Dr Oliver Mathematics
Advance Level Further Mathematics
Further Mathematics 3: Calculator
1 hour 30 minutes

The total number of marks available is 75.
You must write down all the stages in your working.

1. (a) Starting from the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials, show that, for $x \in \mathbb{R}$, (2)

$$\tanh x = \frac{e^{2x} - 1}{e^{2x} + 1}.$$

Solution

Begin with

$$\sinh x = \frac{e^{x} - e^{-x}}{2} \text{ and } \cosh x = \frac{e^{x} + e^{-x}}{2}:$$

$$\begin{aligned} \tanh x &= \frac{\sinh x}{\cosh x} \\ &= \frac{\frac{e^{x} - e^{-x}}{2}}{\frac{e^{x} + e^{-x}}{2}} \\ &= \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} \\ &= \underline{\underline{\frac{e^{2x} - 1}{e^{2x} + 1}}}, \end{aligned}$$

as required.

(b) Hence, given that $-1 < \theta < 1$, prove that (3)

$$\operatorname{artanh} \theta = \tfrac{1}{2} \ln \left(\frac{1 + \theta}{1 - \theta} \right).$$

Solution

Let $y = \operatorname{artanh}\theta$ for some $-1 < \theta < 1$. Now,

$$\begin{aligned} y = \operatorname{artanh}\theta \Rightarrow \theta &= \tanh y \\ \Rightarrow \theta &= \frac{e^{2y}-1}{e^{2y}+1} \\ \Rightarrow \theta(e^{2y}+1) &= e^{2y}-1 \\ \Rightarrow \theta e^{2y}+\theta &= e^{2y}-1 \\ \Rightarrow 1+\theta &= e^{2y}-\theta e^{2y} \\ \Rightarrow 1+\theta &= e^{2y}(1-\theta) \\ \Rightarrow e^{2y} &= \frac{1+\theta}{1-\theta} \\ \Rightarrow 2y &= \ln\left(\frac{1+\theta}{1-\theta}\right) \\ \Rightarrow y &= \tfrac{1}{2}\ln\left(\frac{1+\theta}{1-\theta}\right), \end{aligned}$$

and so

$$\underline{\underline{\operatorname{artanh}\theta = \tfrac{1}{2}\ln\left(\frac{1+\theta}{1-\theta}\right)}}.$$

2. Figure 1 shows a sketch of part of the curve with equation

$$y = 5\cosh x - 6\sinh x.$$

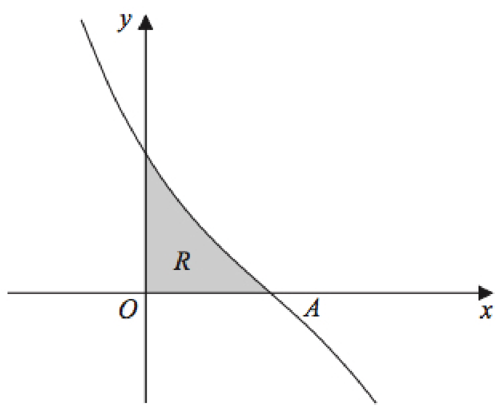

Figure 1: $y = 5\cosh x - 6\sinh x$

The curve crosses the x-axis at the point A.

(a) Find the exact value of the x-coordinate of the point A, giving your answer as a natural logarithm. (3)

Solution

$$\begin{aligned}5\cosh x - 6\sinh x = 0 &\Rightarrow \frac{5}{2}(\mathrm{e}^{x} + \mathrm{e}^{-x}) - \frac{6}{2}(\mathrm{e}^{x} - \mathrm{e}^{-x}) = 0\\ &\Rightarrow 5(\mathrm{e}^{2x} + 1) - 6(\mathrm{e}^{2x} - 1) = 0\\ &\Rightarrow \mathrm{e}^{2x} = 11\\ &\Rightarrow 2x = \ln 11\\ &\Rightarrow \underline{\underline{x = \tfrac{1}{2}\ln 11}}.\end{aligned}$$

(b) Show that (3)

$$(5\cosh x - 6\sinh x)^2 \equiv a\cosh 2x + b\sinh 2x + c,$$

where a, b, and c are constants to be found.

Solution

$$\begin{aligned}(5\cosh x - 6\sinh x)^2 &\equiv 25\cosh^2 x - 60\sinh x\cosh x + 36\sinh^2 x\\ &\equiv \tfrac{25}{2}(\cosh 2x + 1) - 60\sinh x\cosh x + \tfrac{36}{2}(\cosh 2x - 1)\\ &\equiv (\tfrac{25}{2}\cosh 2x + \tfrac{25}{2}) - 30\sinh 2x + (18\cosh 2x - 18)\\ &\equiv \underline{\underline{\tfrac{61}{2}\cosh 2x - 30\sinh 2x - \tfrac{11}{2}}};\end{aligned}$$

hence, $\underline{\underline{a = \frac{61}{2}}}$, $\underline{\underline{b = -30}}$, and $\underline{\underline{c = -\frac{11}{2}}}$.

The finite region R, bounded by the curve and the coordinate axes, is shown shaded in Figure 1.

The region R is rotated through 2π radians about the x-axis.

(c) Use calculus to find the volume of the solid generated, giving your answer as an exact multiple of π. (4)

Solution

$$
\begin{aligned}
\text{Volume} &= \int_0^{\frac{1}{2}\ln 11} \pi(5\cosh x - 6\sinh x)^2 \,\mathrm{d}x \\
&= \pi \int_0^{\frac{1}{2}\ln 11} \left(\tfrac{61}{2}\cosh 2x - 30\sinh 2x - \tfrac{11}{2}\right) \mathrm{d}x \\
&= \pi \left\{ \left[\tfrac{61}{4}\sinh 2x - 15\cosh 2x - \tfrac{11}{2}x\right]_{x=0}^{\frac{1}{2}\ln 11} \right\} \\
&= \pi \left\{ \left[\tfrac{61}{4}\sinh(\ln 11) - 15\cosh(\ln 11) - \tfrac{11}{2}(\tfrac{1}{2}\ln 11)\right] - [0 - 15 - 0] \right\} \\
&= \pi \left\{ \tfrac{61}{8}(\mathrm{e}^{\ln 11} - \mathrm{e}^{-\ln 11}) - \tfrac{15}{2}(\mathrm{e}^{\ln 11} + \mathrm{e}^{-\ln 11}) - \tfrac{11}{4}\ln 11 + 15 \right\} \\
&= \pi \left\{ \tfrac{61}{8}(11 - \tfrac{1}{11}) - \tfrac{15}{2}(11 + \tfrac{1}{11}) - \tfrac{11}{4}\ln 11 + 15 \right\} \\
&= \underline{\underline{(15 - \tfrac{11}{4}\ln 11)\pi}}.
\end{aligned}
$$

3.

$$
\mathbf{M} = \begin{pmatrix} 3 & k & 2 \\ -1 & 0 & 1 \\ 1 & k & 1 \end{pmatrix}, \text{ where } k \text{ is a constant.}
$$

Given that 3 is an eigenvalue of $\mathbf{M}$,

(a) find the value of k. (3)

Solution

$$
\begin{aligned}
\det(\mathbf{M} - 3\mathbf{I}) = 0 &\Rightarrow \begin{vmatrix} 0 & k & 2 \\ -1 & -3 & 1 \\ 1 & k & -2 \end{vmatrix} = 0 \\
&\Rightarrow 0 - k(2 - 1) + 2(-k + 3) = 0 \\
&\Rightarrow -k - 2k + 6 = 0 \\
&\Rightarrow 3k = 6 \\
&\Rightarrow \underline{\underline{k = 2}}.
\end{aligned}
$$

(b) Hence find the other two eigenvalues of $\mathbf{M}$. (4)

Solution

$$\begin{vmatrix} 3-\lambda & 2 & 2 \\ -1 & -\lambda & 1 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$\begin{aligned}
&\Rightarrow\quad (3-\lambda)[-\lambda(1-\lambda)-2] - 2[-(1-\lambda)-1] + 2[-2+\lambda] = 0 \\
&\Rightarrow\quad (3-\lambda)(\lambda^2-\lambda-2) - 2[\lambda-2] + 2[-2+\lambda] = 0 \\
&\Rightarrow\quad (3-\lambda)(\lambda^2-\lambda-2) = 0 \\
&\Rightarrow\quad (3-\lambda)(\lambda-2)(\lambda+1) = 0 \\
&\Rightarrow\quad \underline{\underline{\lambda = -1, 2, \text{ or } 3}}.
\end{aligned}$$

(c) Find an eigenvector corresponding to the eigenvalue 3. (2)

Solution

$\underline{\lambda = 3}$:

$$\begin{pmatrix} 0 & 2 & 2 \\ -1 & -3 & 1 \\ 1 & 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Now, take the first row:

$$2y + 2z = 0 \Rightarrow y = -z.$$

Second, take the third row:

$$x + 2y - 2z = 0 \Rightarrow x + 4y = 0 \Rightarrow x = -4y.$$

Hence, an eigenvector is

$$\underline{\underline{\begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}}},$$

or any scalar multiple.

4. The curve C has equation

$$y = \operatorname{arsinh} x + x\sqrt{x^2+1},\ 0 \leqslant x \leqslant 1.$$

(a) Show that (4)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\sqrt{x^2+1}.$$

Solution

$$
\begin{aligned}
& y = \operatorname{arsinh} x + x\sqrt{x^2+1} \\
\Rightarrow\quad & \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{x^2+1}} + \left(1 \times \sqrt{x^2+1} + x \times \frac{x}{\sqrt{x^2+1}}\right) \\
\Rightarrow\quad & \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{x^2+1}} + \sqrt{x^2+1} + \frac{x^2}{\sqrt{x^2+1}} \\
\Rightarrow\quad & \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 + (x^2+1) + x^2}{\sqrt{x^2+1}} \\
\Rightarrow\quad & \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2 + 2x^2}{\sqrt{x^2+1}} \\
\Rightarrow\quad & \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2(1+x^2)}{\sqrt{x^2+1}} \\
\Rightarrow\quad & \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = 2\sqrt{x^2+1}}},
\end{aligned}
$$

as required.

(b) Hence show that the length of the curve C is given by (2)

$$\int_0^1 \sqrt{4x^2+5}\,\mathrm{d}x.$$

Solution

$$
\begin{aligned}
\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} &= \sqrt{1 + 4(x^2+1)} \\
&= \sqrt{4x^2+5}
\end{aligned}
$$

and, hence, the length of the curve is

$$\underline{\underline{\int_0^1 \sqrt{4x^2+5}\,\mathrm{d}x}},$$

as required.

(c) Using the substitution (6)

$$x = \tfrac{\sqrt{5}}{2}\sinh u,$$

find the exact length of the curve C, giving your answer in the form $a + b \ln c$, where a, b, and c are constants to be found.

Solution

Of course, it should say

$$u = \operatorname{arsinh}(\tfrac{2\sqrt{5}}{5}x)$$

rather than '$x = \ldots$' (why?). Well,

$$\begin{aligned} x = \tfrac{\sqrt{5}}{2}\sinh u \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u} &= \tfrac{\sqrt{5}}{2}\cosh u \\ \Rightarrow \mathrm{d}x &= \tfrac{\sqrt{5}}{2}\cosh u \,\mathrm{d}u \end{aligned}$$

and

$$\begin{aligned} x = 0 &\Rightarrow u = 0, \\ x = 1 &\Rightarrow u = \operatorname{arsinh}(\tfrac{2\sqrt{5}}{5}) \\ &\Rightarrow u = \ln\left(\tfrac{2\sqrt{5}}{5} + \sqrt{(\tfrac{2\sqrt{5}}{5})^2 + 1}\right) \\ &\Rightarrow u = \ln\left(\tfrac{2\sqrt{5}}{5} + \sqrt{\tfrac{4}{5} + 1}\right) \\ &\Rightarrow u = \ln\left(\tfrac{2\sqrt{5}}{5} + \sqrt{\tfrac{9}{5}}\right) \\ &\Rightarrow u = \ln\left(\tfrac{2\sqrt{5}}{5} + \tfrac{3\sqrt{5}}{5}\right) \\ &\Rightarrow u = \ln\sqrt{5}. \end{aligned}$$

Now,

$$\begin{aligned}
\int_0^1 \sqrt{4x^2+5}\,\mathrm{d}x &= \int_0^{\ln\sqrt{5}} \sqrt{4(\tfrac{5}{4}\sinh^2 u)+5}\cdot\tfrac{\sqrt{5}}{2}\cosh u\,\mathrm{d}u \\
&= \int_0^{\ln\sqrt{5}} \sqrt{5\sinh^2 u+5}\cdot\tfrac{\sqrt{5}}{2}\cosh u\,\mathrm{d}u \\
&= \int_0^{\ln\sqrt{5}} \sqrt{5\cosh^2 u}\cdot\tfrac{\sqrt{5}}{2}\cosh u\,\mathrm{d}u \\
&= \int_0^{\ln\sqrt{5}} \tfrac{5}{2}\cosh^2 u\,\mathrm{d}u \\
&= \int_0^{\ln\sqrt{5}} \tfrac{5}{2}(\tfrac{1}{2}+\tfrac{1}{2}\cosh 2u)\,\mathrm{d}u \\
&= \int_0^{\ln\sqrt{5}} (\tfrac{5}{4}+\tfrac{5}{4}\cosh 2u)\,\mathrm{d}u \\
&= \left[\tfrac{5}{4}u+\tfrac{5}{8}\sinh 2u\right]_{u=0}^{\ln\sqrt{5}} \\
&= \left[\tfrac{5}{4}\ln\sqrt{5}+\tfrac{5}{8}\sinh 2(\ln\sqrt{5})\right]-(0+0) \\
&= \tfrac{5}{8}\ln 5+\tfrac{5}{8}\sinh(\ln 5) \\
&= \tfrac{5}{8}\ln 5+\tfrac{5}{16}(\mathrm{e}^{\ln 5}-\mathrm{e}^{-\ln 5}) \\
&= \tfrac{5}{8}\ln 5+\tfrac{5}{16}\left(5-\frac{1}{5}\right) \\
&= \tfrac{5}{8}\ln 5+\tfrac{5}{16}\times\tfrac{12}{5} \\
&= \underline{\underline{\tfrac{5}{8}\ln 5+\tfrac{3}{4}}};
\end{aligned}$$

hence, $\underline{\underline{a=\frac{3}{4}}}$, $\underline{\underline{b=\frac{5}{8}}}$, and $\underline{\underline{c=5}}$.

5. Given that

$$I_n = \int x^n\sqrt{x+8}\,\mathrm{d}x,\ n\geqslant 0,\ x\geqslant 0,$$

(a) show that, for $n \geqslant 1$, (6)

$$I_n = \frac{px^n(x+8)^{\frac{3}{2}}}{2n+3} - \frac{qn}{2n+3}I_{n-1},$$

where p and q are constants to be found.

Solution

We use integration by parts:

$$u = x^n \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = nx^{n-1} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = (x+8)^{\frac{1}{2}} \Rightarrow v = \tfrac{2}{3}(x+8)^{\frac{3}{2}}.$$

So

$$\begin{aligned} I_n &= \tfrac{2}{3}x^n(x+8)^{\frac{3}{2}} - \int nx^{n-1} \cdot \tfrac{2}{3}(x+8)^{\frac{3}{2}}\,\mathrm{d}x \\ &= \frac{2x^n(x+8)^{\frac{3}{2}}}{3} - \tfrac{2}{3}n\int x^{n-1}(x+8)^{\frac{3}{2}}\,\mathrm{d}x \\ &= \frac{2x^n(x+8)^{\frac{3}{2}}}{3} - \tfrac{2}{3}n\int x^{n-1}(x+8)(x+8)^{\frac{1}{2}}\,\mathrm{d}x \\ &= \frac{2x^n(x+8)^{\frac{3}{2}}}{3} - \tfrac{2}{3}n\int \left[x^n(x+8)^{\frac{1}{2}} + 8x^{n-1}(x+8)^{\frac{1}{2}}\right]\mathrm{d}x \\ &= \frac{2x^n(x+8)^{\frac{3}{2}}}{3} - \tfrac{2}{3}nI_n - \tfrac{16}{3}nI_{n-1}. \end{aligned}$$

Then,

$$\begin{aligned} I_n + \tfrac{2}{3}nI_n &= \frac{2x^n(x+8)^{\frac{3}{2}}}{3} - \tfrac{16}{3}nI_{n-1} \\ \Rightarrow \quad (1+\tfrac{2}{3}n)I_n &= \frac{2x^n(x+8)^{\frac{3}{2}}}{3} - \tfrac{16}{3}nI_{n-1} \\ \Rightarrow \quad \tfrac{2n+3}{3}I_n &= \frac{2x^n(x+8)^{\frac{3}{2}}}{3} - \tfrac{16}{3}nI_{n-1} \\ \Rightarrow \quad \underline{\underline{I_n = \frac{2x^n(x+8)^{\frac{3}{2}}}{2n+3} - \frac{16n}{2n+3}I_{n-1}}}; \end{aligned}$$

hence, $\underline{\underline{p = 2}}$ and $\underline{\underline{q = 16}}$.

(b) Use part (a) to find the exact value of (5)

$$\int_0^{10} x^2\sqrt{x+8}\,\mathrm{d}x,$$

giving your answer in the form $k\sqrt{2}$, where k is rational.

Solution

Let
$$J_n = \int_0^{10} x^2\sqrt{x+8}\,\mathrm{d}x.$$
Now,
$$\begin{aligned}
J_2 &= \left[\frac{2x^2(x+8)^{\frac{3}{2}}}{7}\right]_{x=0}^{10} - \tfrac{32}{7}J_1 \\
&= (\tfrac{10\,800}{7}\sqrt{2} - 0) - \tfrac{32}{7}J_1 \\
&= \tfrac{10\,800}{7}\sqrt{2} - \tfrac{32}{7}\left(\left[\frac{2x(x+8)^{\frac{3}{2}}}{5}\right]_{x=0}^{10} - \tfrac{16}{5}J_0\right) \\
&= \tfrac{10\,800}{7}\sqrt{2} - \tfrac{32}{7}(216\sqrt{2} - 0) + \tfrac{512}{35}J_0 \\
&= \tfrac{3\,888}{7}\sqrt{2} + \tfrac{512}{35}\int \sqrt{x+8}\,\mathrm{d}x \\
&= \tfrac{3\,888}{7}\sqrt{2} + \tfrac{512}{35}\left[\tfrac{2}{3}(x+8)^{\frac{3}{2}}\right]_{x=0}^{10} \\
&= \tfrac{3\,888}{7}\sqrt{2} + \tfrac{512}{35}\cdot\tfrac{2}{3}(54\sqrt{2} - 16\sqrt{2}) \\
&= \tfrac{3\,888}{7}\sqrt{2} + \tfrac{38\,912}{105}\sqrt{2} \\
&= \underline{\underline{\tfrac{97\,232}{7}\sqrt{2}}}.
\end{aligned}$$

6. The line l_1 has equation
$$\mathbf{r} = \mathbf{i} + 2\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$$
where λ is a scalar parameter.

The line l_2 has equation
$$\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-1}{3}.$$

(a) Prove that the lines l_1 and l_2 are skew. (4)

Solution

$\underline{l_1}$:
$$\begin{aligned}
&\mathbf{r} = \mathbf{i} + 2\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) \\
\Rightarrow\ &x = 1 + 2\lambda,\ y = 3\lambda,\ \text{and } z = 2 - \lambda.
\end{aligned}$$

$\underline{l_2}$:

$$\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-1}{3} = \mu$$
$$\Rightarrow \quad x = -1+\mu,\ y = 4+\mu, \text{ and } z = 1+3\mu.$$

Both together:

$$\begin{aligned} x:&\quad 1+2\lambda = -1+\mu && (1)\\ y:&\quad 3\lambda = 4+\mu && (2)\\ z:&\quad 2-\lambda = 1+3\mu && (3)\end{aligned}$$

Now, from (1) and (2),

$$\begin{aligned}\mu = 3\lambda - 4 &\Rightarrow 1+2\lambda = -1+(3\lambda-4)\\ &\Rightarrow 1+2\lambda = 3\lambda - 5\\ &\Rightarrow \lambda = 6\\ &\Rightarrow \mu = 14.\end{aligned}$$

Finally, we go to (3): is $2-\lambda = 1+3\mu$? Well,

$$2-6 = -4 \text{ and } 1+3\times 14 = 43,$$

the answer is 'no'. Hence, the lines l_1 and l_2 are skew.

(b) Find the shortest distance between the lines l_1 and l_2. (5)

Solution
For l_2,

$$\mathbf{r} = -\mathbf{i} + 4\mathbf{j} + \mathbf{k} + \mu(\mathbf{i}+\mathbf{j}+3\mathbf{k}).$$

Now, for the shortest distance between the lines l_1 and l_2, we need

$$\left|\frac{(\mathbf{a}-\mathbf{c}).(\mathbf{b}\times\mathbf{d})}{|\mathbf{b}\times\mathbf{d}|}\right|.$$

Now,

$$\begin{aligned}\mathbf{b}\times\mathbf{d} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 3 & -1\\ 1 & 1 & 3\end{vmatrix}\\ &= 10\mathbf{i} - 7\mathbf{j} - \mathbf{k}\end{aligned}$$

and

$$|\mathbf{b} \times \mathbf{d}| = 5\sqrt{6}.$$

Next,

$$\mathbf{a} - \mathbf{c} = 2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$$

and we put it all together:

$$\begin{aligned}\left|\frac{(\mathbf{a} - \mathbf{c}).(\mathbf{b} \times \mathbf{d})}{|\mathbf{b} \times \mathbf{d}|}\right| &= \left|\frac{20 + 28 - 1}{5\sqrt{6}}\right| \\ &= \left|\frac{47}{5\sqrt{6}}\right| \\ &= \underline{\underline{\frac{47\sqrt{6}}{30}}}.\end{aligned}$$

The plane Π contains l_1 and intersects l_2 at the point $(3, 8, 13)$.

(c) Find a cartesian equation for the plane Π. (4)

Solution
Now,

$$(3\mathbf{i} + 8\mathbf{j} + 13\mathbf{k}) - (\mathbf{i} + 2\mathbf{k}) = 2\mathbf{i} + 8\mathbf{j} + 11\mathbf{k}$$

and

$$\begin{aligned}(2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) \times (2\mathbf{i} + 8\mathbf{j} + 11\mathbf{k}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -1 \\ 2 & 8 & 11 \end{vmatrix} \\ &= 41\mathbf{i} - 24\mathbf{j} + 10\mathbf{k}.\end{aligned}$$

They intersect at $(3, 8, 13)$:

$$(41 \times 3) + (-24 \times 8) + (10 \times 13) = 61$$

and so a cartesian equation is

$$\underline{\underline{41x - 24y + 10z = 61}}.$$

7. The ellipse E has foci at the points $(\pm 3, 0)$ and has directrices with equations $x = \pm\frac{25}{3}$.

(a) Find a cartesian equation for the ellipse E. (5)

Solution

$ae = 3$ and $\dfrac{a}{e} = \frac{25}{3}$:

$$\begin{aligned} a^2 &= ae \times \frac{a}{e} \\ &= 3 \times \frac{25}{3} \\ &= 25 \end{aligned}$$

and $a = 5$ with $e = \frac{3}{5}$. Now,

$$\begin{aligned} b^2 &= a^2(1 - e^2) \\ &= 5^2[1 - (\tfrac{3}{5})^2] \\ &= 16. \end{aligned}$$

Hence, the cartesian equation is

$$\underline{\underline{\frac{x^2}{25} + \frac{y^2}{16} = 1}}.$$

The straight line l has equation $y = mx + c$, where m and c are **positive** constants.

(b) Show that the x-coordinates of any points of intersection of l and E satisfy the equation (2)

$$(16 + 25m^2)x^2 + 50mcx + 25(c^2 - 16) = 0.$$

Solution

$$\begin{aligned} y = mx + c \Rightarrow \frac{x^2}{25} + \frac{(mx+c)^2}{16} &= 1 \\ \Rightarrow 16x^2 + 25(mx+c)^2 &= 400 \\ \Rightarrow 16x^2 + 25(m^2x^2 + 2cmx + c^2) - 400 &= 0 \\ \Rightarrow 16x^2 + 25m^2x^2 + 50cmx + 25c^2 - 400 &= 0 \\ \Rightarrow \underline{\underline{(16 + 25m^2)x^2 + 50mcx + 25(c^2 - 16) = 0}}, \end{aligned}$$

as required.

Given that the line l is a tangent to E,

(c) show that (3)

$$c^2 = pm^2 + q,$$

where p and q are constants to be found.

Solution

The discriminant equal zero:

$$\begin{aligned}
& (50mc)^2 - 4(16 + 25m^2) \times 25(c^2 - 16) \\
\Rightarrow\ & 2\,500c^2m^2 = 100(16c^2 + 25c^2m^2 - 400m^2 - 256) \\
\Rightarrow\ & 25c^2m^2 = 16c^2 + 25c^2m^2 - 400m^2 - 256 \\
\Rightarrow\ & 16c^2 = 400m^2 + 256 \\
\Rightarrow\ & \underline{\underline{c^2 = 25m^2 + 16}};
\end{aligned}$$

hence, $\underline{\underline{p = 25}}$ and $\underline{\underline{q = 16}}$.

The line l intersects the x-axis at the point A and intersects the y-axis at the point B.

(d) Show that the area of triangle OAB, where O is the origin, is (3)

$$\frac{25m^2 + 16}{2m}.$$

Solution

Now,

$$c = \sqrt{25m^2 + 16}$$

and

$$\begin{aligned}
0 = mx + c \Rightarrow mx &= -\sqrt{25m^2 + 16} \\
\Rightarrow x &= -\frac{\sqrt{25m^2 + 16}}{m}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\text{area} &= \tfrac{1}{2}bh \\
&= \tfrac{1}{2} \times \frac{\sqrt{25m^2 + 16}}{m} \times \sqrt{25m^2 + 16} \\
&= \underline{\underline{\frac{25m^2 + 16}{2m}}},
\end{aligned}$$

as required.

(e) Find the minimum area of triangle OAB. (2)

Solution

Let

$$A = \frac{25m^2 + 16}{2m}.$$

Now,

$$A = \frac{25m^2 + 16}{2m} \Rightarrow A = \tfrac{25}{2}m + 8m^{-1}$$
$$\Rightarrow \frac{\mathrm{d}A}{\mathrm{d}m} = \tfrac{25}{2} - 8m^{-2}.$$

Next,

$$\frac{\mathrm{d}A}{\mathrm{d}m} = 0 \Rightarrow \frac{25}{2} - \frac{8}{m^2} = 0$$
$$\Rightarrow \frac{25}{2} = \frac{8}{m^2}$$
$$\Rightarrow 25m^2 = 16$$
$$\Rightarrow m^2 = \tfrac{16}{25}$$
$$\Rightarrow m = \tfrac{4}{5}$$
$$\Rightarrow A = \frac{25(\frac{4}{5})^2 + 16}{2(\frac{4}{5})}$$
$$\Rightarrow \underline{\underline{A = 20}}.$$