# Core Mathematics 2

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Mathematics

## Factor theorem

$(ax-b)$ is a factor of a polynomial $\mathrm{f}(x)$ if and only if $\mathrm{f}(\frac{b}{a})=0$. When showing that, for example, $(x+4)$ is a factor, it is not enough merely to show that $\mathrm{f}(-4)=0$: you must go on to state that $(x+4)$ is a factor by the factor theorem.

## Remainder theorem

The remainder upon dividing a polynomial $\mathrm{f}(x)$ by $(ax-b)$ is $\mathrm{f}(\frac{b}{a})$.

## Polynomial division

Once you know that $(x-a)$ is a factor of a polynomial $\mathrm{f}(x)$ you can then use polynomial division to write $\mathrm{f}(x)$ as a product of $(x-a)$ and a polynomial of smaller degree. For example, let $\mathrm{f}(x)=x^3-7x-6$. Since $\mathrm{f}(-1)=0$ we know that $(x+1)$ is a factor using the factor theorem. Then

$$\begin{array}{r|rrrr}
 & x^2 & -x & -6 & \\
x+1 & x^3 & & -7x & -6 \\
 & x^3 & +x^2 & & \\ \hline
 & & -x^2 & -7x & -6 \\
 & & -x^2 & -x & \\ \hline
 & & & -6x & -6 \\
 & & & -6x & -6 \\ \hline
 & & & & 0
\end{array}$$

and hence $x^3-7x-6 \equiv (x+1)(x^2-x-6)$.

## Logarithms

For any $a>0$ and any $x>0$,

$$a^b = x \Leftrightarrow \log_a x = b.$$

$$\log_a(xy) = \log_a x + \log_a y$$

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a(x^n) = n\log_a x$$

$$\log_a a = 1$$

$$\log_a 1 = 0$$

$$\log_a x = \frac{\log_b x}{\log_b a} \text{ for any } b>0.$$

## Circles

The equation of a circle, centre $(a,b)$ and radius $r$ is given by

$$(x-a)^2+(y-b)^2=r^2.$$

The rest of the work in chapter 4 is essentially GCSE…

## Binomial coefficients

The binomial coefficents are defined by

$$\binom{n}{r} = \frac{n!}{r!(n-r)!},$$

and you need to able to recognise that

$$\binom{n}{r} = \binom{n}{n-r} \qquad \binom{n}{0} = 1$$

$$\binom{n}{1} = n \qquad \binom{n}{2} = \tfrac{1}{2}n(n-1)$$

$$\binom{n}{3} = \tfrac{1}{6}n(n-1)(n-2),$$

and so on.

## Binomial expansion

The binomial expansion is given by

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n-1}ab^{n-1} + b^n$$

## Geometric sequence

A sequence of the form $a$, $ar$, $ar^2$, … is said to be a *geometric sequence* with *first term* $a$ and *common ratio* $r$. The $n^{\text{th}}$ term of the sequence is given by $ar^{n-1}$ and the sum of the first $n$ terms given by

$$S_n = \frac{a(r^n-1)}{r-1} = \frac{a(1-r^n)}{1-r};$$

the version that you use depends on whether $|r|>1$ or $|r|<1$ — you need to be able to prove this result.

## Sum to infinity

Given the first term $a$ and a common ratio $r$ such that $|r|<1$, the *sum to infinity* is given by

$$S_\infty = \frac{a}{1-r}.$$

## Radians

There are $2\pi$ radians in a circle and, from this point on, the radian is the default angle measure that you use in mathematics: if you want to use degrees then you have to put the degree symbol in, otherwise $\sin 3$ is taken to mean the sine of 3 radians. For a circle of radius $r$ and an angle $\theta$ in radians,

$$\text{length of an arc} = r\theta,$$
$$\text{area of a sector} = \tfrac{1}{2}r^2\theta,$$

## Trigonometry

You need to be able to

- sketch the graphs of the three trigonometric functions, as well as related functions,
- know which quadrants each function is either positive or negative in,
- use the properties of the graph to find all solutions to a trigonometric equation in a given range,
- know the exact values of the sine, cosine, and tangent of 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$,
- recognise and use the identities
  $$\sin^2\theta + \cos^2\theta \equiv 0 \text{ and } \tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$
  to both prove identities and to solve equations.

## Increasing and decreasing functions

Let $\mathrm{f}(x)$ be a function on the interval $(a,b)$. The function is *increasing* if, for all $a \le x_1 < x_2 \le b$, $\mathrm{f}(x_1) < \mathrm{f}(x_2)$; in that case, $\mathrm{f}'(x)>0$ for all $a \le x \le b$. The function is *decreasing* if, for all $a \le x_1 < x_2 \le b$, $\mathrm{f}(x_1) > \mathrm{f}(x_2)$; in that case, $\mathrm{f}'(x)<0$ for all $a \le x \le b$.

## Higher-order derivatives

Just as $\mathrm{f}'(x)$ and $\frac{\mathrm{d}y}{\mathrm{d}x}$ represent the first derivative, the second derivative (which is what you get by differentiating the first derivative) are denoted by $\mathrm{f}''(x)$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$; the third and fourth derivatives and so on all follow in the same way.

## Local maxima and minima

A point at which $\mathrm{f}'(x)=0$ is a *candidate* for being either a local maximum or a local minimum. If, in addition, $\mathrm{f}''(x)<0$ then we have a *local maximum*; if $\mathrm{f}''(x)>0$ then we have a *local minimum*.

## Points of inflexion

A *point of inflexion* is a point across which the second derivative of the function changes sign. (The second derivative need not even be defined at this point: if, for example, it goes from negative to undefined to positive then we still have a point of inflexion.) In Core Mathematics 2 the only points of inflexion that are examined are those where the gradient of the tangent to the curve is zero but that is not true in later modules and you should be aware that, for example, the graphs of the sine, cosine, and tangent functions all have non-stationary points of inflexion where they cross the $x$-axis.

## Definite integration

Let $\mathrm{F}(x)$ be any anti-derivative of $\mathrm{f}(x)$, i.e., $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{F}(x)) = \mathrm{f}(x)$. Then the *definite integral* is defined as

$$\int_a^b \mathrm{f}(x)\,\mathrm{d}x = [\mathrm{F}(x)]_{x=a}^{b} = \mathrm{F}(b) - \mathrm{F}(a).$$

Note that *any* anti-derivative will do and so we do not need the '$+c$'.

## Integration to find areas

A definite integral *does not* give you the area bounded by the function and the $x$-axis: it gives you the signed area where areas above the $x$-axis are assigned positive values and areas below the $x$-axis are assigned negative values. Thus

$$\int_1^2 x^2\,\mathrm{d}x = \left[\tfrac{1}{3}x^3\right]_{x=1}^{2} = \tfrac{1}{3}(8-1) = 2\tfrac{1}{3}$$

does tell you that there are $2\frac{1}{3}$ units$^2$ of area bounded by the $x$-axis, the lines $x=1$ and $x=2$ and the curve $y=x^2$ but

$$\int_0^2 (x-1)\,dx = \left[\tfrac{1}{2}x^2 - x\right]_{x=0}^{2} = 0$$

only tells you that the there is the same area above and below the $x$-axis bounded the lines $x=0$, $x=2$, and the line $y=x-1$.