Dr Oliver Mathematics
Double Counting

In this note, we will explore double counting.

If the same set is counted in two different ways, you get the same answer.

Example 1
Prove, by double counting, that

$$r\binom{n}{r} = n\binom{n-1}{r-1}.$$

Solution
Consider a community of n people from which you have to select a committee of r members with a designated leader.

<u>*LHS*</u>: Choose the r members first in $\binom{n}{r}$ ways and then choose a leader from them in r ways. This gives a total of $r\binom{n}{r}$ ways.

<u>*RHS*</u>: First choose the leader, in n ways and then from the remaining $(n-1)$ people choose $(r-1)$ to complete the committee. This gives a total of $n\binom{n-1}{r-1}$ ways.

<u>*LHS* = *RHS*</u>: Well, we see that

$$r\binom{n}{r} = n\binom{n-1}{r-1}. \quad \blacksquare$$

Example 2
At a party of 11 people, every person claims that they shook hands with exactly 5 other people. Show that someone is not telling the truth.

Solution
Firstly, since everyone participated in 5 handshakes, there are

$$11 \times 5 = 55.$$

Secondly, since every handshake involves only 2 people, there are

$$n \times 2 = 2n.$$

Then

$$2n = 55 \Rightarrow n = 27\tfrac{1}{2},$$

which is a contradiction.

Thus, someone is not telling the truth. ■

Here are some examples for you to try.

1. Prove, by double counting, that

$$\binom{n}{r} = \binom{n}{n-r}.$$

Solution

<u>*LHS*</u>: It counts the number of ways to select r people from a group of n people to receive a, say, chocolate.
<u>*RHS*</u>: It counts the number of ways to select $(n-r)$ people from a group of n people who *do not* receive a chocolate.

2. Prove, by double counting, that

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

Solution

<u>*LHS*</u>: It counts the number of ways to select r people from a group of n people to receive a, say, chocolate.
<u>*RHS*</u>: Now, either a specific person of the n is in the group selected, or he is not selected.
(i) In the first case the remaining $(r-1)$ in the group must be selected from the remaining $(n-1)$ to receive a chocolate and the number of ways to do this is $\binom{n-1}{r-1}$.
(ii) In the second case all r in the group $(n-1)$ must be selected from the remaining $(n-1)$ and the number of ways to do this is $\binom{n-1}{r}$.
Put them both together and we have

$$\binom{n-1}{r-1} + \binom{n-1}{r}$$

ways.

3. Prove, by double counting, that

$$\binom{n}{s}\binom{s}{r} = \binom{n}{r}\binom{n-r}{s-r}.$$

Solution

Consider a group of n people. This time, we count the number of ways of selecting a team of s members, among which r are designated as captains.

LHS: One way to do this is to start by selecting the team, which can be done in $\binom{n}{s}$ ways. For each team, we then select the captains, which can be done in $\binom{s}{r}$ ways. The total is therefore

$$\binom{n}{s}\binom{s}{r}$$

ways.

RHS: Another way to count is to start by selecting the captains first, which can be done in $\binom{n}{r}$ ways. Then, we must select the rest of the team. There remains $(n-r)$ people and we must choose $(s-r)$ to round out the team. So the total count is

$$\binom{n}{r}\binom{n-r}{s-r}$$

ways.

4. Prove, by double counting, that

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \ldots + n\binom{n}{n} = n\,2^{n-1}.$$

Solution

Given a group of n people, count the number of ways of forming a committee of any size where one person is appointed as the chairperson.

LHS: One way is to let k be the size of the committee. There are $\binom{n}{k}$ possible committees of size k, and k possible choices of chairperson. This leads to the left-hand side of the equation.

RHS: Another way to count is to start by appointing the chairperson, which can be done in n ways. Once a chairperson is selected, we must assemble the rest of the committee, which is any subset of the remaining $(n-1)$ people. There are 2^{n-1} possible subsets, which leads us to the right-hand side of the equation.

5. (Vandermonde's Identity) Prove, by double counting, that

$$\binom{m+w}{k} = \sum_{i=0}^{k} \binom{m}{i}\binom{w}{k-i}.$$

Solution

<u>*LHS*</u>: It counts the number of ways to choose a committee of k people from a group of m men and w women.
<u>*RHS*</u>: If there are i men, there must be $(k-i)$ women. Now, there are $\binom{m}{i}$ ways to choose the men and $\binom{w}{k-i}$ ways to choose the women and we multiply: $\binom{m}{i}\binom{w}{k-i}$. And, finally, we can form a sum from 0 to k:

$$\sum_{i=0}^{k} \binom{m}{i}\binom{w}{k-i}.$$

[In fact, $m = w = k = n$ and we get

$$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2.]$$

6. (Christmas Stocking Identity or Hockey-Stick Identity) Prove, by double counting, that

$$\binom{m+n+1}{n+1} = \sum_{k=0}^{m} \binom{n+k}{n}.$$

Solution

<u>*LHS*</u>: Suppose we have a set of $(m+n+1)$ marbles and we would like to choose a subset of size $(n+1)$. There are $\binom{m+n+1}{n+1}$ ways of doing this.
<u>*RHS*</u>: Suppose the marbles are numbered 1, 2, $\ldots$, $(m+n+1)$. Since we are choosing $(n+1)$ marbles, the largest-numbered marble in our subset must have label $(n+1)$ or larger. If the largest-numbered marble in our subset has label $(n+k+1)$ (where

$0 \leqslant k \leqslant m$), then the remaining n marbles must be selected among the $(n+k)$ smaller-numbered marbles, which can be done in $\binom{n+k}{n}$ ways.

7. Prove, by double counting, that

$$n^2 = 2\binom{n}{2} + \binom{n}{1}.$$

Solution

<u>LHS</u>: Let $S = \{1, 2, 3, \ldots, n-1, n\}$, and consider all ordered pairs (a, b) such that $a,\ b \in S$. There are clearly n^2 such pairs.
<u>RHS</u>: The number of ordered pairs (a, b), where $a \neq b$, is twice the number of two-element subsets of S; hence it is equal to $2\binom{n}{2}$. The number of ordered pairs (a, b), where $a = b$, is equal to the number of elements of S, which is $n = \binom{n}{1}$. And, finally, we can form a sum:

$$2\binom{n}{2} + \binom{n}{1}.$$

8. On a 8×8 chessboard, how many squares are there, of all possible sizes?

Solution

Well, the number of squares of size 1×1 is 64; the number of squares of size 2×2 is 49; the number of squares of size 3×3 is 36; and so on, down to the number of squares of size 8×8 is 1.

Size	All possible
1×1	$64 = 8^2$
2×2	$49 = 7^2$
3×3	$36 = 6^2$
4×4	$25 = 5^2$
5×5	$16 = 4^2$
6×6	$9 = 3^2$
7×7	$4 = 2^2$
8×8	$1 = 1^2$
All sizes	$\underline{\underline{204}}$

9. 15 students join a summer course. Every day, 3 students are on duty after school to clean the classroom. After the course, it was found that every pair of students has been on duty together exactly once. How many days does the course last for?

Solution

We count the total number P of pairs of students who work together for all days. There are n days and each day there are 3 pairs. So $P = 3n$. On the other hand, $P = \binom{15}{2}$. Hence,

$$n = \tfrac{1}{3}\binom{15}{2} = \underline{\underline{35}}.$$