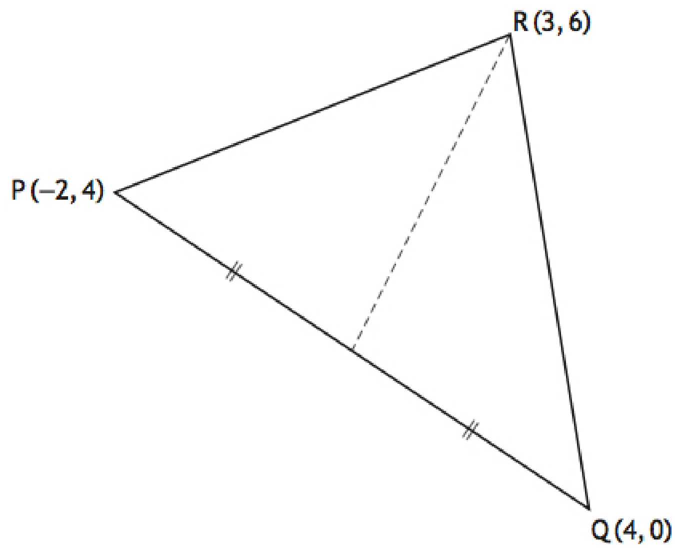
Dr Oliver Mathematics
Mathematics: Higher
2018 Paper 1: Non-Calculator
1 hour 10 minutes

The total number of marks available is 60.
You must write down all the stages in your working.

1. PQR is a triangle with vertices $P(-2, 4)$, $Q(4, 0)$, and $R(3, 6)$. (3)



Find the equation of the median through R.

Solution

The midpoint – let us call it$S$ – of PQ is

$$\left(\frac{-2+4}{2}, \frac{4+0}{2}\right) = (1, 2).$$

Now, the gradient of PS is

$$\begin{aligned}\frac{6-2}{3-1} &= \frac{4}{2} \\ &= 2\end{aligned}$$

and the equation of the median through R is

$$y - 6 = 2(x - 3) \Rightarrow y - 6 = 2x - 6$$
$$\Rightarrow \underline{\underline{y = 2x}}.$$

2. A function $\mathrm{g}(x)$ is defined on $\mathbb{R}$, the set of real numbers, by (3)

$$\mathrm{g}(x) = \tfrac{1}{5}x - 4.$$

Find the inverse function, $\mathrm{g}^{-1}(x)$.

Solution

$$y = \tfrac{1}{5}x - 4 \Rightarrow y + 4 = \tfrac{1}{5}x$$
$$\Rightarrow x = 5(y + 4)$$

hence,

$$\underline{\underline{\mathrm{g}^{-1}(x) = 5(x + 4)}}.$$

3. Given (3)

$$\mathrm{h}(x) = 3\cos 2x,$$

find the value of $\mathrm{h}'(\tfrac{1}{6}\pi)$.

Solution

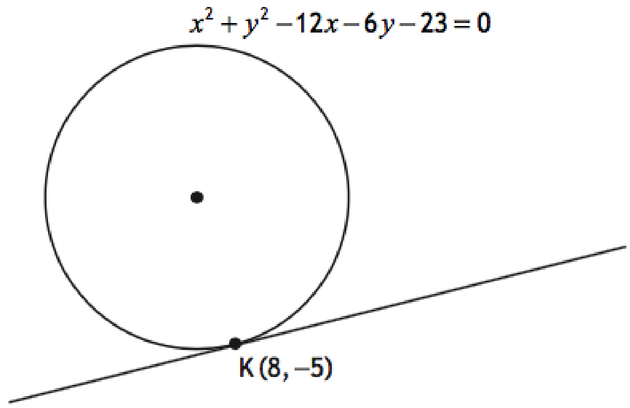
$$\mathrm{h}(x) = 3\cos 2x \Rightarrow \mathrm{h}'(x) = 3(-\sin 2x) \cdot 2$$
$$\Rightarrow \mathrm{h}'(x) = -6\sin 2x$$

and

$$\mathrm{h}'(\tfrac{1}{6}\pi) = \underline{\underline{-3\sqrt{3}}}.$$

4. The point $K(8, -5)$ lies on the circle with equation (4)

$$x^2 + y^2 - 12x - 6y - 23 = 0.$$



Find the equation of the tangent to the circle at K.

Solution

$$\begin{aligned} x^2 + y^2 - 12x - 6y - 23 = 0 &\Rightarrow x^2 - 12x + y^2 - 6y = 23 \\ &\Rightarrow (x^2 - 12x + 36) + (y^2 - 6y + 9) = 23 + 36 + 9 \\ &\Rightarrow (x-6)^2 + (y-3)^2 = 68; \end{aligned}$$

hence, the centre C is $(6, 3)$. Now,

$$\begin{aligned} m_{CK} &= \frac{3 - (-5)}{6 - 8)} \\ &= -\tfrac{8}{2} \\ &= -4 \end{aligned}$$

and the gradient of the tangent is $\frac{1}{4}$. Finally, the equation of the tangent to the circle is

$$\begin{aligned} y + 5 = \tfrac{1}{4}(x - 8) &\Rightarrow y + 5 = \tfrac{1}{4}x - 2 \\ &\Rightarrow \underline{\underline{y = \tfrac{1}{4}x - 7}}. \end{aligned}$$

5. $A(-3, 4, -7)$, $B(5, t, 5)$, and $C(7, 9, 8)$ are collinear.

   (a) State the ratio in which B divides AC. (1)

   Solution

We take the x-component:

$$AB = 5 - (-3) = 8 \text{ and } BC = 7 - 5 = 2;$$

hence, the ratio in which B divides AC is

$$8 : 2 = \underline{\underline{4 : 1}}.$$

(b) State the value of t. (1)

Solution

We take the y-component:

$$AB = t - 4 \text{ and } BC = 9 - t;$$

now,

$$\begin{aligned} \frac{4}{t-4} = \frac{1}{9-t} &\Rightarrow 4(9-t) = t - 4 \\ &\Rightarrow 36 - 4t = t - 4 \\ &\Rightarrow 40 = 5t \\ &\Rightarrow \underline{\underline{t = 8}}. \end{aligned}$$

6. Find the value of (3)

$$\log_5 250 - \tfrac{1}{3}\log_5 8.$$

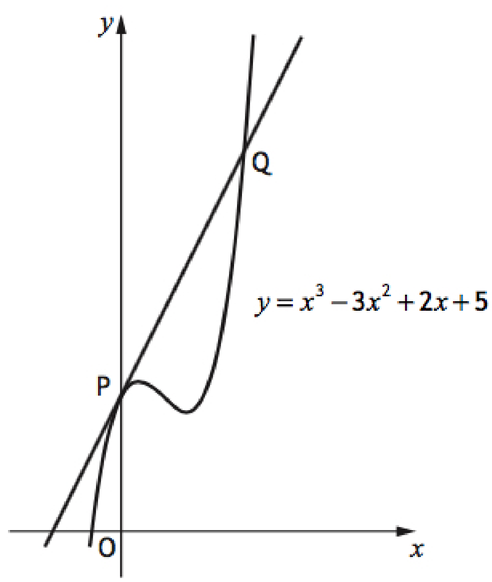
Solution

$$\begin{aligned}\log_5 250 - \tfrac{1}{3}\log_5 8 &= \log_5 250 - \log_5 8^{\frac{1}{3}} \\ &= \log_5 250 - \log_5 2 \\ &= \log_5\left(\frac{250}{2}\right) \\ &= \log_5 125 \\ &= \log_5 5^3 \\ &= 3\log_5 5 \\ &= \underline{\underline{3}}.\end{aligned}$$

7. The curve with equation

$$y = x^3 - 3x^2 + 2x + 5$$

is shown on the diagram.



(a) Write down the coordinates of P, the point where the curve crosses the y-axis . (1)

Solution
$\underline{\underline{P(0,5)}}$.

(b) Determine the equation of the tangent to the curve at P. (3)

Solution

$$y = x^3 - 3x^2 + 2x + 5 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 6x + 2$$

and

$$x = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2.$$

Finally,

$$y - 5 = 2(x - 0) \Rightarrow \underline{\underline{y = 2x + 5}}.$$

(c) Find the coordinates of Q, the point where this tangent meets the curve again. (4)

Solution

$$\begin{aligned} 2x + 5 = x^3 - 3x^2 + 2x + 5 &\Rightarrow x^3 - 3x^2 = 0 \\ &\Rightarrow x^2(x - 3) = 0 \\ &\Rightarrow x = 0 \text{ or } x - 3 = 0 \\ &\Rightarrow x = 0 \text{ or } x = 3; \end{aligned}$$

hence,

$$x = 3 \Rightarrow y = 2(3) + 5 = 11;$$

the the coordinates are $\underline{\underline{Q(3, 11)}}$.

8. A line has equation (2)

$$y - \sqrt{3}x + 5 = 0.$$

Determine the angle this line makes with the positive direction of the x-axis.

Solution

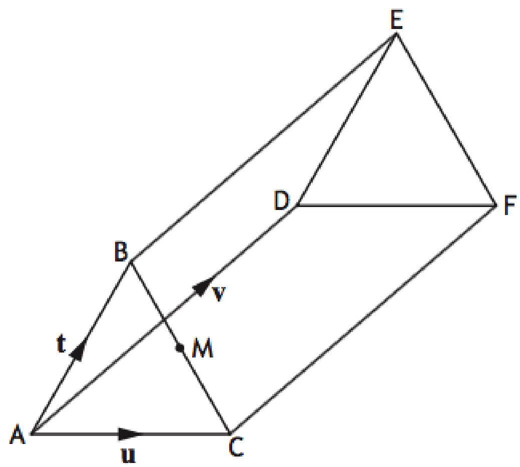
$$y - \sqrt{3}x + 5 = 0 \Rightarrow y = \sqrt{3}x - 5$$

and this makes an angle of

$$\tan^{-1}\sqrt{3} = \underline{\underline{60^\circ}}$$

with the positive direction of the x-axis.

9. The diagram shows a triangular prism $ABCDEF$.



$\overrightarrow{AB} = \mathbf{t}$, $\overrightarrow{AC} = \mathbf{u}$, and $\overrightarrow{AD} = \mathbf{v}$.

(a) Express $\overrightarrow{BC}$ in terms of $\mathbf{u}$ and $\mathbf{t}$. (1)

Solution

$$\begin{aligned}\overrightarrow{BC} &= \overrightarrow{BA} + \overrightarrow{AC} \\ &= \underline{\underline{-\mathbf{t} + \mathbf{u}}}.\end{aligned}$$

M is the midpoint of BC.

(b) Express MD in terms of $\mathbf{t}$, $\mathbf{u}$, and $\mathbf{v}$. (2)

Solution

$$\begin{aligned}\overrightarrow{MD} &= \overrightarrow{MC} + \overrightarrow{CA} + \overrightarrow{AD} \\ &= \tfrac{1}{2}\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AD} \\ &= \tfrac{1}{2}(-\mathbf{t} + \mathbf{u}) - \mathbf{u} + \mathbf{v} \\ &= -\tfrac{1}{2}\mathbf{t} + \tfrac{1}{2}\mathbf{u} - \mathbf{u} + \mathbf{v} \\ &= \underline{\underline{-\tfrac{1}{2}\mathbf{t} - \tfrac{1}{2}\mathbf{u} + \mathbf{v}}}.\end{aligned}$$

10. Given that (4)

- $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 - 3x + 4$ and

- $y = 14$ when $x = 2$,

express y in terms of x.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 - 3x + 4 \Rightarrow y = 2x^3 - \tfrac{3}{2}x^2 + 4x + c$$

for some constant c. Now,

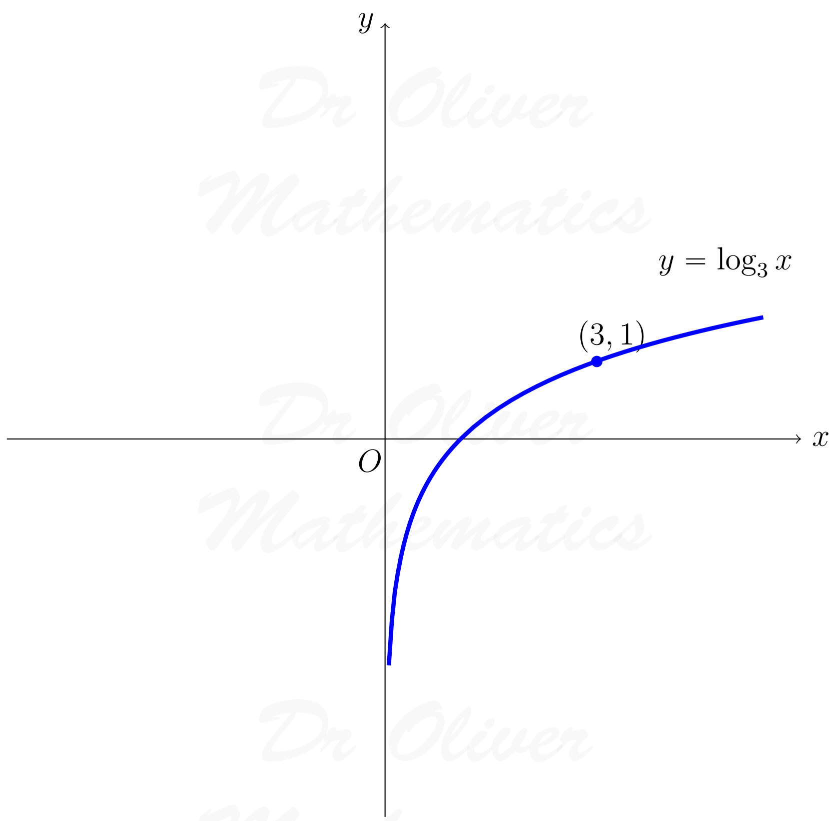
$$\begin{aligned} x = 2, y = 14 &\Rightarrow 14 = 16 - 6 + 8 + c \\ &\Rightarrow c = -4 \end{aligned}$$

and

$$\underline{\underline{y = 2x^3 - \tfrac{3}{2}x^2 + 4x - 4.}}$$

11. The diagram shows the curve with equation

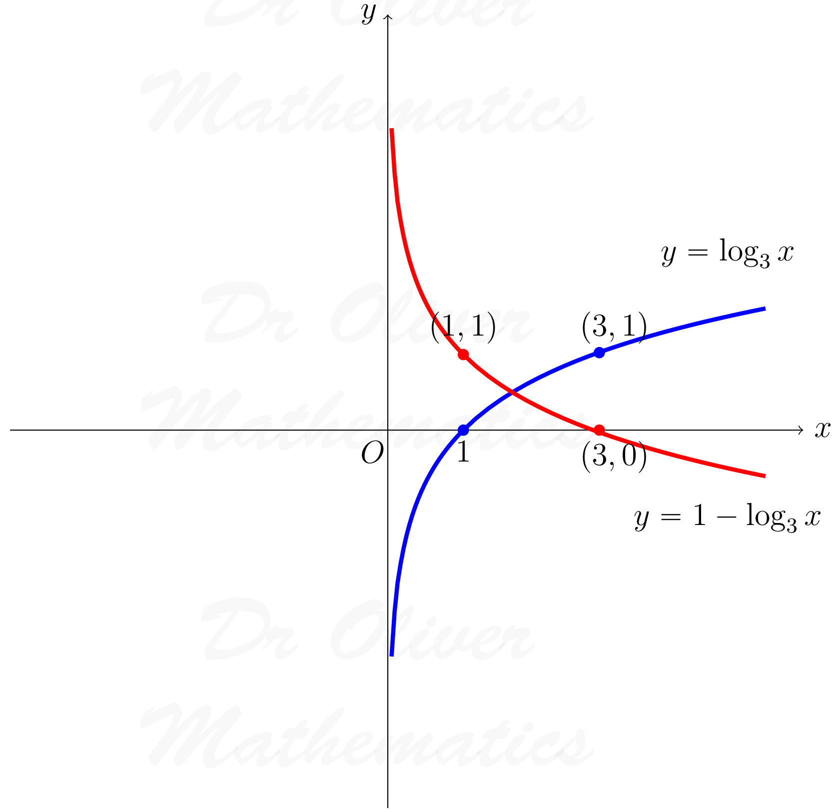
$$y = \log_3 x.$$



(a) Sketch the curve with equation (2)

$$y = 1 - \log_3 x.$$

Solution



(b) Determine the exact value of the x-coordinate of the point of intersection of the two curves. (3)

Solution

$$\begin{aligned}\log_3 x = 1 - \log_3 x &\Rightarrow 2\log_3 x = 1\\ &\Rightarrow \log_3 x = \tfrac{1}{2}\\ &\Rightarrow x = 3^{\frac{1}{2}}\\ &\Rightarrow \underline{\underline{x = \sqrt{3}}}.\end{aligned}$$

12. Vectors **a** and **b** are such that

$$\mathbf{a} = 4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} \text{ and } \mathbf{b} = -2\mathbf{i} + \mathbf{j} + p\mathbf{k}.$$

(a) Express $2\mathbf{a} + \mathbf{b}$ in component form. (1)

Solution

$$\begin{aligned} 2\mathbf{a} + \mathbf{b} &= 2(4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) + (-2\mathbf{i} + \mathbf{j} + p\mathbf{k}) \\ &= \underline{\underline{6\mathbf{i} - 3\mathbf{j} + (4+p)\mathbf{k}}}. \end{aligned}$$

(b) Hence find the values of p for which (3)

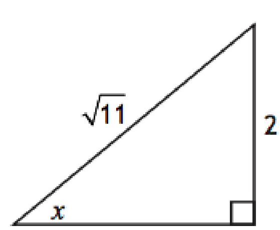
$$|2\mathbf{a} + \mathbf{b}| = 7.$$

Solution

$$\begin{aligned} |2\mathbf{a} + \mathbf{b}| = 7 &\Rightarrow |6\mathbf{i} - 3\mathbf{j} + (4+p)\mathbf{k}| = 7 \\ &\Rightarrow |6\mathbf{i} - 3\mathbf{j} + (4+p)\mathbf{k}|^2 = 7^2 \\ &\Rightarrow 6^2 + (-3)^2 + (4+p)^2 = 49 \\ &\Rightarrow 36 + 9 + (4+p)^2 = 49 \\ &\Rightarrow (4+p)^2 = 4 \\ &\Rightarrow 4 + p = \pm 2 \\ &\Rightarrow \underline{\underline{p = -6 \text{ or } p = -2}}. \end{aligned}$$

13. The right-angled triangle in the diagram is such that

$$\sin x = \frac{2}{\sqrt{11}} \text{ and } 0 < x < \tfrac{1}{4}\pi.$$



(a) Find the exact value of:

(i) $\sin 2x$, (3)

Solution

$$\begin{aligned}\text{adj} &= \sqrt{(\sqrt{11})^2 - 2^2}\\ &= \sqrt{11-4}\\ &= \sqrt{7}.\end{aligned}$$

Now,

$$\begin{aligned}\sin 2x &= 2\sin x\cos x\\ &= 2\cdot\frac{2}{\sqrt{11}}\cdot\frac{\sqrt{7}}{\sqrt{11}}\\ &= \underline{\underline{\tfrac{4\sqrt{7}}{11}}}.\end{aligned}$$

(ii) $\cos 2x$ (1)

Solution

$$\begin{aligned}\cos 2x &= 2\cos^2 x - 1\\ &= 2\left(\frac{\sqrt{7}}{\sqrt{11}}\right)^2 - 1\\ &= 2\left(\tfrac{7}{11}\right) - 1\\ &= \tfrac{14}{11} - 1\\ &= \underline{\underline{\tfrac{3}{11}}}.\end{aligned}$$

(b) By expressing $\sin 3x$ as $\sin(2x + x)$, find the exact value of $\sin 3x$. (3)

Solution

$$
\begin{aligned}
\sin 3x &= \sin 2x \cos x + \sin x \cos 2x \\
&= \left(\frac{4\sqrt{7}}{11} \cdot \frac{\sqrt{7}}{\sqrt{11}}\right) + \left(\frac{2}{\sqrt{11}} \cdot \frac{3}{11}\right) \\
&= \frac{28}{11\sqrt{11}} + \frac{6}{11\sqrt{11}} \\
&= \frac{34}{11\sqrt{11}} \\
&= \frac{34}{11\sqrt{11}} \times \frac{\sqrt{11}}{\sqrt{11}} \\
&= \underline{\underline{\frac{34\sqrt{11}}{121}}}.
\end{aligned}
$$

14. Evaluate (5)

$$\int_{-4}^{9} \frac{1}{\sqrt[3]{(2x+9)^2}}\, \mathrm{d}x.$$

Solution

$$
\begin{aligned}
\int_{-4}^{9} \frac{1}{\sqrt[3]{(2x+9)^2}}\, \mathrm{d}x &= \int_{-4}^{9} (2x+9)^{-\frac{2}{3}}\, \mathrm{d}x \\
&= \left[\tfrac{3}{2}(2x+9)^{\frac{1}{3}}\right]_{x=-4}^{9} \\
&= \tfrac{3}{2}\,(3-1) \\
&= \underline{\underline{3}}.
\end{aligned}
$$

15. A cubic function, f, is defined on the set of real numbers. (4)

- $(x+4)$ is a factor of $\mathrm{f}(x)$,
- $x = 2$ is a repeated root of $\mathrm{f}(x)$,
- $\mathrm{f}'(-2) = 0$, and
- $\mathrm{f}'(x) > 0$ where the graph with equation $y = \mathrm{f}(x)$ crosses the y-axis.

Sketch a possible graph of $y = \mathrm{f}(x)$.

Solution

First, is

$$\mathrm{f}(x) = (x+4)(x-2)^2$$

(incorporating the first two items)? If it is, then

$$\mathrm{f}(x) = (x+4)(x-2)^2 \Rightarrow \mathrm{f}(x) = (x+4)(x^2-4x+4)$$

$\times$	x^2	$-4x$	$+4$
x	x^3	$-4x^2$	$+4x$
$+4$	$+4x^2$	$-16x$	$+16$

$$\Rightarrow \mathrm{f}(x) = x^3 - 12x + 16$$
$$\Rightarrow \mathrm{f}'(x) = 3x^2 - 12.$$

Now,

$$\mathrm{f}'(-2) = 3 \cdot (-2) - 12 = 0 \text{ (third item)}.$$

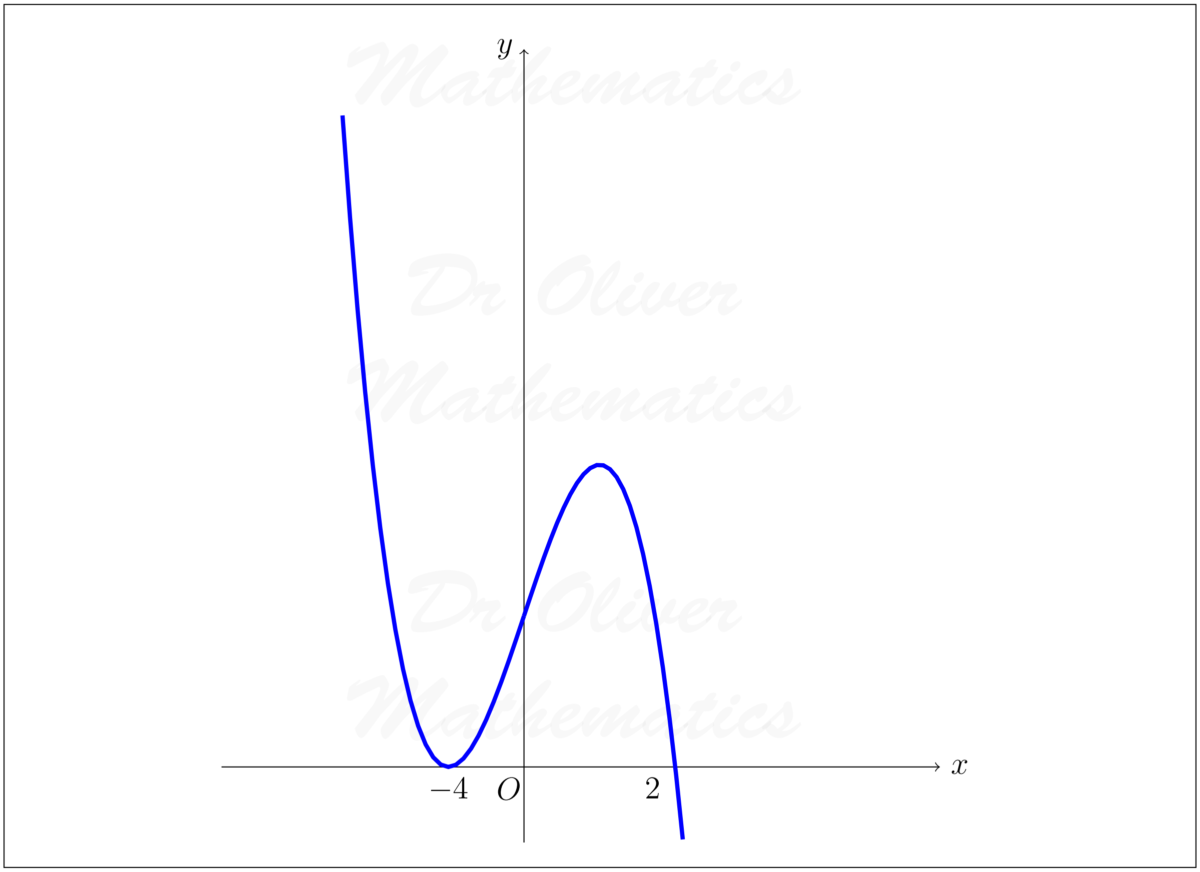
Finally,

$$\mathrm{f}'(0^+) = -12^+ \text{ (fourth item)}$$

so, no – but we increase by 12:

$$\mathrm{f}(x) = (x+4)(x-2)^2 + 12.$$

Hence,


y
x
−4
O
2