Dr Oliver Mathematics
Further Mathematics: Further Pure Mathematics 1 (Paper 3A)
June 2022: Calculator
1 hour 30 minutes

The total number of marks available is 75.
You must write down all the stages in your working.
Inexact answers should be given to three significant figures unless otherwise stated.

1. An ellipse has equation

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$

and eccentricity e_1.

A hyperbola has equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

and eccentricity e_2.

Given that

$$e_1 \times e_2 = 1,$$

(a) show that (4)

$$a^2 = 3b^2.$$

Solution
For the ellipse, $a^2 = 16$ and $b^2 = 4$:

$$\begin{aligned} b^2 = a^2(1 - e_1^2) &\Rightarrow 4 = 16(1 - e_1^2) \\ &\Rightarrow 1 - e_1^2 = \tfrac{1}{4} \\ &\Rightarrow e_1^2 = \tfrac{3}{4}. \end{aligned}$$

For the hyperbola,

$$\begin{aligned} b^2 = a^2(e_2^2 - 1) &\Rightarrow b^2 = a^2\left(\frac{1}{e_1^2} - 1\right) \\ &\Rightarrow b^2 = a^2\left(\frac{1}{\frac{3}{4}} - 1\right) \\ &\Rightarrow b^2 = a^2\left(\tfrac{4}{3} - 1\right) \\ &\Rightarrow b^2 = \tfrac{1}{3}a^2 \\ &\Rightarrow \underline{\underline{a^2 = 3b^2}}, \end{aligned}$$

as required.

Given also that the coordinates of the foci of the ellipse are the same as the coordinates of the foci of the hyperbola,

(b) determine the equation of the hyperbola. (3)

Solution

Well, for the focus of the ellipse,

$$x = 4 \times \tfrac{\sqrt{3}}{2} = 2\sqrt{3},$$

and, the focus of the hyperbola,

$$\begin{aligned} x = a \times \tfrac{2}{\sqrt{3}} &\Rightarrow 2\sqrt{3} = a \times \tfrac{2}{\sqrt{3}} \\ &\Rightarrow a = 3 \\ &\Rightarrow a^2 = 9 \\ &\Rightarrow b^2 = \tfrac{1}{3} \times 9 \\ &\Rightarrow b^2 = 3. \end{aligned}$$

Hence, the equation of the hyperbola is

$$\underline{\underline{\frac{x^2}{9} - \frac{y^2}{3} = 1.}}$$

2. During 2029, the number of hours of daylight per day in London, H, is modelled by the equation

$$H = 0.3\sin(\tfrac{x}{60}) - 4\cos(\tfrac{x}{60}) + 11.5,\ 0 \leqslant x < 365,$$

where x is the number of days after 1st January 2029 and the angle is in radians.

(a) Show that, according to the model, the number of hours of daylight in London on the 31st January 2029 will be 8.13 to 3 significant figures. (1)

Solution

$$
\begin{aligned}
x = 30 \Rightarrow H &= 0.3\sin(\tfrac{30}{60}) - 4\cos(\tfrac{30}{60}) + 11.5 \\
\Rightarrow H &= 8.133\,497\,414 \text{ (FCD)} \\
\Rightarrow \underline{\underline{H = 8.13 \text{ (3 sf)}}}&,
\end{aligned}
$$

as required.

(b) Use the substitution (2)

$$\tan(\tfrac{x}{120})$$

to show that H can be written as

$$H = \frac{at^2 + bt + c}{1 + t^2},$$

where a, b, and c are constants to be determined.

Solution

Here, we now have to substitute

$$\sin(\tfrac{x}{60}) = \frac{2t}{1+t^2} \text{ and } \cos(\tfrac{x}{60}) = \frac{1-t^2}{1+t^2}:$$

$$
\begin{aligned}
H &= 0.3\sin(\tfrac{x}{60}) - 4\cos(\tfrac{x}{60}) + 11.5 \\
&= \frac{0.3(2t)}{1+t^2} - \frac{4(1-t^2)}{1+t^2} + 11.5 \\
&= \frac{0.3(2t) - 4(1-t^2) + 11.5(1+t^2)}{1+t^2} \\
&= \frac{0.6t - 4 + 4t^2 + 11.5 + 11.5t^2}{1+t^2} \\
&= \underline{\underline{H = \frac{15.5t^2 + 0.6t + 7.5}{1+t^2}}};
\end{aligned}
$$

hence, $\underline{\underline{a = 15.5}}$, $\underline{\underline{b = 0.6}}$, and $\underline{\underline{c = 7.5}}$

(c) Hence determine, according to the model, the date of the first day of 2029 when there will be at least 12 hours of daylight in London. (4)

Solution

$$\begin{aligned} 12 = \frac{15.5t^2 + 0.6t + 7.5}{1 + t^2} &\Rightarrow 12(1 + t^2) = 15.5t^2 + 0.6t + 7.5 \\ &\Rightarrow 12 + 12t^2 = 15.5t^2 + 0.6t + 7.5 \\ &\Rightarrow 3.5t^2 + 0.6t - 4.5 = 0. \end{aligned}$$

Quadratic formula: $a = 3.5$, $b = 0.6$, $c = -4.5$:

$$\begin{aligned} t &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-0.6 \pm \sqrt{0.6^2 - 4 \times 3.5 \times (-4.5)}}{2 \times 3.5} \\ &= \frac{-0.6 \pm \sqrt{63.36}}{7} \\ &= -1.222\,842\,785,\ 1.051\,414\,214 \text{ (FCD)}; \end{aligned}$$

now, pick the second one (why?):

$$\begin{aligned} t = 1.051\,414\ldots &\Rightarrow x = 120 \tan^{-1}(1.051\,414\ldots) \\ &\Rightarrow x = 97.254\,687\,86 \text{ (FCD)}. \end{aligned}$$

Finally, there will be at least 12 hours of daylight in London is 8 April or 9 April.

3. With respect to a fixed origin O, the points A and B have coordinates $(2, 2, -1)$ and $(4, 2p, 1)$, respectively, where p is a constant.

For each of the following, determine the possible values of p for which,

(a) OB makes an angle of 45° with the positive x-axis, (3)

Solution

Well,

$$\begin{aligned} OB &= \sqrt{4^2 + (2p)^2 + 1^2} \\ &= \sqrt{4p^2 + 17} \end{aligned}$$

and

$$\cos 45^\circ = \frac{4}{\sqrt{4p^2+17}} \Rightarrow \sqrt{4p^2+17} = \frac{4}{\frac{1}{\sqrt{2}}}$$
$$\Rightarrow \sqrt{4p^2+17} = 4\sqrt{2}$$
$$\Rightarrow 4p^2 + 17 = 32$$
$$\Rightarrow 4p^2 = 15$$
$$\Rightarrow p^2 = \tfrac{15}{4}$$
$$\Rightarrow \underline{\underline{p = \pm\tfrac{\sqrt{15}}{4}}}.$$

(b) $\overrightarrow{OA} \times \overrightarrow{OB}$ is parallel to $\begin{pmatrix} 4 \\ -p \\ 2 \end{pmatrix}$, (3)

Solution

Now,

$$\overrightarrow{OA} \times \overrightarrow{OB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -1 \\ 4 & 2p & 1 \end{vmatrix}$$
$$= (2+2p)\mathbf{i} - 6\mathbf{j} + (4p-8)\mathbf{k}.$$

Next,

$$(2+2p)\mathbf{i} - 6\mathbf{j} + (4p-8)\mathbf{k} = (4\lambda)\mathbf{i} + (-p\lambda)\mathbf{j} + (2\lambda)\mathbf{k},$$

for some constant λ. So,

$$2 + 2p = 4\lambda \quad (1)$$
$$-6 = -p\lambda \quad (2)$$
$$4p - 8 = 2\lambda \quad (3).$$

$2 \times (3)$:

$$8p - 16 = 4\lambda \quad (4)$$

and do $(4) - (1)$:

$$(8p-16) - (2+2p) = 0 \Rightarrow 6p = 18$$
$$\Rightarrow \underline{\underline{p = 3}}.$$
$$(\Rightarrow \lambda = 2)$$

(c) the area of triangle OAB is $3\sqrt{2}$. (3)

Solution

$$
\begin{aligned}
& \tfrac{1}{2}|\overrightarrow{OA} \times \overrightarrow{OB}| = 3\sqrt{2} \\
\Rightarrow\ & |\overrightarrow{OA} \times \overrightarrow{OB}| = 6\sqrt{2} \\
\Rightarrow\ & (2+2p)^2 + (-6)^2 + (4p-8)^2 = (6\sqrt{2})^2 \\
\Rightarrow\ & (4+8p+4p^2) + 36 + (16p^2 - 64p + 64) = 72 \\
\Rightarrow\ & 20p^2 - 56p + 32 = 0 \\
\Rightarrow\ & 20p^2 - 56p + 32 = 0 \\
\Rightarrow\ & 4(5p^2 - 14p + 8) = 0
\end{aligned}
$$

$$
\left.\begin{array}{ll}
\text{add to:} & -14 \\
\text{multiply to:} & (+5) \times (+8) = +40
\end{array}\right\} -10,\ -4
$$

$$
\begin{aligned}
\Rightarrow\ & 4[5p^2 - 10p - 4p + 8] = 0 \\
\Rightarrow\ & 4[5p(p-2) - 4(p-2)] = 0 \\
\Rightarrow\ & 4(5p-4)(p-2) = 0 \\
\Rightarrow\ & \underline{\underline{p = \tfrac{4}{5} \text{ or } p = 2.}}
\end{aligned}
$$

4. The velocity v ms^{-1}, of a raindrop, t seconds after it falls from a cloud, is modelled by the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -0.1v^2 + 10,\ t \geqslant 0.$$

Initially the raindrop is at rest.

(a) Use two iterations of the approximation formula (5)

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n \approx \frac{y_{n+1} - y_n}{h},$$

to estimate the velocity of the raindrop 1 second after it falls from the cloud.

Solution

Well,

$$\begin{aligned} t_0 &= 0, \\ v_0 &= 0, \\ \left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)_0 &= 10, \text{ and} \\ h &= 0.5, \end{aligned}$$

as we halve h in to two parts. Now,

$$\begin{aligned} v_1 &= v_0 + h\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)_0 \\ &= 0 + 0.5 \times 10 \\ &= 5; \end{aligned}$$

$$\begin{aligned} \left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)_1 &= -0.1(5^2) + 10 \\ &= -2.5 + 10 \\ &= 7.5; \end{aligned}$$

$$\begin{aligned} v_2 &= v_1 + h\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)_1 \\ &= 5 + 0.5 \times 7.5 \\ &= 5 + 3.75 \\ &= \underline{\underline{8.75 \text{ ms}^{-1}}}. \end{aligned}$$

Given that the initial acceleration of the raindrop is found to be smaller than is suggested by the current model,

(b) refine the model by changing the value of one constant. (1)

Solution
E.g.,

$$\underline{\underline{\frac{\mathrm{d}v}{\mathrm{d}t} = -0.1v^2 + A,}}$$

where $0 < A < 10$.

5. The rectangular hyperbola H has equation

$$xy = 36.$$

(a) Use calculus to show that the equation of the tangent to H at the point $P\left(6t, \frac{6}{t}\right)$ is (3)

$$yt^2 + x = 12t.$$

Solution

Implicit differentiation:

$$\begin{aligned} xy = 36 &\Rightarrow x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0 \\ &\Rightarrow x\frac{\mathrm{d}y}{\mathrm{d}x} = -y \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x}. \end{aligned}$$

Now,

$$\begin{aligned} x = 6t, y = \frac{6}{t} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{\frac{6}{t}}{6t} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{1}{t^2} \end{aligned}$$

and the equation of the tangent is

$$\begin{aligned} y - \frac{6}{t} = -\frac{1}{t^2}(x - 6t) &\Rightarrow t^2\left(y - \frac{6}{t}\right) = -(x - 6t) \\ &\Rightarrow yt^2 - 6t = -x + 6t \\ &\Rightarrow \underline{\underline{yt^2 + x = 12t}}, \end{aligned}$$

as required.

The point $Q\left(12t, \frac{3}{t}\right)$ also lies on H.

(b) Find the equation of the tangent to H at the point Q. (2)

Solution

$$x = 12t, y = \frac{3}{t} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\frac{3}{t}}{12t}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{4t^2}$$

and the equation of the tangent is

$$y - \frac{3}{t} = -\frac{1}{4t^2}(x - 12t) \Rightarrow 4t^2\left(y - \frac{3}{t}\right) = -(x - 12t)$$
$$\Rightarrow 4yt^2 - 12t = -x + 12t$$
$$\Rightarrow \underline{\underline{4yt^2 + x = 24t}}.$$

The tangent at P and the tangent at Q meet at the point R.

(c) Show that as t varies the locus of R is also a rectangular hyperbola. (4)

Solution

Now,

$$yt^2 + x = 12t \quad (1)$$
$$4yt^2 + x = 24t \quad (2).$$

Do (2) − (1):

$$3yt^2 = 12t \Rightarrow 3yt^2 - 12t = 0$$
$$\Rightarrow 3t(yt - 4) = 0$$
$$\Rightarrow t = 0 \text{ or } y = \frac{4}{t}.$$

Ignore $t = 0$ (why?) and we get

$$y = \frac{4}{t} \Rightarrow x = 8t.$$

Finally,

$$xy = (8t)\left(\frac{4}{t}\right) \Rightarrow \underline{\underline{xy = 32}}.$$

What shape is this? It is a rectangular hyperbola, of course!

6. The points P, Q, and R have position vectors

$$\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix}, \text{ and } \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}.$$

respectively.

(a) Determine a vector equation of the plane that passes through the points P, Q, and R, giving your answer in the form (2)

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{b} + \mu \mathbf{c},$$

where λ and μ are scalar parameters.

Solution

Well,

$$\overrightarrow{PQ} = \begin{pmatrix} 2 \\ 3 \\ -9 \end{pmatrix} \text{ and } \overrightarrow{PR} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

and a vector equation is

$$\underline{\underline{\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -9 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}}}.$$

(b) Determine the coordinates of the point of intersection of the plane with the x-axis. (4)

Solution

We do this by setting $y = z = 0$: this gives us

$$\begin{aligned} -2 + 3\lambda + 2\mu &= 0 \quad (1) \\ 4 - 9\lambda - \mu &= 0 \quad (2). \end{aligned}$$

Do $2 \times (2)$:

$$8 - 18\lambda - 2\mu = 0 \quad (3)$$

and do $(1) + (3)$:

$$\begin{aligned} 6 - 15\lambda = 0 \Rightarrow 15\lambda &= 6 \\ \Rightarrow \lambda &= \tfrac{2}{5} \\ \Rightarrow \mu &= \tfrac{2}{5}. \end{aligned}$$

Next,

$$x = 1 + 2(\tfrac{2}{5}) + (\tfrac{2}{5}) = 2\tfrac{1}{5}$$

and the coordinates of the point is

$$\underline{\underline{(2\tfrac{1}{5}, 0, 0)}}.$$

7. Figure 1 shows a sketch of the curve with equation

$$y = |x^2 - 8|$$

and a sketch of the straight line with equation $y = mx + c$, where m and c are positive constants.

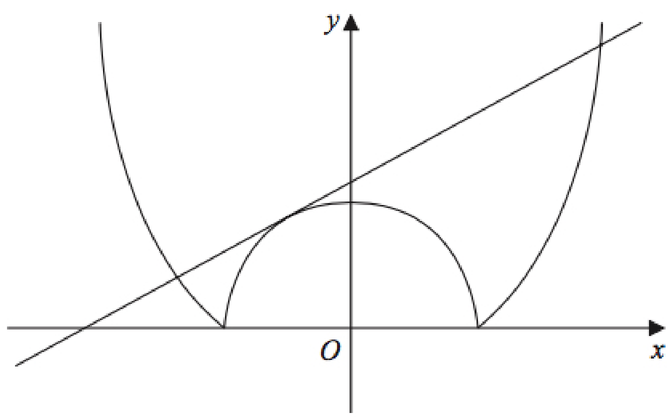

Figure 1: $y = |x^2 - 8|$ and $y = mx + c$

The equation

$$|x^2 - 8| = mx + c$$

has exactly 3 roots, as shown in Figure 1.

(a) Show that (2)

$$m^2 - 4c + 32 = 0.$$

Solution

Well,

$$x^2 - 8 = -(mx + c) \Rightarrow x^2 - 8 = -mx - c$$
$$\Rightarrow x^2 + mx + (c - 8) = 0$$

and, the setting the discriminant equal to zero, is

$$m^2 - 4(1)(c - 8) = 0 \Rightarrow \underline{\underline{m^2 - 4c + 32 = 0}},$$

as required.

Given that $c = 3m$,

(b) determine the value of m and the value of c. (3)

Solution

$$c = 3m \Rightarrow m^2 - 4(3m) + 32 = 0$$
$$\Rightarrow m^2 - 12m + 32 = 0$$

$$\left.\begin{array}{ll} \text{add to:} & -12 \\ \text{multiply to:} & +32 \end{array}\right\} -8,\ -4$$

$$\Rightarrow (m - 8)(m - 4) = 0$$
$$\Rightarrow m = 8 \text{ or } m = 4$$
$$\Rightarrow c = 24 \text{ or } c = 12.$$

Is it $y = 8x + 24$ or $y = 4x + 8$ or both? Well, for $y = 8x + 24$,

$$y = 0 \Rightarrow x = -3$$

and, for $y = 4x + 8$,

$$y = 0 \Rightarrow x = -2.$$

We can rule the first case, as

$$x^2 - 8 \Rightarrow x = \pm 2\sqrt{2} = \pm 2.828\ldots.$$

Hence,

$$\underline{\underline{m = 4}} \text{ and } \underline{\underline{c = 8}}.$$

(c) Hence solve (3)

$$|x^2 - 8| \geqslant mx + c.$$

Solution

$$\begin{aligned} x^2 - 8 = 4x + 12 &\Rightarrow x^2 - 4x = 20 \\ &\Rightarrow x^2 - 4x + 4 = 20 + 4 \\ &\Rightarrow (x-2)^2 = 24 \\ &\Rightarrow x - 2 = \pm 2\sqrt{6} \\ &\Rightarrow x = 2 \pm 2\sqrt{6} \end{aligned}$$

and

$$\begin{aligned} x^2 - 8 = -(4x + 12) &\Rightarrow x^2 - 8 = -4x - 12 \\ &\Rightarrow x^2 + 4x + 4 = 0 \\ &\Rightarrow (x+2)^2 = 0 \\ &\Rightarrow x = -2. \end{aligned}$$

We want the region where the curve is **not below** the straight line and, hence,

$$\underline{\underline{x \leqslant 2 - 2\sqrt{6}}},\ \underline{\underline{x = -2}},\ \text{or}\ \underline{\underline{x \geqslant 2 + 2\sqrt{6}}}.$$

8. (a) (i) Use differentiation to determine the Taylor series expansion of $\ln x$, in ascending powers of $(x-1)$, up to and including the term in $(x-1)^2$. (4)

Solution
Well,

$$\begin{aligned} \mathrm{f}(x) = \ln x &\Rightarrow \mathrm{f}(1) = 0 \\ \mathrm{f}'(x) = x^{-1} &\Rightarrow \mathrm{f}'(1) = 1 \\ \mathrm{f}'(x) = -x^{-2} &\Rightarrow \mathrm{f}''(1) = -1 \end{aligned}$$

and the Taylor series expansion is

$$\begin{aligned} \ln x &= 0 + (1)(x-1) + \tfrac{1}{2!}(-1)(x-1)^2 + \ldots \\ &= \underline{\underline{(x-1) - \tfrac{1}{2}(x-1)^2 + \ldots}}. \end{aligned}$$

(ii) Hence prove that (2)

$$\lim_{x\to 1}\left(\frac{\ln x}{x-1}\right) = 1.$$

Solution
Now,

$$\begin{aligned}\lim_{x\to 1}\left(\frac{\ln x}{x-1}\right) &= \lim_{x\to 1}\left(\frac{(x-1)-\frac{1}{2}(x-1)^2+\ldots}{x-1}\right)\\ &= \lim_{x\to 1}\left(1-\tfrac{1}{2}(x-1)+\ldots\right)\\ &= \underline{\underline{1}},\end{aligned}$$

as required.

(b) Use L'Hospital's rule to determine (4)

$$\lim_{x\to 0}\left(\frac{1}{(x+3)\tan(6x)\operatorname{cosec}(2x)}\right).$$

Solutions relying entirely on calculator technology are not acceptable.

Solution
We need an indeterminate form:

$$\begin{aligned}&\lim_{x\to 0}\left(\frac{1}{(x+3)\tan(6x)\operatorname{cosec}(2x)}\right)\\ =\ &\lim_{x\to 0}\left(\frac{\sin(2x)}{(x+3)\tan(6x)}\right)\end{aligned}$$

We differentiate both the numerator and the denominator:

$$\begin{aligned}=\ &\lim_{x\to 0}\left(\frac{2\cos(2x)}{\tan(6x)+6(x+3)\sec^2(6x)}\right)\\ =\ &\frac{2}{1+6(0+3)(1)}\\ =\ &\tfrac{2}{18}\\ =\ &\underline{\underline{\tfrac{1}{9}}}.\end{aligned}$$

9. A particle P moves along a straight line.

At time t minutes, the displacement, x metres, of P from a fixed point O on the line is modelled by the differential equation

$$t^2\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 2t\frac{\mathrm{d}x}{\mathrm{d}t} + 2x + 16t^2x = 4t^3\sin 2t \quad (1).$$

(a) Show that the transformation (5)

$$x = ty$$

transforms equation (1) into the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 16y = 4\sin 2t.$$

Solution

Now,

$$x = ty \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = y + t\frac{\mathrm{d}y}{\mathrm{d}t}$$

and

$$\begin{aligned}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= \frac{\mathrm{d}y}{\mathrm{d}t} + \left(\frac{\mathrm{d}y}{\mathrm{d}t} + t\frac{\mathrm{d}^2y}{\mathrm{d}t^2}\right) \\ &= 2\frac{\mathrm{d}y}{\mathrm{d}t} + t\frac{\mathrm{d}^2y}{\mathrm{d}t^2}.\end{aligned}$$

Next,

$$\begin{aligned}& t^2\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 2t\frac{\mathrm{d}x}{\mathrm{d}t} + 2x + 16t^2x = 4t^3\sin 2t \\ \Rightarrow\ & t^2\left(2\frac{\mathrm{d}y}{\mathrm{d}t} + t\frac{\mathrm{d}^2y}{\mathrm{d}t^2}\right) - 2t\left(y + t\frac{\mathrm{d}y}{\mathrm{d}t}\right) + 2(ty) + 16t^2(ty) = 4t^3\sin 2t \\ \Rightarrow\ & 2t^2\frac{\mathrm{d}y}{\mathrm{d}t} + t^3\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2ty - 2t^2\frac{\mathrm{d}y}{\mathrm{d}t} + 2ty + 16t^3y = 4t^3\sin 2t \\ \Rightarrow\ & t^3\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 16t^3y = 4t^3\sin 2t \\ \Rightarrow\ & \underline{\underline{\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 16y = 4\sin 2t}},\end{aligned}$$

as required.

(b) Hence find a general solution for the displacement of P from O at time t minutes. (8)

Solution

Complementary Function:

$$m^2 + 16 = 0 \Rightarrow m = \pm 4\mathrm{i}$$

which means the complementary function is

$$y = A\sin 4t + B\cos 4t.$$

Particular Integral: Now,

$$\begin{aligned} y &= C\sin 2t + D\cos 2t \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= 2C\cos 2t - 2D\sin 2t \\ \frac{\mathrm{d}^2y}{\mathrm{d}t^2} &= -4C\sin 2t - 4D\cos 2t \end{aligned}$$

and

$$\begin{aligned} & \frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 16y = 4\sin 2t \\ \Rightarrow\quad & (-4C\sin 2t - 4D\cos 2t) + 16(C\sin 2t + D\cos 2t) = 4\sin 2t \\ \Rightarrow\quad & 12C\sin 2t + 12D\cos 2t = 4\sin 2t. \end{aligned}$$

Now,

$$12C = 4 \Rightarrow C = \tfrac{1}{3}$$

and, clearly, $D = 0$. So the particular integral is

$$y = \tfrac{1}{3}\sin 2t.$$

Putting them both together:

$$\begin{aligned} & y = \mathrm{CF} + \mathrm{PI} \\ \Rightarrow\quad & y = A\sin 4t + B\cos 4t + \tfrac{1}{3}\sin 2t \\ \Rightarrow\quad & \underline{\underline{x = t\left(A\sin 4t + B\cos 4t + \tfrac{1}{3}\sin 2t\right)}}. \end{aligned}$$