Dr Oliver Mathematics
Further Mathematics
First Order Differential Equations
Past Examination Questions

This booklet consists of 34 questions across a variety of examination topics.
The total number of marks available is 312.

1. (a) Find the general solution of the differential equation (8)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y \tan x = \cos^3 x,$$

expressing y in terms of x.

(b) Find the particular solution for which $y = 2$ when $x = \pi$. (2)

2. (a) Use the substitution $z = x + y$ to show that the differential equation (3)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x + y + 3}{x + y - 1} \qquad (\dagger)$$

may be written in the form

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{2(z+1)}{z-1}.$$

(b) Hence find the general solution of the differential equation (†). (4)

3. (a) Show that (2)

$$\sqrt{\frac{1+x}{1-x}}$$

is an integrating factor for the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{1-x^2} = \sqrt{1-x},\ |x| < 1.$$

(b) Hence find the solution of the differential equation for which $y = 2$ when $x = 0$. Give your answer in the form $y = \mathrm{f}(x)$. (6)

4. (a) Use the substitution $y = xz$ to find the general solution of the differential equation (6)

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - y = x \cos\left(\frac{y}{x}\right).$$

(b) Find the solution of the differential equation for which $y = \pi$ when $x = 4$. (2)

5. The substitution $y = u^k$, where k is an integer, is used to solve the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = x^2y^2 \qquad (\dagger)$$

by changing it into an equation ($\ddagger$) in the variables u and x.

(a) Show that equation ($\ddagger$) may be written in the form (4)

$$\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{3u}{kx} = \frac{1}{k}xu^{k+1}.$$

(b) Write down the value of k for which the integrating factor method may be used to solve equation ($\ddagger$). (1)

(c) Using this value of k, solve equation ($\ddagger$) and hence find the general solution of equation ($\dagger$), giving your answer in the form $y = \mathrm{f}(x)$. (4)

6. The variables x and y are related by the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x^2 + y^2}{xy}. \qquad (\dagger)$$

(a) Use the substitution $y = ux$, where u is a function of x to transform the differential equation ($\dagger$) into (3)

$$x\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2}{u}.$$

(b) Hence find the general solution of differential equation ($\dagger$), giving your answer in the form $y^2 = \mathrm{f}(x)$. (4)

7. Find the solution of the differential equation (9)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y\cot x = 2x$$

for which $y = 2$ when $x = \frac{\pi}{6}$. Give your answer in the form $y = \mathrm{f}(x)$.

8. Solve the differential equation (8)

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = x^4\mathrm{e}^{2x}$$

for y in terms of x, given that $y = 0$ when $x = 1$.

9. The differential equation (8)

$$3xy^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y^3 = \frac{\cos x}{x}$$

is to be solved for $x > 0$. Use the substitution $u = y^3$ to find the general solution for y in terms of x.

10. (a) By using an integrating factor, find the general solution of the differential equation (7)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{4y}{2x+1} = 4(2x+1)^5.$$

(b) The gradient of a curve at any point (x, y) on the curve is given by the differential equation (3)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{4y}{2x+1} = 4(2x+1)^5.$$

The point whose x-coordinate is zero is a stationary point of the curve. Using your answer to part (a), find the equation of the curve.

11. (a) Differentiate $\ln(\ln x)$ with respect to x. (1)

(b) (i) Show that $\ln x$ is an integrating factor for the first-order differential equation (2)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x \ln x} y = 9x^2, \ x > 1.$$

(ii) Hence find the solution of this differential equation, given that $y = 4\mathrm{e}^3$ when $x = \mathrm{e}$. (6)

12. By using an integrating factor, find the general solution of the differential equation (6)

$$\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{2x}{x^2+4} u = 3(x^2+4),$$

giving your answer in the form $u = \mathrm{f}(x)$.

13. (a) Find the particular values of the constants a, b, and c for which $a + b \sin 2x + c \cos 2x$ satisfies the differential equation (4)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 20 - 20 \cos 2x.$$

(b) Hence find the solution of this differential equation, given that $y = 4$ when $x = 0$. (4)

14. Find the general solution of the differential equation (7)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y \cot 2x = \sin x, \ 0 < x < \tfrac{\pi}{2},$$

giving your answer in the form $y = \mathrm{f}(x)$.

15. Find the general solution of the differential equation (7)

$$(x+1)\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \frac{1}{x},$$

giving your answer in the form $y = \mathrm{f}(x)$.

16. During an industrial process, the mass of salt, S kg, dissolved in a liquid t minutes after the process begins is modelled by the differential equation

$$\frac{\mathrm{d}S}{\mathrm{d}t} + \frac{2S}{120 - t} = \tfrac{1}{4},\ 0 \leqslant t < 120.$$

Given that $S = 6$ when $t = 0$,

(a) find S in terms of t, (8)

(b) calculate the maximum mass of salt that the model predicts will be dissolved in the liquid at any one time during the process. (4)

17. Obtain the general solution of the differential equation (8)

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \cos x,\ x > 0,$$

giving your answer in the form $y = \mathrm{f}(x)$.

18. (7)

$$\frac{\mathrm{d}y}{\mathrm{d}x} - y\tan x = 2\sec^3 x.$$

Giving that $y = 3$ and $x = 0$, find y in terms of x.

19. Solve the differential equation (7)

$$\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = x,$$

to obtain y as a function of x.

20. (a) Show that the substitution $y = vx$ transforms the differential equation (3)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} + \frac{3y}{x},\ x > 0,\ y > 0 \quad (\dagger)$$

into the differential equation

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = 2v + \frac{1}{v} \quad (\ddagger).$$

(b) By solving differential equation (‡), find a general solution of the differential equation (†) in the form $y = \mathrm{f}(x)$. (7)

Given that $y = 3$ and $x = 1$,

(c) find the particular solution of differential equation (†). (2)

21. (a) Show that the substitution $y = \frac{1}{t}$ transforms the differential equation (4)

$$\sin x \frac{\mathrm{d}y}{\mathrm{d}x} + y \cos x = y^2,\ 0 < x < \pi \quad (\dagger)$$

into the differential equation

$$\frac{\mathrm{d}t}{\mathrm{d}x} - t \cot x = -\operatorname{cosec} x,\ 0 < x < \pi \quad (\ddagger).$$

(b) Solve the differential equation (‡). (5)

(c) Hence show that (2)

$$y = \frac{1}{\cos x + c \sin x},$$

where c is an arbitrary constant, is a general solution of the differential equation (†).

Given that $y = \frac{\sqrt{2}}{3}$ at $x = \frac{\pi}{4}$,

(d) find the value of y at $x = \frac{\pi}{2}$. (3)

22. (a) Find, in the form $y = \mathrm{f}(x)$, the general solution of the equation (6)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y \cot x = \sin x,\ 0 < x < \pi.$$

Given that $y = 1$ at $x = \frac{\pi}{2}$,

(b) show that, at $x = \frac{\pi}{4}$, (4)

$$y = \frac{(6 - \pi)\sqrt{2}}{8}.$$

23. (a) Show that the substitution $z = \dfrac{1}{y^2}$ transforms the differential equation (4)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y = 4xy^3,\ y > 0 \quad (\dagger)$$

into the differential equation

$$\frac{\mathrm{d}z}{\mathrm{d}x} - 2z = -8x \quad (\ddagger).$$

(b) Hence find the solution of the differential equation (†) in the form $y = \mathrm{f}(x)$. (7)

The stationary point of the graph of a particular solution of the differential equation (†) is (x_1, y_1), $x_1 > 0$.

(c) Show that $y_1 = \dfrac{1}{2\sqrt{x_1}}$. (2)

24. Find the general solution of the differential equation (7)

$$\sin \frac{\mathrm{d}y}{\mathrm{d}x} - y \cos x = \sin 2x \sin x,$$

giving your answer in the form $y = \mathrm{f}(x)$.

25. (a) Show that the substitution $z = y^{\frac{1}{2}}$ transforms the differential equation (5)

$$\frac{\mathrm{d}y}{\mathrm{d}x} - 4y \tan x = 2y^{\frac{1}{2}} \quad (\dagger)$$

into the differential equation

$$\frac{\mathrm{d}z}{\mathrm{d}x} - 2z \tan x = 1 \quad (\ddagger).$$

(b) Solve the differential equation ($\ddagger$) to find z as a function of x. (6)

(c) Hence obtain the general solution of the differential equation ($\dagger$). (1)

26. Find the general solution of the differential equation (8)

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = \frac{\ln x}{x},\ x > 0,$$

giving your answer in the form $y = \mathrm{f}(x)$.

27. (a) Show that the substitution $y = vx$ transforms the differential equation (3)

$$3xy^2\frac{\mathrm{d}y}{\mathrm{d}x} = x^3 + y^3 \quad (\dagger)$$

into the differential equation

$$3v^2x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 - 2v^3 \quad (\ddagger).$$

(b) By solving differential equation ($\ddagger$), find a general solution of the differential equation ($\dagger$) in the form $y = \mathrm{f}(x)$. (6)

Given that $y = 2$ at $x = 1$,

(c) find the value of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at $x = 1$. (2)

28. (a) Find, in the form $y = \mathrm{f}(x)$, the general solution of the equation (6)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\tan x = \sin 2x, 0 < x < \tfrac{\pi}{2}.$$

Given that $y = 2$ at $x = \frac{\pi}{3}$,

(b) find the value of y at $x = \frac{\pi}{6}$, giving your answer in the form $a + k\ln b$ where a and b are integers and k is rational. (4)

29. (a) Find the general solution of the differential equation (5)

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 4x^2.$$

(b) Find the particular solution for which $y = 5$ at $x = 1$, giving your answer in the form $y = \mathrm{f}(x)$. (2)

(c) (i) Find the exact values of the coordinates of the turning points of the curve with equation $y = \mathrm{f}(x)$, making your method clear. (2)

(ii) Sketch the curve with equation $y = \mathrm{f}(x)$, showing the coordinates of the turning points. (3)

30. (a) Show that the substitution $v = y^{-3}$ transforms the differential equation (5)

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 2x^4y^4 \qquad (\dagger)$$

into the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{3v}{x} = -6x^3. \qquad (\ddagger)$$

(b) By solving the differential equation (‡), find a general solution of differential equation (†) in the form $y^3 = \mathrm{f}(x)$. (6)

31. (a) Find the general solution of the differential equation (6)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\tan x = \mathrm{e}^{4x}\cos^2 x, -\tfrac{\pi}{2} < x < \tfrac{\pi}{2},$$

giving your answer in the form $y = \mathrm{f}(x)$.

(b) Find the particular solution for which $y = 1$ at $x = 0$. (2)

32. Find, in the form $y = \mathrm{f}(x)$, he general solution of the differential equation (6)

$$\tan x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 3\cos 2x\tan x,\ 0 < x < \tfrac{\pi}{2}.$$

33.

$$p\frac{\mathrm{d}x}{\mathrm{d}t} + qx = r, \text{ where } p, q, \text{ and } r \text{ are constants.}$$

(a) Given that $x = 0$ when $t = 0$,

(i) find x in terms of t, (4)

(ii) find the limiting value of x as $t \to \infty$. (1)

(b) (7)

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} + 2y = \sin\theta.$$

Given that $y = 0$ when $\theta = 0$, find y in terms of θ.

34. (a) Find, in the form $y = \mathrm{f}(x)$, the general solution of the equation (8)

$$\cos x\frac{\mathrm{d}y}{\mathrm{d}x} + 2\sin x = 2\cos^3 x\sin x + 1, 0 < x < \tfrac{\pi}{2}.$$

Given that $y = 5\sqrt{2}$ when $x = \frac{\pi}{4}$,

(b) find the value of y when $x = \frac{\pi}{6}$, giving your answer in the form $a + b\sqrt{3}$, where a and b are rational numbers to be found. (3)