix} 0.3 & 0.3 & 0.2 & 0.2 \end{pmatrix} \begin{pmatrix} 8 & 12 & 4 \\ 7 & 10 & 2 \\ 10 & 12 & 0 \\ 6 & 8 & 4 \end{pmatrix} \begin{pmatrix} 300 \\ 500 \\ 800 \end{pmatrix}}}.$$

(b) Hence evaluate this total amount. (4)

Solution
Well,

$$\begin{aligned} & \begin{pmatrix} 0.3 & 0.3 & 0.2 & 0.2 \end{pmatrix} \begin{pmatrix} 8 & 12 & 4 \\ 7 & 10 & 2 \\ 10 & 12 & 0 \\ 6 & 8 & 4 \end{pmatrix} \begin{pmatrix} 300 \\ 500 \\ 800 \end{pmatrix} \\ = & \begin{pmatrix} 0.3 & 0.3 & 0.2 & 0.2 \end{pmatrix} \begin{pmatrix} 11\,600 \\ 8\,700 \\ 9\,000 \\ 9\,000 \end{pmatrix} \\ = & \begin{pmatrix} 9\,690 \end{pmatrix} \end{aligned}$$

and the amount is £9 690.

5. (a) Expand (1)

$$(1 + x)^5.$$

Solution

$$\begin{aligned} (1+x)^5 &= 1^5 + \binom{5}{1}(1)^4(x) + \binom{5}{2}(1)^3(x)^2 + \binom{5}{3}(1)^2(x)^3 + \binom{5}{4}(1)(x)^4 + (x)^5 \\ &= \underline{\underline{1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5}}. \end{aligned}$$

(b) Hence express (3)

$$(1 + \sqrt{2})^5$$

in the form

$$a + b\sqrt{2},$$

where a and b are integers.

Solution

$$\begin{aligned} (1+\sqrt{2})^5 &= 1 + 5(\sqrt{2}) + 10(\sqrt{2})^2 + 10(\sqrt{2})^3 + 5(\sqrt{2})^4 + (\sqrt{2})^5 \\ &= (1 + 20 + 20) + (5 + 20 + 4)\sqrt{2} \\ &= \underline{\underline{41 + 29\sqrt{2}}}; \end{aligned}$$

hence, $\underline{\underline{a = 41}}$ and $\underline{\underline{b = 29}}$.

(c) Obtain the corresponding result for (2)

$$(1-\sqrt{2})^5$$

and **hence** evaluate

$$(1+\sqrt{2})^5 + (1-\sqrt{2})^5.$$

Solution

$$\begin{aligned}(1-\sqrt{2})^5 &= 1 - 5(\sqrt{2}) + 10(\sqrt{2})^2 - 10(\sqrt{2})^3 + 5(\sqrt{2})^4 - (\sqrt{2})^5 \\ &= (1+20+20) - (5+20+4)\sqrt{2} \\ &= \underline{\underline{41 - 29\sqrt{2}}}\end{aligned}$$

and

$$\begin{aligned}(1+\sqrt{2})^5 + (1-\sqrt{2})^5 &= 41 + 29\sqrt{2} + (41 - 29\sqrt{2}) \\ &= \underline{\underline{82}}.\end{aligned}$$

6. Two circular flower beds have a combined area of $\frac{29}{2}\pi$ m^2. (6)

The sum of the circumferences of the two flower beds is 10π m.

Determine the radius of each flower bed.

Solution

Well, let the radii be r and R m. Then

$$\pi r^2 + \pi R^2 = \tfrac{29}{2}\pi \Rightarrow r^2 + R^2 = \tfrac{29}{2} \quad (1)$$

and

$$\begin{aligned}2\pi r + 2\pi R = 10\pi &\Rightarrow 2(r+R) = 10 \\ &\Rightarrow r + R = 5 \\ &\Rightarrow R = 5 - r \quad (2).\end{aligned}$$

Now, insert (2) into (1):

$$\begin{aligned} r^2 + (5-r)^2 = \tfrac{29}{2} &\Rightarrow r^2 + (25 - 10r + r^2) = \tfrac{29}{2} \\ &\Rightarrow 2r^2 - 10r + 25 = \tfrac{29}{2} \\ &\Rightarrow 4r^2 - 20r + 50 = 29 \\ &\Rightarrow 4r^2 - 20r + 21 = 0 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & -20 \\ \text{multiply to:} & (+4) \times (+21) = +84 \end{array}\right\} -14, -6$$

e.g.,

$$\begin{aligned} &\Rightarrow 4r^2 - 14r - 6r + 21 = 0 \\ &\Rightarrow 2r(2r-7) - 3(2r-7) = 0 \\ &\Rightarrow (2r-3)(2r-7) = 0 \\ &\Rightarrow 2r - 3 = 0 \text{ or } 2r - 7 = 0 \\ &\Rightarrow \underline{\underline{r = 1\tfrac{1}{2} \text{ or } r = 3\tfrac{1}{2}}}. \end{aligned}$$

7. The position vectors of points A and B, relative to an origin O, are $2\mathbf{i} + 4\mathbf{j}$ and $6\mathbf{i} + 10\mathbf{j}$ respectively.

The position vector of C, relative to O, is $k\mathbf{i} + 25\mathbf{j}$, where k is a positive constant.

(a) Find the value of k for which the length of BC is 25 units. (3)

Solution

Well,

$$\begin{aligned} 25^2 = (k-6)^2 + (25-10)^2 &\Rightarrow 625 = (k-6)^2 + 225 \\ &\Rightarrow (k-6)^2 = 400 \\ &\Rightarrow k - 6 = \pm 20 \\ &\Rightarrow k = 6 \pm 20; \end{aligned}$$

hence, as k is a positive constant, $\underline{\underline{k = 26}}$.

(b) Find the value of k for which ABC is a straight line. (3)

Solution

Well, the ratios are equal:

$$\frac{6-2}{10-4} = \frac{k-6}{25-10} \Rightarrow \tfrac{2}{3} = \frac{k-6}{15}$$
$$\Rightarrow k - 6 = \tfrac{2}{3} \times 15$$
$$\Rightarrow k - 6 = 10$$
$$\Rightarrow \underline{\underline{k = 16}}.$$

8. Given that $x \in \mathbb{R}$ and that (7)

- $\mathscr{E} = \{x : 2 < x < 10\}$,
- $A = \{x : 3x + 2 < 20\}$, and
- $B = \{x : x^2 < 11x - 28\}$,

find the set of values of x which define

(a) $A \cap B$,

Solution

$\underline{A}$:

$$3x + 2 < 20 \Rightarrow 3x < 18$$
$$\Rightarrow x < 6.$$

$\underline{B}$:

$$x^2 < 11x - 28 \Rightarrow x^2 - 11x + 28 < 0$$

$$\left.\begin{array}{ll}\text{add to:} & -11 \\ \text{multiply to:} & +28\end{array}\right\} -7, -4$$

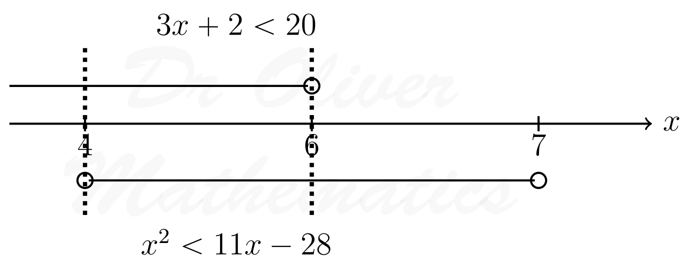
$$\Rightarrow (x-7)(x-4) < 0$$

we need a 'table of signs':

	$x < 4$	$x = 4$	$4 < x < 7$	$x = 7$	$x > 7$
$x - 4$	$-$	0	$+$	$+$	$+$
$x - 7$	$-$	$-$	$-$	0	$+$
$(x-4)(x-7)$	$+$	0	$-$	0	$+$

$$\Rightarrow 4 < x < 7.$$

We draw a number line:



Hence,

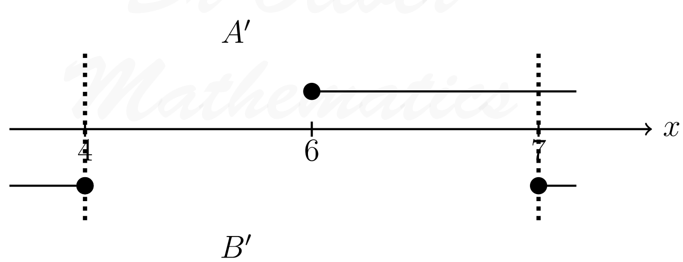
$$A \cap B = \underline{\underline{\{x : 4 < x < 6\}}}.$$

(b) $(A \cup B)'$.

Solution

Well,

$$(A \cup B)' = A' \cap B'.$$



Finally, since $\mathscr{E} = \{x : 2 < x < 10\}$,

$$(A \cup B)' = \underline{\underline{\{x : 7 \leqslant x < 10\}}}.$$

9. A particle travels in a straight line so that, t s after passing through a fixed point O, its speed, v ms^{-1}, is given by

$$v = 8\cos(\tfrac{1}{2}t).$$

(a) Find the acceleration of the particle when $t = 1$. (3)

Solution
Now,

$$v = 8\cos(\tfrac{1}{2}t) \Rightarrow a = -4\sin(\tfrac{1}{2}t)$$

and

$$\begin{aligned} t = 1 &\Rightarrow a = -4\sin(\tfrac{1}{2}) \\ &\Rightarrow a = -1.917\,702\,154 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{a = -1.92 \text{ ms}^{-2} \text{ (3 sf)}}}. \end{aligned}$$

The particle first comes to instantaneous rest at the point P.

(b) Find the distance OP. (4)

Solution
Well,

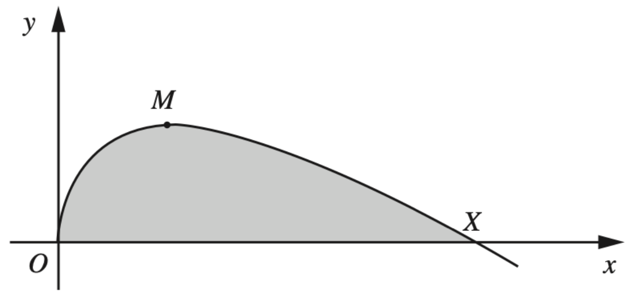
$$\begin{aligned} v = 0 &\Rightarrow 8\cos(\tfrac{1}{2}t) = 0 \\ &\Rightarrow \cos(\tfrac{1}{2}t) = 0 \\ &\Rightarrow \tfrac{1}{2}t = \tfrac{1}{2}\pi \\ &\Rightarrow t = \pi. \end{aligned}$$

Next,

$$\begin{aligned} OP &= \int_0^{\pi} 8\cos(\tfrac{1}{2}t)\,\mathrm{d}t \\ &= \left[16\sin(\tfrac{1}{2}t)\right]_{t=0}^{\pi} \\ &= 16 - 0 \\ &= \underline{\underline{16 \text{ m}}}. \end{aligned}$$

10. The diagram shows part of the curve

$$y = 4\sqrt{x} - x.$$



The origin O lies on the curve and the curve intersects the positive x-axis at X.

The maximum point of the curve is at M.

Find

(a) the coordinates of X and of M, (5)

Solution

Well,

$$\begin{aligned} y = 0 &\Rightarrow 4\sqrt{x} - x = 0 \\ &\Rightarrow \sqrt{x}(4 - \sqrt{x}) = 0 \\ &\Rightarrow \sqrt{x} = 0 \text{ or } \sqrt{x} = 4 \\ &\Rightarrow x = 0 \text{ or } x = 16. \end{aligned}$$

Hence, the point X is $\underline{\underline{(16, 0)}}$.

Now,

$$\begin{aligned} y = 4\sqrt{x} - x &\Rightarrow y = 4x^{\frac{1}{2}} - x \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2x^{-\frac{1}{2}} - 1 \end{aligned}$$

and

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 2x^{-\frac{1}{2}} - 1 = 0 \\
&\Rightarrow 2x^{-\frac{1}{2}} = 1 \\
&\Rightarrow x^{-\frac{1}{2}} = \tfrac{1}{2} \\
&\Rightarrow x^{\frac{1}{2}} = 2 \\
&\Rightarrow x = 2^2 \\
&\Rightarrow x = 4 \\
&\Rightarrow y = 4;
\end{aligned}$$

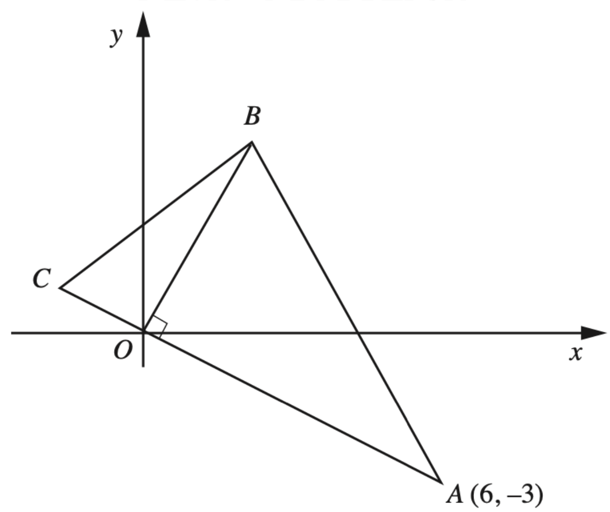
hence, the point M is $\underline{\underline{(4, 4)}}$.

(b) the area of the shaded region. (4)

Solution

$$\begin{aligned}
\int_0^{16} (4x^{\frac{1}{2}} - x)\,\mathrm{d}x &= \left[\tfrac{8}{3}x^{\frac{3}{2}} - \tfrac{1}{2}x^2\right]_{x=0}^{16} \\
&= \left(170\tfrac{2}{3} - 128\right) - (0 - 0) \\
&= \underline{\underline{42\tfrac{2}{3}\text{ m}}}.
\end{aligned}$$

11. **Solutions to this question by accurate drawing will not be accepted.**
The diagram shows a triangle ABC in which A is the point $(6, -3)$.



The line AC passes through the origin O.

The line OB is perpendicular to AC.

(a) Find the equation of OB. (2)

Solution

Well,

$$m_{OA} = \frac{-3-0}{6-0} = -\frac{1}{2}$$

and

$$m_{OB} = -\frac{1}{-\frac{1}{2}} = 2.$$

Finally, the equation of OB is

$$\underline{\underline{y = 2x}}.$$

The area of triangle AOB is 15 units^2.

(b) Find the coordinates of B. (3)

Solution

Well,

$$\begin{aligned}\tfrac{1}{2} \times OA \times OB = 15 &\Rightarrow \sqrt{6^2 + (-3)^2} \times OB = 30 \\ &\Rightarrow \sqrt{36+9} \times OB = 30 \\ &\Rightarrow 3\sqrt{5}OB = 30 \\ &\Rightarrow OB = 2\sqrt{5}.\end{aligned}$$

Now, $(x, 2x, 2\sqrt{x})$ is a Pythagorean triple:

$$\begin{aligned}x^2 + (2x)^2 = (2\sqrt{x})^2 &\Rightarrow x^2 + 4x^2 = 20 \\ &\Rightarrow 5x^2 = 20 \\ &\Rightarrow x^2 = 4 \\ &\Rightarrow x = 2 \\ &\Rightarrow y = 4;\end{aligned}$$

hence, $\underline{\underline{B(2,4)}}$.

The length of AO is 3 times the length of OC.

(c) Find the coordinates of C. (2)

Solution
Well,

$$\begin{aligned}\overrightarrow{OC} &= \tfrac{1}{3}\overrightarrow{AO} \\ &= \tfrac{1}{3}\begin{pmatrix} 0-6 \\ 0-(-3) \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 1 \end{pmatrix};\end{aligned}$$

hence, $\underline{\underline{C(-2,1)}}$.

The point D is such that the quadrilateral $ABCD$ is a kite.

(d) Find the area of $ABCD$. (2)

Solution

$$OC = \sqrt{(-2-0)^2 + (1-0)^2} = \sqrt{5}$$

and

$$\text{area}_{OBC} = \tfrac{1}{2} \times \sqrt{5} \times 2\sqrt{5} = 5.$$

Finally,

$$\text{kite}_{ABCD} = 2(5+15) = \underline{\underline{40}}.$$

EITHER

12. The function f is defined, for $x > 0$, by

$$\mathrm{f} : x \mapsto \ln x.$$

(a) State the range of f. (1)

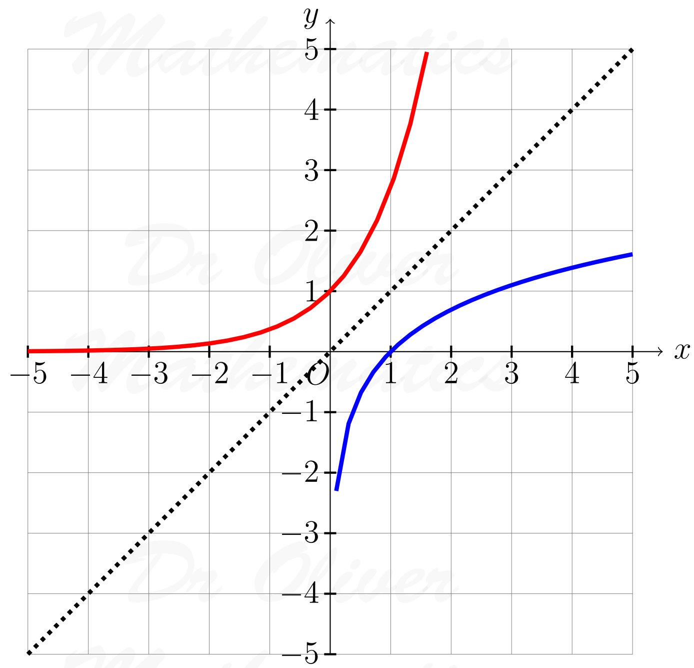
Solution
$\underline{\underline{\mathrm{f}(x) \in \mathbb{R}}}$.

(b) State the range of f^{-1}. (1)

Solution
$\underline{\underline{\mathrm{f}^{-1}(x) > 0}}$.

(c) On the same diagram, sketch and label the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$. (2)

Solution



The function g is defined, for $x > 0$, by

$$\mathrm{g} : x \mapsto 3x + 2.$$

(d) Solve the equation (2)

$$\mathrm{f}\,\mathrm{g}(x) = 3.$$

Solution

$$\begin{aligned}\mathrm{f}\,\mathrm{g}(x) = 3 &\Rightarrow \mathrm{f}(\mathrm{g}(x)) = 3\\ &\Rightarrow \mathrm{f}(3x+2) = 3\\ &\Rightarrow \ln(3x+2) = 3\\ &\Rightarrow 3x + 2 = \mathrm{e}^3\\ &\Rightarrow 3x = \mathrm{e}^3 - 2\\ &\Rightarrow x = \underline{\underline{\tfrac{1}{3}(\mathrm{e}^3 - 2) \text{ or } 6.03 \text{ (3 sf)}}}.\end{aligned}$$

(e) Solve the equation (4)

$$\mathrm{f}^{-1}\,\mathrm{g}^{-1}(x) = 7.$$

Solution

$$\begin{aligned} \mathrm{f}^{-1}\,\mathrm{g}^{-1}(x) = 7 &\Rightarrow \mathrm{g}^{-1}(x) = \mathrm{f}(7) \\ &\Rightarrow \mathrm{g}^{-1}(x) = \ln 7 \\ &\Rightarrow x = \mathrm{g}(\ln 7) \\ &\Rightarrow \underline{\underline{x = 3\ln 7 + 2 \text{ or } 7.84 \text{ (3 sf)}}}. \end{aligned}$$

OR

13. (a) Find the values of k for which (4)

$$y = kx + 2$$

is a tangent to the curve

$$y = 4x^2 + 2x + 3.$$

Solution

Well, we will eliminate y:

$$4x^2 + 2x + 3 = kx + 2 \Rightarrow 4x^2 + (2-k)x + 1 = 0$$

and 'equal roots' suggest $b^2 - 4ac = 0$:

$$\begin{aligned} (2-k)^2 - 4(4)(1) = 0 &\Rightarrow (2-k)^2 = 16 \\ &\Rightarrow 2 - k = \pm 4 \\ &\Rightarrow k = 2 \pm 4 \\ &\Rightarrow \underline{\underline{k = 6 \text{ or } k = -2}}. \end{aligned}$$

(b) Express (3)

$$4x^2 + 2x + 3$$

in the form

$$a(x+b)^2 + c,$$

where a, b, and c are constants.

Solution

Well,

$$\begin{aligned} 4x^2 + 2x + 3 &= 4(x^2 + \tfrac{1}{2}x) + 3 \\ &= 4[(x^2 + \tfrac{1}{2}x + \tfrac{1}{16}) - \tfrac{1}{16}] + 3 \\ &= 4[(x + \tfrac{1}{4})^2 - \tfrac{1}{16}] + 3 \\ &= 4(x + \tfrac{1}{4})^2 - \tfrac{1}{4} + 3 \\ &= \underline{\underline{4(x + \tfrac{1}{4})^2 + \tfrac{11}{4}}}; \end{aligned}$$

hence, $\underline{\underline{a = 4}}$, $\underline{\underline{b = \frac{1}{4}}}$, and $\underline{\underline{c = \frac{11}{4}}}$.

(c) Determine, with explanation, whether or not the curve (2)

$$y = 4x^2 + 2x + 3$$

meets the x-axis.

Solution
Does

$$4x^2 + 2x + 3 = 0?$$

Well,

$$2^2 - 4(4)(3) = -44 < 0,$$

so $\underline{\underline{\text{no}}}$: the curve does $\underline{\underline{\text{not}}}$ meets the x-axis.

The function f is defined by

$$\mathrm{f} : x \mapsto 4x^2 + 2x + 3,$$

where $x \geqslant p$.

(d) Determine the smallest value of p for which f has an inverse. (1)

Solution
$\underline{\underline{p = -\frac{1}{4}}}$.