Dr Oliver Mathematics
Further Mathematics: Further Pure Mathematics 2 (Paper 4A)
June 2022: Calculator
1 hour 30 minutes

The total number of marks available is 75.
You must write down all the stages in your working.
Inexact answers should be given to three significant figures unless otherwise stated.

1. The group S_4 is the set of all possible permutations that can be performed on the four numbers 1, 2, 3, and 4, under the operation of composition. For the group S_4,

   (a) write down the identity element, (1)

   Solution
   The identity element is
   $$\underline{\underline{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}}}.$$

   (b) write down the inverse of the element a, where (1)
   $$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}.$$

   Solution
   The inverse is
   $$\underline{\underline{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}}}.$$

   (c) demonstrate that the operation of composition is associative using the following elements (2)
   $$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}, c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}.$$

Solution

Well,

$$\begin{aligned}(a \circ b) \circ c &= \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}\right] \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}a \circ (b \circ c) &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} \circ \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}\right] \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix};\end{aligned}$$

so

$$(a \circ b) \circ c = a \circ (b \circ c),$$

demonstrating that operation of composition is associative.

(d) Explain why it is possible for the group S_4 to have a subgroup of order 4. (2)
You do not need to find such a subgroup.

Solution

Well, the order of the group is

$$4! = 24.$$

Now,

$$\frac{24}{4} = 6$$

and 4 is a factor of 24.
Hence, it is possible for a subgroup to have order 4.

2. Matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & a \\ -3 & b & 1 \\ 0 & 1 & a \end{pmatrix},$$

where a and b are integers, such that $a < b$.

Given that the characteristic equation for $\mathbf{M}$ is

$$\lambda^3 - 7\lambda^2 + 13\lambda + c = 0,$$

where c is a constant,

(a) determine the values of a, b, and c. (5)

Solution

$$\begin{aligned} & \det(\mathbf{M} - \lambda I) = 0 \\ \Rightarrow\ & (1-\lambda)[(b-\lambda)(a-\lambda) - 1] - 0 + a[-3-0] = 0 \\ \Rightarrow\ & (1-\lambda)[(ab - (a+b)\lambda + \lambda^2) - 1] - 3a = 0 \\ \Rightarrow\ & (1-\lambda)[(ab-1) - (a+b)\lambda + \lambda^2] - 3a = 0 \end{aligned}$$

$\times$	$ab-1$	$-(a+b)\lambda$	$+\lambda^2$
1	$ab-1$	$-(a+b)\lambda$	$+\lambda^2$
$-\lambda$	$-(ab-1)\lambda$	$+(a+b)\lambda^2$	$-\lambda^3$

$$\begin{aligned} \Rightarrow\ & ab - 1 + [-(a+b) - (ab-1)]\lambda + [1 + (a+b)]\lambda^2 - \lambda^3 - 3a = 0 \\ \Rightarrow\ & \lambda^3 - [1 + (a+b)]\lambda^2 - [-(a+b) - (ab-1)]\lambda - (ab - 1 - 3a) = 0 \\ \Rightarrow\ & \lambda^3 - [1 + (a+b)]\lambda^2 + [a + b + (ab-1)]\lambda - (ab - 1 - 3a) = 0; \end{aligned}$$

so,

$$\begin{aligned} \lambda^2 :&\quad a + b + 1 = 7 \Rightarrow a + b = 6 \quad (1) \\ \lambda :&\quad a + b + (ab-1) = 13 \Rightarrow a + b + ab = 14 \quad (2). \end{aligned}$$

From (2),

$$\begin{aligned} a + b + ab = 14 &\Rightarrow 6 + ab = 14 \\ &\Rightarrow ab = 8. \end{aligned}$$

Now, a and b are integers, such that $a < b$. Thus,

$$\underline{\underline{a = 2}} \text{ and } \underline{\underline{b = 4}}.$$

What about c? Well,

$$\begin{aligned} c &= -(ab - 1 - 3a) \\ &= -(8 - 1 - 6) \\ &= \underline{\underline{-1}}. \end{aligned}$$

(b) Hence, using the Cayley-Hamilton theorem, determine the matrix $\mathbf{M}^{-1}$. (3)

Solution

Well, the Cayley-Hamilton theorem states:

$$\mathbf{M}^3 - 7\mathbf{M}^2 + 13\mathbf{M} - \mathbf{I} = \mathbf{0}.$$

Now,

$$\begin{aligned}
& \mathbf{I} = \mathbf{M}^3 - 7\mathbf{M}^2 + 13\mathbf{M} \\
\Rightarrow\quad & \mathbf{M}^{-1}\mathbf{I} = \mathbf{M}^{-1}(\mathbf{M}^3 - 7\mathbf{M}^2 + 13\mathbf{M}) \\
\Rightarrow\quad & \mathbf{M}^{-1} = \mathbf{M}^2 - 7\mathbf{M} + 13\mathbf{I} \\
\Rightarrow\quad & \mathbf{M}^{-1} = \begin{pmatrix} 1 & 0 & 2 \\ -3 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix}^2 - 7\begin{pmatrix} 1 & 0 & 2 \\ -3 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix} + 13\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ -3 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ -3 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 6 \\ -15 & 17 & 0 \\ -3 & 6 & 5 \end{pmatrix}$$

$$\begin{aligned}
\Rightarrow\quad & \mathbf{M}^{-1} = \begin{pmatrix} 1 & 2 & 6 \\ -15 & 17 & 0 \\ -3 & 6 & 5 \end{pmatrix} - 7\begin{pmatrix} 1 & 0 & 2 \\ -3 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix} + 13\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\Rightarrow\quad & \mathbf{M}^{-1} = \underline{\underline{\begin{pmatrix} 7 & 2 & -8 \\ 6 & 2 & -7 \\ -3 & -1 & 4 \end{pmatrix}}}.
\end{aligned}$$

3. There are three lily pads on a pond.

A frog hops repeatedly from one lily pad to another.

The frog starts on lily pad A, as shown in Figure 1.

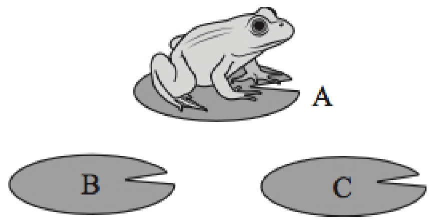


Figure 1: a frog hops repeatedly from one lily pad to another

In a model, the frog hops from its position on one lily pad to either of the other two lily pads with equal probability.

Let p_n be the probability that the frog is on lily pad A after n hops.

(a) Explain, with reference to the model, why $p_1 = 0$. (1)

Solution
E.g., the frog **leaves** A and it **ends up** in either B or C.

The probability p_n satisfies the recurrence relation

$$p_{n+1} = \tfrac{1}{2}(1 - p_n),\ n \geqslant 1 \text{ where } p_1 = 0.$$

(b) Prove by induction that, for $n \geqslant 1$, (6)

$$p_n = \tfrac{2}{3}(-\tfrac{1}{2})^n + \tfrac{1}{3}.$$

Solution
$\underline{n = 1}$:

$$\begin{aligned} p_1 &= \tfrac{2}{3}(-\tfrac{1}{2})^1 + \tfrac{1}{3} \\ &= -\tfrac{1}{3} + \tfrac{1}{3} \\ &= 0, \end{aligned}$$

so the result is true for $n = 1$.

Suppose it is true for $n = k$, i.e.,

$$p_k = \tfrac{2}{3}(-\tfrac{1}{2})^k + \tfrac{1}{3}.$$

Then

$$\begin{aligned} p_{k+1} &= \tfrac{1}{2}\left[1 - (\tfrac{2}{3}(-\tfrac{1}{2})^k + \tfrac{1}{3})\right] \\ &= \tfrac{1}{2} - \tfrac{2}{3}(\tfrac{1}{2})(-\tfrac{1}{2})^k - \tfrac{1}{6} \\ &= +\tfrac{2}{3}(-\tfrac{1}{2})(-\tfrac{1}{2})^k + \tfrac{1}{3} \\ &= \tfrac{2}{3}(-\tfrac{1}{2})^{k+1} + \tfrac{1}{3}, \end{aligned}$$

so the statement is true for $n = k + 1$.

Hence, by mathematical induction, the statement is true for all n.

(c) Use the result in part (b) to explain why, in the long term, the probability that the frog is on lily pad A is $\frac{1}{3}$. (1)

Solution
Well, as $n \to \infty$, $(-\frac{1}{2})^n \to 0$ which means

$$p_n \to \tfrac{1}{3}.$$

4. (a) Use the Euclidean algorithm to show that 124 and 17 are relatively prime (coprime) (2)

Solution
Now,

$$\begin{aligned} 124 &= 17 \times 7 + 5, \\ 17 &= 3 \times 5 + 2, \\ 5 &= 2 \times 2 + 1. \end{aligned}$$

The HCF is 1 so 124 and 17 are relatively prime (coprime).

(b) Hence solve the equation (3)

$$124x + 17y = 10.$$

Solution

Well,

$$
\begin{aligned}
1 &= 5 - 2 \times 2 \\
&= 5 - 2(17 - 3 \times 5) \\
&= 7 \times 5 - 2 \times 17 \\
&= 7(124 - 17 \times 7) - 2 \times 17 \\
&= 7 \times 124 - 51 \times 7
\end{aligned}
$$

and

$$
\begin{aligned}
10 &= 10(7 \times 124 - 51 \times 7) \\
&= \underline{\underline{70 \times 124 - 510 \times 7}};
\end{aligned}
$$

hence, $\underline{\underline{x = 70}}$ and $\underline{\underline{y = 510}}$.

(c) Solve the congruence equation (2)

$$124x \equiv 6 (\text{mod } 17).$$

Solution

E.g.,

$$
\begin{aligned}
124x \equiv 6 (\text{mod } 17) &\Rightarrow 7 \times 124x \equiv 7 \times 6 (\text{mod } 17) \\
&\Rightarrow 868x \equiv 42 (\text{mod } 17) \\
&\Rightarrow (17 \times 51x + x) \equiv (17 \times 2 + 8)(\text{mod } 17) \\
&\Rightarrow \underline{\underline{x \equiv 8 (\text{mod } 17)}}.
\end{aligned}
$$

5. The locus of points z satisfies

$$|z + a\mathrm{i}| = 3|z - a|,$$

where a is an integer.

The locus is a circle with its centre in the third quadrant and radius $\frac{3}{2}\sqrt{2}$.

Determine

(a) the value of a, (4)

Solution

Let $z = x + \mathrm{i}y$. Then

$$\begin{aligned}
& |z + a\mathrm{i}| = 3|z - a| \\
\Rightarrow\quad & |x + (y + a)\mathrm{i}|^2 = 9|(x - a) + \mathrm{i}y|^2 \\
\Rightarrow\quad & x^2 + (y + a)^2 = 9\left[(x - a)^2 + y^2\right] \\
\Rightarrow\quad & x^2 + (y^2 + 2ay + a^2) = 9\left[(x^2 - 2ax + a^2) + y^2\right] \\
\Rightarrow\quad & x^2 + (y^2 + 2ay + a^2) = 9(x^2 - 2ax + a^2) + 9y^2 \\
\Rightarrow\quad & x^2 + (y^2 + 2ay + a^2) = 9x^2 - 18ax + 9a^2 + 9y^2 \\
\Rightarrow\quad & 8x^2 - 18ax + 8y^2 - 2ay + 8a^2 = 0 \\
\Rightarrow\quad & x^2 - \tfrac{9}{4}ax + y^2 - \tfrac{1}{4}ay = -a^2 \\
\Rightarrow\quad & (x^2 - \tfrac{9}{4}ax + \tfrac{81}{64}a^2) + (y^2 - \tfrac{1}{4}ay + \tfrac{1}{64}a^2) = -a^2 + \tfrac{81}{64}a^2 + \tfrac{1}{16}a^2 \\
\Rightarrow\quad & (x - \tfrac{9}{8}a)^2 + (y - \tfrac{1}{8}a)^2 = \tfrac{9}{32}a^2.
\end{aligned}$$

Finally,

$$\begin{aligned}
\sqrt{\tfrac{9}{32}a^2} = \tfrac{3}{2}\sqrt{2} &\Rightarrow \tfrac{3\sqrt{2}}{8}a = \pm\tfrac{3}{2}\sqrt{2} \\
&\Rightarrow a = \pm 4.
\end{aligned}$$

Which is it? “The locus is a circle with its centre in the third quadrant” so

$$\underline{\underline{a = -4}}.$$

(b) the coordinates of the centre of the circle. (2)

Solution

$$\begin{aligned}
& (x - \tfrac{9}{8}(-4))^2 + (y - \tfrac{1}{8}(-4))^2 = \tfrac{9}{32}[(-4)]^2 \\
= \quad & (x + \tfrac{9}{2})^2 + (y + \tfrac{1}{2})^2 = \tfrac{9}{2};
\end{aligned}$$

hence, the centre of the circle is $\underline{\underline{(-\tfrac{9}{2}, -\tfrac{1}{2})}}$.

6. (a) Determine the general solution of the recurrence relation (4)

$$u_n = 2u_{n-1} - u_{n-2} + 2^n, \geqslant 2.$$

Solution
Well,

$$\begin{aligned} m^2 = 2m - 1 &\Rightarrow m^2 - 2m + 1 = 0 \\ &\Rightarrow (m-1)^2 = 0 \\ &\Rightarrow m = 1 \text{ (repeated)} \end{aligned}$$

so

$$u_n = an + b,$$

for some constant a and b.

Now, consider $u_n = c(2^n)$:

$$\begin{aligned} c(2^n) = 2c(2^{n-1}) - c(2^{n-2}) + 2^n &\Rightarrow 2^2c(2^{n-2}) = 2^2c(2^{n-2}) - c(2^{n-2}) + 2^n 2^{n-2} \\ &\Rightarrow 4c = 4c - c + 4 \\ &\Rightarrow c = 4. \end{aligned}$$

So,

$$u_n = 4(2^n)$$

and putting it altogether,

$$\underline{\underline{u_n = an + b + 4(2^n)}}.$$

(b) Hence solve this recurrence relation given that $u_0 = 2u_1$ and $u_4 = 3u_2$. (2)

Solution
Well,

$$\begin{aligned} u_0 = 2u_1 &\Rightarrow b + 4 = 2(a + b + 8) \\ &\Rightarrow b + 4 = 2a + 2b + 16 \\ &\Rightarrow 2a + b = -12 \quad (1) \end{aligned}$$

and

$$\begin{aligned} u_4 = 3u_2 &\Rightarrow 4a + b + 64 = 3(2a + b + 16) \\ &\Rightarrow 4a + b + 64 = 6a + 3b + 48 \\ &\Rightarrow 2a + 2b = 16 \quad (2). \end{aligned}$$

Do (2) − (1):

$$\begin{aligned} b = 28 &\Rightarrow 2a + 28 = -12 \\ &\Rightarrow 2a = -40 \\ &\Rightarrow a = -20. \end{aligned}$$

Hence,

$$\underline{\underline{u_n = -20n + 28 + 4(2^n)}}.$$

7. The polynomial $\mathrm{F}(x)$ is a quartic such that

$$\mathrm{F}(x) = px^4 + qx^3 + 2x^2 + rx + s,$$

where p, q, r, and s are distinct constants.

Determine the number of possible quartics given that

(a) (i) the constants p, q, r, and s belong to the set $\{-4, -2, 1, 3, 5\}$, (1)

Solution

$$5 \times 4 \times 3 \times 2 = \underline{\underline{120}}.$$

(ii) the constants p, q, r, and s belong to the set $\{-4, -2, 0, 1, 3, 5\}$. (1)

Solution
Well, we have a quartic so $p \neq 0$:

$$5 \times 5 \times 4 \times 3 = \underline{\underline{300}}.$$

A 3-digit positive integer $N = abc$ has the following properties

- N is divisible by 11,
- the sum of the digits of N is even, and
- $N \equiv 8 (\mathrm{mod}\ 9)$.

(b) (i) Use the first two properties to show that (2)

$$a - b + c = 0.$$

Solution
First statement: As N divisible by 11 implies

$$a - b + c = 11d,$$

where d is an integer. Now, e.g., $a = 9$, $b = 0$, and $c = 9$ we can see that

$$a - b + c = 18$$

and $a = 0$, $b = 9$, and $c = 0$ we can see that

$$a - b + c = -9;$$

so

$$a - b + c = 0 \text{ or } a - b + c = 11.$$

Second statement:

$$a + b + c = \text{is a even number}$$

and

$$\begin{aligned} a - b + c &= a + b + c - (2b) \\ &= \text{is a even number.} \end{aligned}$$

Hence,

$$\underline{\underline{a - b + c = 0}}. \quad (1)$$

(ii) Hence determine all possible integers N, showing all your working and reasoning. (4)

Solution
Well,

$$N = 100a + 10b + c$$

and

$$100a + 10b + c = a + b + c + 9(11a + b)$$

which means

$$a + b + c \equiv 8 (\text{mod } 9).$$

Now, either

$$a + b + c = 8 \quad (2)$$

or

$$a + b + c = 26 \quad (3).$$

(Wait! Can this

$$a + b + c = 26$$

happens? It means two of a, b, or c need to be 9 and 9 and the third one needs to be 8. No, it cannot!)

Do (2) − (1):

$$2b = 8 \Rightarrow b = 4.$$

So

$$a = 1, c = 3 \quad a = 2, c = 2 \quad a = 3, c = 1 \quad a = 4, c = 0$$

and, finally, all possible integers are

$$\underline{\underline{143,\ 242,\ 341,\ \text{and } 440}}.$$

8. The locus of points $z = x + \mathrm{i}y$ that satisfy

$$\arg\left(\frac{z - 8 - 5\mathrm{i}}{z - 2 - 5\mathrm{i}}\right)$$

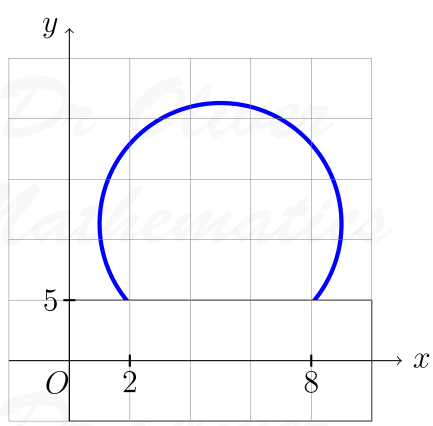
is an arc of a circle C.

(a) On an Argand diagram sketch the locus of z. (2)

Solution

$$\arg\left(\frac{z - 8 - 5\mathrm{i}}{z - 2 - 5\mathrm{i}}\right) = \arg\left(\frac{z - (8 + 5\mathrm{i})}{z - (2 + 5\mathrm{i})}\right)$$

and we want the major arc (why?).



(b) Explain why the centre of C has x-coordinate 5. (1)

Solution

The centre lies on the perpendicular bisector between 2 + 5i and 8 + 5i:

$$x = \frac{2+8}{2} = \underline{\underline{5}}.$$

(c) Determine the radius of C. (2)

Solution

$$\begin{aligned}\sin \tfrac{1}{3}\pi = \frac{3}{r} &\Rightarrow r = \frac{3}{\frac{\sqrt{3}}{2}} \\ &\Rightarrow \underline{\underline{r = 2\sqrt{3}}}.\end{aligned}$$

(d) Determine the y-coordinate of the centre of C. (2)

Solution

Finally,

$$\begin{aligned}y &= 5 + \sqrt{(2\sqrt{3})^2 - 3^2} \\ &= 5 + \sqrt{12 - 9} \\ &= \underline{\underline{5 + \sqrt{3}}}.\end{aligned}$$

9.

$$I_n = \int_0^{\frac{1}{2}\pi} \sin^n 2x \, \mathrm{d}x.$$

(a) Prove that, for $n \geqslant 2$, (4)

$$I_n = \frac{(n-1)}{n} I_{n-2}.$$

Solution

$$\begin{aligned}u = \sin^{n-1} 2x &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2(n-1)\sin^{n-2} 2x \cos 2x \\ \frac{\mathrm{d}v}{\mathrm{d}x} = \sin 2x &\Rightarrow v = -\tfrac{1}{2}\cos 2x.\end{aligned}$$

We use integration by parts:

$$\begin{aligned}
& I_n \\
= & \int_0^{\frac{1}{2}\pi} \sin^n 2x \,\mathrm{d}x \\
= & \int_0^{\frac{1}{2}\pi} (\sin^{n-1} 2x)(\sin 2x) \,\mathrm{d}x \\
= & \left[-\tfrac{1}{2}\sin^{n-1} 2x \cos 2x\right]_{x=0}^{\frac{1}{2}\pi} - \int_0^{\frac{1}{2}\pi} (2(n-1)\sin^{n-2} 2x \cos 2x)(-\tfrac{1}{2}\cos 2x) \,\mathrm{d}x \\
= & (0-0) + (n-1)\int_0^{\frac{1}{2}\pi} \sin^{n-2} 2x \cos^2 2x \,\mathrm{d}x \\
= & (n-1)\int_0^{\frac{1}{2}\pi} \sin^{n-2} 2x (1-\sin^2 2x) \,\mathrm{d}x \\
= & (n-1)\int_0^{\frac{1}{2}\pi} \sin^{n-2} 2x \,\mathrm{d}x - (n-1)\int_0^{\frac{1}{2}\pi} \sin^n 2x \\
= & (n-1)I_{n-2} - (n-1)I_n.
\end{aligned}$$

Now,

$$I_n = (n-1)I_{n-2} - (n-1)I_n \Rightarrow nI_n = (n-1)I_{n-2}$$
$$\Rightarrow \underline{\underline{I_n = \frac{(n-1)}{n} I_{n-2}}},$$

as required.

(b) Hence determine the exact value of (3)

$$\int_0^{\frac{1}{2}\pi} 64 \sin^5 x \cos^5 x \,\mathrm{d}x.$$

Solution

Well,

$$\begin{aligned} 64\sin^5 x\cos^5 x &= 2(2^5\sin^5 x\cos^5 x) \\ &= 2(2\sin x\cos x)^5 \\ &= 2(\sin 2x)^5 \\ &= 2\sin^5 2x. \end{aligned}$$

Now,

$$\begin{aligned} I_5 &= \tfrac{4}{5}I_3 \\ &= \tfrac{4}{5}(\tfrac{2}{3}I_1) \\ &= \tfrac{8}{15}I_1 \end{aligned}$$

and

$$\begin{aligned} I_1 &= \int_0^{\frac{1}{2}\pi} \sin 2x \,\mathrm{d}x \\ &= \left[-\tfrac{1}{2}\cos 2x\right]_{x=0}^{\frac{1}{2}\pi} \\ &= \tfrac{1}{2} - (-\tfrac{1}{2}) \\ &= 1. \end{aligned}$$

So,

$$\begin{aligned} I_5 &= \tfrac{8}{15}\times 1 \\ &= \tfrac{8}{15} \end{aligned}$$

and

$$\begin{aligned} \int_0^{\frac{1}{2}\pi} 64\sin^5 x\cos^5 x \,\mathrm{d}x &= \int_0^{\frac{1}{2}\pi} 2\sin^5 2x \,\mathrm{d}x \\ &= 2\int_0^{\frac{1}{2}\pi} \sin^5 2x \,\mathrm{d}x \\ &= 2\times\tfrac{8}{15} \\ &= \underline{\underline{\tfrac{16}{15}}}. \end{aligned}$$

10. Figure 2 shows a picture of a plant pot.

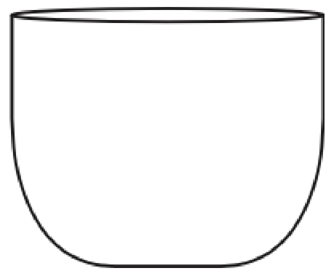


Figure 2: a plant pot

The plant pot has

- a flat circular base of radius 10 cm and
- a height of 15 cm

Figure 3 shows a sketch of the curve C with parametric equations

$$x = 10 + 15t - 5t^3,\ y = 15t^2,\ 0 \leqslant t \leqslant 1.$$

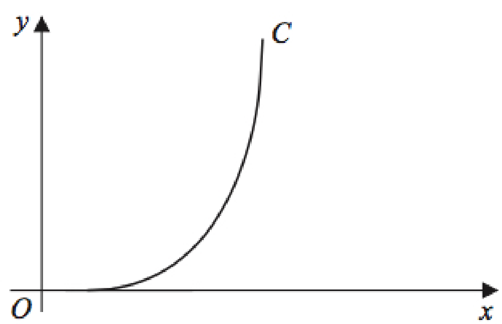


Figure 3: $x = 10 + 15t - 5t^3$, $y = 15t^2$

The curved inner surface of the plant pot is modelled by the surface of revolution formed by rotating curve C through 2π radians about the y-axis.

(a) Show that, according to the model, the area of the curved inner surface of the plant pot is given by (5)

$$150\pi \int_0^1 (2 + 3t + 2t^2 + 2t^3 - t^5)\, \mathrm{d}t.$$

Solution

Well,

$$x = 10 + 15t - 5t^3 \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = 15 - 15t^2$$
$$y = 15t^2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = 30t$$

and

$$\begin{aligned}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} &= \sqrt{(15 - 15t^2)^2 + (30t)^2} \\ &= \sqrt{(225 - 450t^2 + 225t^4) + 900t^2} \\ &= \sqrt{225 + 450t^2 + 225t^4} \\ &= \sqrt{(15 + 15t^2)^2} \\ &= 15 + 15t^2.\end{aligned}$$

Finally,

$$\begin{aligned}\text{surface of revolution} &= 2\pi \int x \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t \\ &= 2\pi \int_0^1 (10 + 15t - 5t^3)(15 + 15t^2)\,\mathrm{d}t\end{aligned}$$

$\times$	10	$+15t$	$-5t^3$
15	150	$+225t$	$-75t^3$
$+15t^2$	$+150t^2$	$+225t^3$	$-75t^5$

$$\begin{aligned}&= 2\pi \int_0^1 (150 + 225t + 150t^2 + 225t^3 - 75t^5)\,\mathrm{d}t \\ &= 2\pi \int_0^1 75(2 + 3t + 2t^2 + 2t^3 - t^5)\,\mathrm{d}t \\ &= \underline{\underline{150\pi \int_0^1 (2 + 3t + 2t^2 + 2t^3 - t^5)\,\mathrm{d}t}},\end{aligned}$$

as required.

(b) Determine, according to the model, the total area of the inner surface of the plant pot. (4)

Solution

Now,

$$\begin{aligned}\text{surface of revolution} &= 150\pi \int_0^1 (2 + 3t + 2t^2 + 2t^3 - t^5)\,\mathrm{d}t \\ &= 150\pi \left[2t + \tfrac{3}{2}t^2 + \tfrac{2}{3}t^3 + \tfrac{1}{2}t^4 - \tfrac{1}{6}t^6\right]_{t=0}^{1} \\ &= 150\pi \left\{\left(2 + \tfrac{3}{2} + \tfrac{2}{3} + \tfrac{1}{2} - \tfrac{1}{6}\right) - (0 + 0 + 0 + 0 - 0)\right\} \\ &= 150\pi \times 4.5 \\ &= 675\pi\end{aligned}$$

and

$$\begin{aligned}\text{area of the circular base} &= \pi \times 10^2 \\ &= 100\pi.\end{aligned}$$

Hence,

$$\begin{aligned}\text{total area} &= \text{surface of revolution} + \text{area of the circular base} \\ &= 675\pi + 100\pi \\ &= \underline{\underline{775\pi \text{ cm}^2}}.\end{aligned}$$

Each plant pot will be painted with one coat of paint, both inside and outside.
The paint in one tin will cover an area of 12 m^2.

(c) Use the answer to part (b) to estimate how many plant pots can be painted using one tin of paint. (2)

Solution

Well,

$$\begin{aligned}12 \text{ m}^2 &= 12 \times 1 \text{ m}^2 \\ &= 12 \times 100 \text{ cm} \times 100 \text{ cm} \\ &= 12 \times 10\,000 \text{ cm}^2 \\ &= 120\,000 \text{ cm}^2\end{aligned}$$

and we paint both **inside** and **outside**:

$$2 \times 775\pi = 1\,550\pi.$$

Now,

$$\frac{120\,000}{1\,550\pi} = 26.643\,346\,03 \text{ (FCD)}.$$

Hence, the number of plant pots is $\underline{\underline{24}}$.

(d) Give a reason why the model might not give an accurate answer to part (c). (1)

Solution

E.g., there will be a small rim on the plant pot which is not considered, the surface area of the pot may not perfectly fit the curve, the surface area of the outside of the plant pot will be more than the inside.