Dr Oliver Mathematics
AQA Further Maths Level 2
June 2021 Paper 2
2 hours

The total number of marks available is 80.
You must write down all the stages in your working.
You are permitted to use a scientific or graphical calculator in this paper.

1. Expand and simplify (3)

$$5(2x - 1) + 4(11 - x).$$

Give your answer in the form

$$a(bx + c),$$

where a, b, and c are integers greater than 1

Solution

$$\begin{aligned} 5(2x-1) + 4(11-x) &= 10x - 5 + 44 - 4x \\ &= 6x + 39 \\ &= \underline{\underline{3(2x+13)}}; \end{aligned}$$

hence, $\underline{\underline{a = 3}}$, $\underline{\underline{b = 2}}$, and $\underline{\underline{c = 13}}$.

2. $5m$ is decreased by 40%.
The answer is $(m + 1)$.

(a) Work out the value of m. (2)

Solution

Well,

$$\begin{aligned} (1 - 0.4)(5m) = m + 1 &\Rightarrow 0.6(5m) = m + 1 \\ &\Rightarrow 3m = m + 1 \\ &\Rightarrow 2m = 1 \\ &\Rightarrow \underline{\underline{m = 0.5}}. \end{aligned}$$

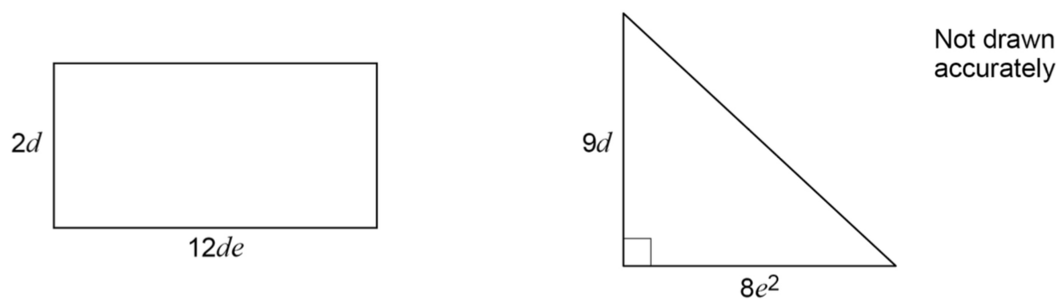
(b) Solve (2)

$$\sqrt[3]{2w - 10} = 18.$$

Solution

$$\begin{aligned}\sqrt[3]{2w - 10} = 18 &\Rightarrow 2w - 10 = 18^3 \\ &\Rightarrow 2w - 10 = 5\,832 \\ &\Rightarrow 2w = 5\,842 \\ &\Rightarrow \underline{\underline{w = 2\,921}}.\end{aligned}$$

3. The rectangle and triangle shown have equal areas. (3)



Work out the value of

$$\frac{d}{e}.$$

Give your answer in its simplest form.

Solution

Well,

$$\begin{aligned}\text{area}_{\text{ square}} &= 2d \times 12de \\ &= 24d^2e\end{aligned}$$

and

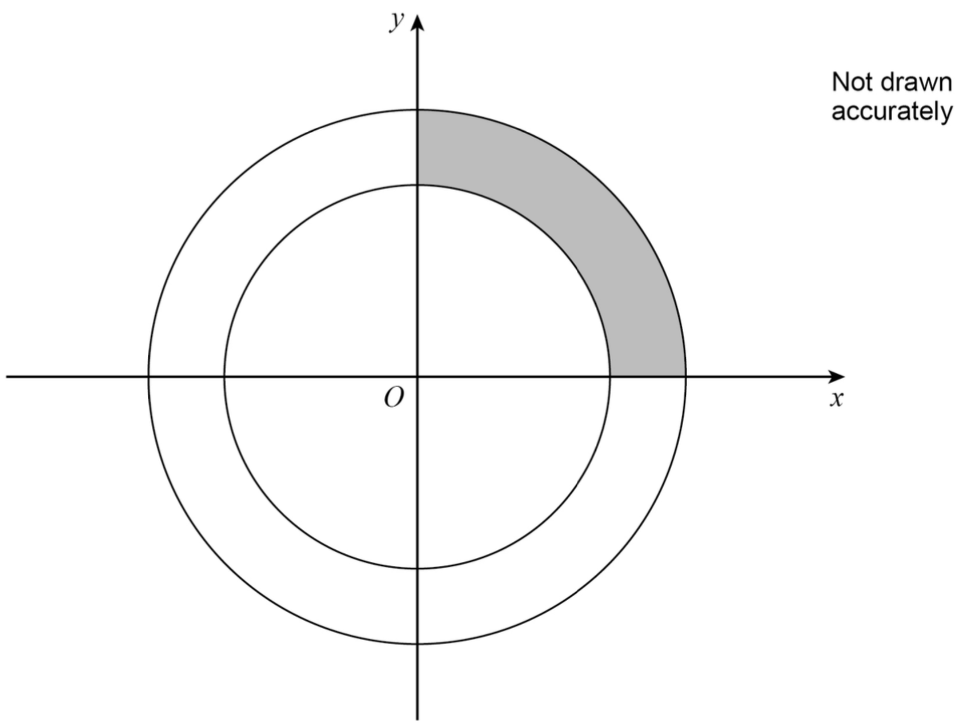
$$\begin{aligned}\text{area}_{\text{ triangle}} &= \tfrac{1}{2} \times 9d \times 8e^2 \\ &= 36de^2.\end{aligned}$$

They are the same:

$$24d^2e = 36de^2 \Rightarrow 2d = 3e$$
$$\Rightarrow \frac{d}{e} = \underline{\underline{\frac{3}{2}}}.$$

4. The equations of the two circles shown are (3)

$$x^2 + y^2 = 100 \text{ and } x^2 + y^2 = 36.$$



Work out the shaded area.
Give your answer as an integer multiple of *pi*.

Solution

What are the radii of the two circles?

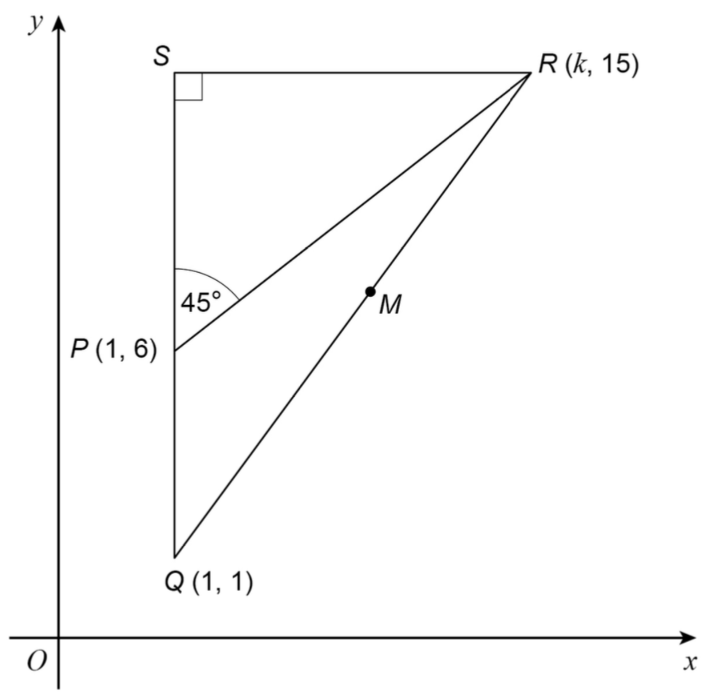
$$\sqrt{100} = 10 \text{ and } \sqrt{36} = 6.$$

Now,

$$\begin{aligned}\text{shaded area} &= \tfrac{1}{4}\left[(\pi \times 10^2) - (\pi \times 6^2)\right]\\ &= \tfrac{1}{4}(100\pi - 36\pi)\\ &= \tfrac{1}{4}(64\pi)\\ &= \underline{\underline{16\pi}}.\end{aligned}$$

5. SQR is a right-angled triangle. (3)

- P is a point on SQ.
- Angle $SPR = 45°$.
- M is the midpoint of QR.
- k is a constant.



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Work out the coordinates of M.

Solution

Well, $S(1, 15)$.
Next, $PS = 15 - 6 = 9$ and R is the point $(1 + 9, 15) = (10, 15)$.
Finally, the coordinates of M are

$$\left(\frac{1+10}{2}, \frac{1+15}{2}\right) = \underline{\underline{(5\tfrac{1}{2}, 8)}}.$$

6. Rearrange (3)

$$y = \sqrt{\frac{x+2w}{3}}$$

to make w the subject.

Solution

$$\begin{aligned} y = \sqrt{\frac{x+2w}{3}} &\Rightarrow y^2 = \frac{x+2w}{3} \\ &\Rightarrow 3y^2 = x + 2w \\ &\Rightarrow 3y^2 - x = 2w \\ &\Rightarrow \underline{\underline{w = \frac{3y^2 - x}{2}}}. \end{aligned}$$

7. a is a value greater than 1.

(a) Work out the value of m for which (2)

$$(a^m)^4 = (a^5)^{2m}.$$

Solution

$$(a^m)^4 = (a^5)^{2m} \Rightarrow a^{4m} = a^{10m}$$

as the exponent is the same:

$$\begin{aligned} &\Rightarrow 4m = 10m \\ &\Rightarrow 6m = 0 \\ &\Rightarrow \underline{\underline{m = 0}}. \end{aligned}$$

$$w^3x^2y^5 = w^{13}x^7.$$

(b) Write y in terms of w and x. (2)
Give your answer in its simplest form.

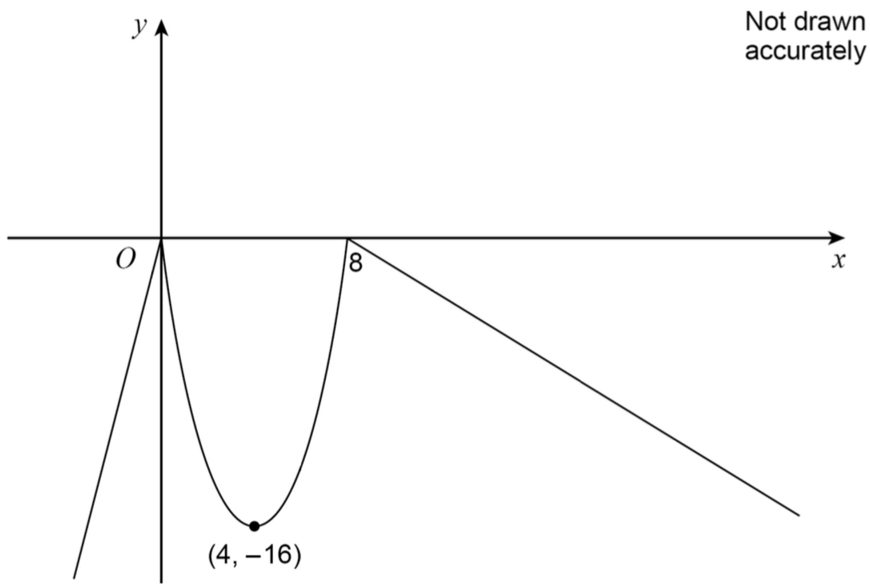
Solution

$$\begin{aligned} w^3x^2y^5 = w^{13}x^7 &\Rightarrow y^5 = w^{10}x^5 \\ &\Rightarrow y = \sqrt[5]{w^{10}x^5} \\ &\Rightarrow y = \sqrt[5]{(w^2x)^5} \\ &\Rightarrow \underline{\underline{y = w^2x}}. \end{aligned}$$

8. A function f is given by (4)

$$f(x) = \begin{cases} 4x, & x < 0, \\ x^2 - 8x, & 0 \leqslant x \leqslant 8, \\ 16 - 2x, & x > 8. \end{cases}$$

A sketch of $y = f(x)$ is shown.



Work out **all** the values of x for which

$$f(x) = -12.$$

Solution

$\underline{y = 4x,\ x < 0}$:

$$4x = -12 \Rightarrow x = -3.$$

$\underline{y = x^2 - 8x,\ 0 \leqslant x \leqslant 8}$:

$$x^2 - 8x = -12 \Rightarrow x^2 - 8x + 12 = 0$$

$$\left.\begin{array}{ll} \text{add to:} & -8 \\ \text{multiply to:} & +12 \end{array}\right\} -6,\ -2$$

$$\begin{aligned} &\Rightarrow (x-6)(x-2) = 0 \\ &\Rightarrow x - 6 = 0 \text{ or } x - 2 = 0 \\ &\Rightarrow x = 6 \text{ or } x = 2. \end{aligned}$$

$\underline{y = 16 - 2x,\ x > 8}$:

$$\begin{aligned} 16 - 2x = -12 &\Rightarrow -2x = -28 \\ &\Rightarrow x = 14. \end{aligned}$$

Hence, the values of x are

$$\underline{\underline{-3,\ 2,\ 6,\ \text{or } 14}}.$$

9. (a) Circle the expression that is equivalent to (1)

$$\frac{1}{a} + \frac{1}{b}:$$

$$\frac{2}{a+b} \quad \frac{ab}{b+a} \quad \frac{2}{ab} \quad \frac{b+a}{ab}.$$

Solution

Well,

$$\begin{aligned} \frac{1}{a} + \frac{1}{b} &= \frac{b}{ab} + \frac{a}{ab} \\ &= \frac{b+a}{ab} \end{aligned}$$

so

$$\frac{2}{a+b} \quad \frac{ab}{b+a} \quad \frac{2}{ab} \quad \underline{\underline{\frac{b+a}{ab}}}.$$

(b) Simplify fully (3)

$$\frac{6c^4 - c^3}{36c^2 - 1}.$$

Solution

Difference of two squares:

$$\begin{aligned} 36c^2 - 1 &= (6c)^2 - 1^2 \\ &= (6c-1)(6c+1). \end{aligned}$$

Finally,

$$\begin{aligned} \frac{6c^4 - c^3}{36c^2 - 1} &= \frac{c^3(6c-1)}{(6c-1)(6c+1)} \\ &= \underline{\underline{\frac{c^3}{6c+1}}}. \end{aligned}$$

10. The radius of a sphere, in cm, is $\frac{3}{2}k$. (3)

The volume of the sphere, in cm^3, is 972π.

Volume of a sphere = $\frac{4}{3}\pi r^3$ where r is the radius

Work out the value of k.

Solution

$$\begin{aligned} V = \tfrac{4}{3}\pi r^3 \Rightarrow 972\pi &= \tfrac{4}{3}\pi \times (\tfrac{3}{2}k)^3 \\ \Rightarrow (\tfrac{3}{2}k)^3 &= 729 \\ \Rightarrow (\tfrac{3}{2}k)^3 &= 9^3 \\ \Rightarrow \tfrac{3}{2}k &= 9 \\ \Rightarrow \underline{\underline{k = 6}}. \end{aligned}$$

11. Expand and simplify fully (3)

$$(5x + 3y^2)(4x - y^2).$$

Solution

$\times$	$5x$	$+3y^2$
$4x$	$20x^2$	$+12xy^2$
$-y^2$	$-5xy^2$	$-3y^4$

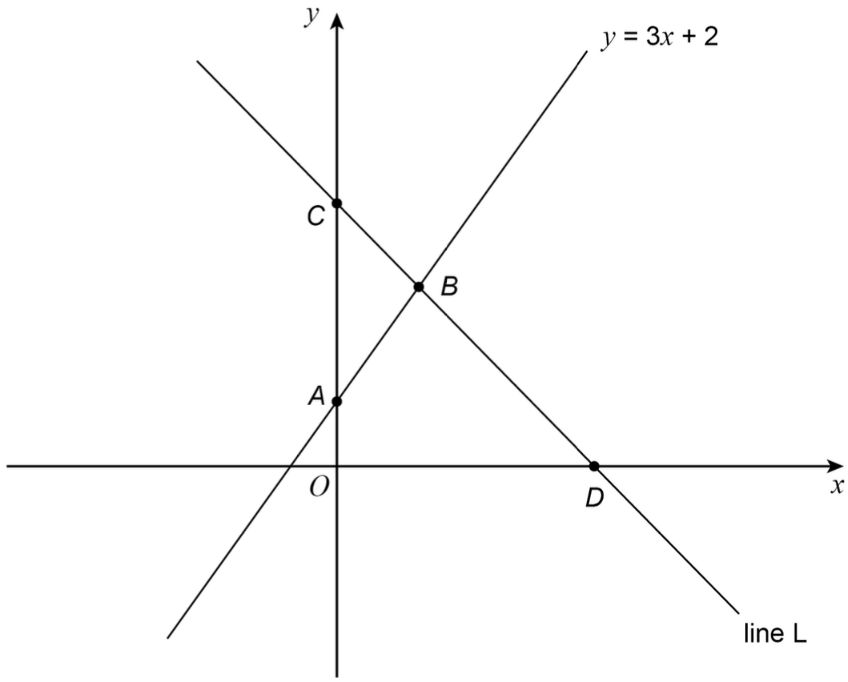
Hence,

$$(5x + 3y^2)(4x - y^2) = \underline{\underline{20x^2 + 7xy^2 - 3y^4}}.$$

12. A and B are points on the line (5)

$$y = 3x + 2.$$

- B, C, and $D(5, 0)$ are points on the line L.
- $OA : AC = 1 : 4$.



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Work out the x-coordinate of B.

Solution

Well, $A(0, 2)$. Now,

$$\begin{aligned}\overrightarrow{OC} &= 5\overrightarrow{OA} \\ &= 5\begin{pmatrix} 0 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 10 \end{pmatrix}\end{aligned}$$

so $C(0, 10)$. Next,

$$\begin{aligned}m_{CD} &= \frac{10-0}{0-5} \\ &= -2\end{aligned}$$

and the equation of the line L is

$$y - 0 = -2(x - 5) \Rightarrow y = -2x + 10.$$

Equate the two terms:

$$\begin{aligned} 3x + 2 = -2x + 10 &\Rightarrow 5x = 8 \\ &\Rightarrow \underline{\underline{x = 1\tfrac{3}{5}}}. \end{aligned}$$

13. P is the point on the curve (4)

$$y = ax^3 + 10x^2,$$

where $x = 2$.

The gradient of the **normal** to the curve at P is $-\frac{1}{4}$.

Work out the value of a.

Solution

$$y = ax^3 + 10x^2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3ax^2 + 20x.$$

Now,

$$m_{\text{normal}} = -\tfrac{1}{4} \Rightarrow m_{\text{tangent}} = 4.$$

Next,

$$\begin{aligned} x = 2, m_{\text{tangent}} = 4 &\Rightarrow 3a(2^2) + 20(2) = 4 \\ &\Rightarrow 12a + 40 = 4 \\ &\Rightarrow 12a = -36 \\ &\Rightarrow \underline{\underline{a = -3}}. \end{aligned}$$

14.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Describe geometrically the single transformation represented by $\mathbf{A}$. (1)

Solution

It is a reflection in the x-axis.

$$\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(b) Describe geometrically the single transformation represented by $\mathbf{B}^2$. (2)

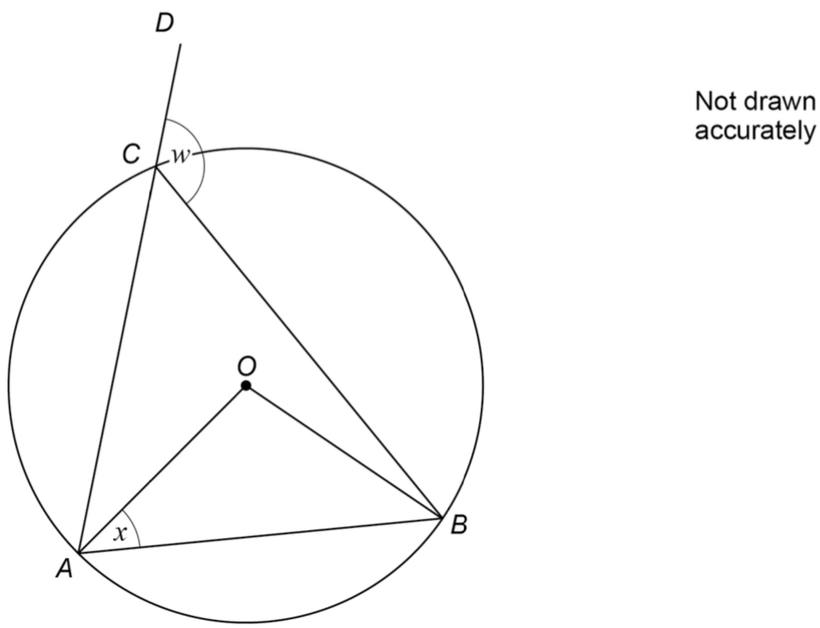
Solution

$$\mathbf{B}^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix};$$

it is a rotation, through 180°, centre $(0, 0)$).

15. A, B, and C are points on a circle, centre O. (5)

- ACD is a straight line.
- Angle $BCD = w$.



Prove that

$$w = x + 90^\circ.$$

Solution

Well, $\angle ACB = 180 - w$ (supplementary angles)
$\angle AOB = 2(180 - w) = 360 - 2w$ (angle at the centre is twice the angle at the circumference)
$\angle OAB = \angle OBA = x$ (base angles).

Now, as there are 180° in a triangle,

$$\begin{aligned} x + (360 - 2w) + x = 180 &\Rightarrow 2x - 2w + 360 = 180 \\ &\Rightarrow 2(x - w + 180) = 180 \\ &\Rightarrow x - w + 180 = 90 \\ &\Rightarrow \underline{\underline{w = x + 90}}, \end{aligned}$$

as required.

16. The coefficient of x^4 in the expansion of (3)

$$(a + 2x)^6$$

is 1 500.

Work out the two possible values of a.

Solution

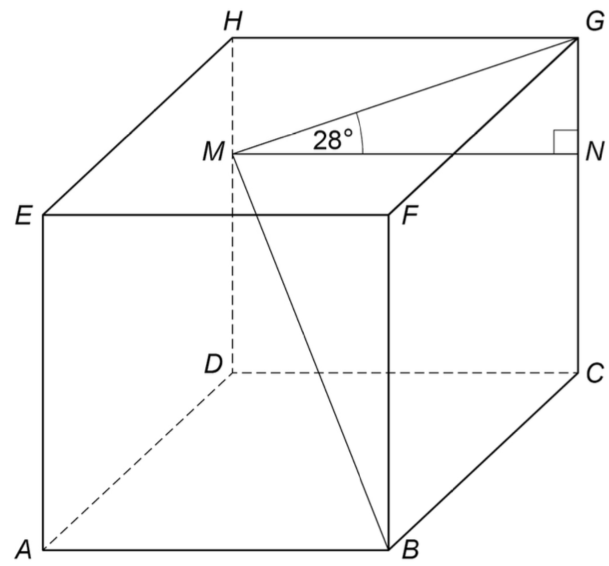
The coefficient of x^4 is

$$\binom{6}{4}(a)^2(2)^4 = 240a^2.$$

So,

$$\begin{aligned} 240a^2 = 1\,500 &\Rightarrow a^2 = 6\tfrac{1}{4} \\ &\Rightarrow \underline{\underline{a = \pm 2\tfrac{1}{2}}}. \end{aligned}$$

17. $ABCDEFGH$ is a cube with side length 32 cm. (5)
M and N are points on DH and CG respectively.



Work out the size of the angle that the line BM makes with the plane $ABCD$.

Solution

$$\tan = \frac{\text{opp}}{\text{adj}} \Rightarrow \tan 28^\circ = \frac{GN}{32}$$
$$\Rightarrow GN = 32 \tan 28^\circ$$
$$\Rightarrow DM = 32 - 32 \tan 28^\circ.$$

Now,

$$BM^2 = AB^2 + AD^2 + DM^2 \Rightarrow BM^2 = 32^2 + 32^2 + (32 - 32 \tan 28^\circ)^2$$
$$\Rightarrow BM^2 = 2\,272.559\,162 \text{ (FCD)}$$
$$\Rightarrow BM = 47.671\,366\,27 \text{ (FCD)}.$$

Finally, let x be the required angle. Then

$$\sin = \frac{\text{opp}}{\text{hyp}} \Rightarrow \sin x = \frac{32(1 - \tan 28^\circ)}{47.671\,366\ldots}$$
$$\Rightarrow x = 18.321\,330\,46 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{x = 18.3^\circ \text{ (3 sf)}}}.$$

18. (5)

$$y = 12x + \frac{3}{x}.$$

Show that y has a minimum value when $x = 0.5$.

Solution

Well, surely there's a mistake in the question: what happens if $x = 0$? Anyway,

$$\begin{aligned} y = 12x + \frac{3}{x} &\Rightarrow y = 12x + 3x^{-1} \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 12 + 3(-x^{-2}) \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 12 - 3x^{-2} \\ &\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -3(-2x^{-3}) \\ &\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x^{-3}. \end{aligned}$$

Now,

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 12 - 3x^{-2} = 0 \\ &\Rightarrow 3x^{-2} = 12 \\ &\Rightarrow x^{-2} = 4 \\ &\Rightarrow x^2 = \tfrac{1}{4} \\ &\Rightarrow x = \pm\tfrac{1}{2} \end{aligned}$$

and

$$x = \tfrac{1}{2} \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 48.$$

So, we established that there is a candidate for the critical point at $x = \frac{1}{2}$ and then we the second derivate to prove the value is 48.

y has a minimum value when $x = 0.5$.

19.

$$\mathrm{f}(x) = (x + 2)^3.$$

g is a function such that

$$\mathrm{g}\,\mathrm{f}(x) = (x + 2)^{12}.$$

(a) Work out an expression for $\mathrm{g}(x)$. (1)

Solution

$$\mathrm{g}(x) = \underline{\underline{x^4}}.$$

$$\mathrm{h}(x) = x^2 + 5.$$

m is a function such that

$$\mathrm{h}\,\mathrm{m}(x) = 4x^2 + 5.$$

(b) Work out an expression for $\mathrm{m}\,\mathrm{h}(x)$. (2)

Solution

Well, $\mathrm{m}(x) = 2x$ and

$$\begin{aligned}\mathrm{m}\,\mathrm{h}(x) &= \mathrm{m}(\mathrm{h}(x)) \\ &= \mathrm{m}(x^2 + 5) \\ &= \underline{\underline{2(x^2 + 5)}}.\end{aligned}$$

20. Show that (4)

$$\frac{2\sin x + \cos x}{\tan x} - \frac{1}{\sin x}$$

can be written in the form

$$a\cos x + b\sin x,$$

where a and b are integers.

Solution

Well,

$$\begin{aligned}\frac{2\sin x+\cos x}{\tan x}-\frac{1}{\sin x} &\equiv \frac{2\sin x+\cos x}{\frac{\sin x}{\cos x}}-\frac{1}{\sin x}\\ &\equiv \frac{\cos x(2\sin x+\cos x)}{\sin x}-\frac{1}{\sin x}\\ &\equiv \frac{2\sin x\cos x+\cos^2 x-1}{\sin x}\\ &\equiv \frac{2\sin x\cos x-(1-\cos^2 x)}{\sin x}\\ &\equiv \frac{2\sin x\cos x-\sin^2 x}{\sin x}\\ &\equiv \frac{\sin x(2\cos x-\sin x)}{\sin x}\\ &\equiv \underline{\underline{2\cos x-\sin x}};\end{aligned}$$

hence, $\underline{\underline{a=2}}$ and $\underline{\underline{b=-1}}$.

21. (6)

$$3x^2+2bx+8a$$

can be written in the form

$$3(x+a)^2+b+2.$$

Work out the two possible pairs of values of a and b.

Solution

$\times$	x	$+a$
x	x^2	$+ax$
$+a$	$+ax$	$+a^2$

So

$$\begin{aligned}3(x+a)^2+b+2 &= 3(x^2+2ax+a^2)+b+2\\ &= 3x^2+6ax+(a^2+b+2).\end{aligned}$$

Coefficient of x:

$$2b=6a \Rightarrow b=3a \quad (1).$$

Coefficient of the constant term:

$$8a = 3a^2 + b + 2 \Rightarrow b = -3a^2 + 8a - 2 \quad (2).$$

Do (1) = (2):

$$3a = -3a^2 + 8a - 2 \Rightarrow 3a^2 - 5a + 2 = 0$$

$$\left.\begin{array}{ll} \text{add to:} & -5 \\ \text{multiply to:} & (+3) \times (+2) = +6 \end{array}\right\} -3,\ -2$$

e.g.,

$$\begin{aligned} &\Rightarrow 3a^2 - 3a - 2a + 2 = 0 \\ &\Rightarrow 3a(a-1) - 2(a-1) = 0 \\ &\Rightarrow (3a-2)(a-1) = 0 \\ &\Rightarrow 3a - 2 = 0 \text{ or } a - 1 = 0 \\ &\Rightarrow a = \tfrac{2}{3} \text{ or } a = 1 \\ &\Rightarrow b = 2 \text{ or } b = 3; \end{aligned}$$

hence, the two possible values are

$$\underline{\underline{a = \tfrac{2}{3}, b = 2}} \text{ or } \underline{\underline{a = 1, b = 3}}.$$