Dr Oliver Mathematics
Mathematics
Arithmetic Series
Past Examination Questions

This booklet consists of 25 questions across a variety of examination topics.
The total number of marks available is 215.

1. What is the value for n such that (4)

$$2000 = 4 + 8 + 12 + \ldots ?$$

Solution

$a = 4$, $d = 4$, and $n = ?$:

$$\begin{aligned}
& 2000 > \tfrac{n}{2}[2 \times 4 + (n-1) \times 4] \\
\Rightarrow\ & 2000 > \tfrac{n}{2}[8 + 4(n-1)] \\
\Rightarrow\ & 4000 > n(4n+4) \\
\Rightarrow\ & 4000 > 4n^2 + 4n \\
\Rightarrow\ & 1000 > n^2 + n \\
\Rightarrow\ & 1000\tfrac{1}{4} > n^2 + n + \tfrac{1}{4} \\
\Rightarrow\ & 1000\tfrac{1}{4} > (n + \tfrac{1}{2})^2 \\
\Rightarrow\ & n + \tfrac{1}{2} < -\sqrt{1000\tfrac{1}{4}} \text{ or } n + \tfrac{1}{2} > \sqrt{1000\tfrac{1}{4}} \\
\Rightarrow\ & n < -\tfrac{1}{2} - \sqrt{1000\tfrac{1}{4}} \text{ or } n > -\tfrac{1}{2} + \sqrt{1000\tfrac{1}{4}} \\
\Rightarrow\ & n < -32.126\,729\,2 \text{ or } n > 31.126\,729\,2 \text{ (FCD)}
\end{aligned}$$

and so the answer is $\underline{\underline{n = 32}}$.

2. The rth term of an arithmetic series is $(2r - 5)$.

   (a) Write down the first three terms of this series. (2)

Solution

$\underline{\underline{-3}}$, $\underline{\underline{-1}}$, and $\underline{\underline{1}}$.

(b) State the value of the common difference. (1)

Solution

2.

(c) Show that (3)

$$\sum_{r=1}^{n}(2r-5) = n(n-4).$$

Solution

$$\begin{aligned}\sum_{r=1}^{n}(2r-5) &= 2\sum_{r=1}^{n} r - 5 \\ &= 2 \times \tfrac{1}{2}n(n+1) - 5 \\ &= n(n+1) - 5 \\ &= n^2 - 4n \\ &= n(n-1).\end{aligned}$$

3. A girl saves money over a period of weeks. She saves 5p in Week 1, 7p in Week 2, 9p in Week 3, and so on until Week 200. Her weekly savings form an arithmetic sequence.

(a) Find the amount she saves in Week 200. (3)

Solution

$a = 5$, $d = 2$, and $n = 200$:

$$5 + 199 \times 2 = 5 + 398 = 403\text{p or } £4.03.$$

(b) Calculate her total savings over the complete 200 week period. (3)

Solution

$$S_{200} = \frac{200}{2}[2 \times 5 + 199 \times 2] = 100(10 + 398) = 100 \times 408 = £408.$$

4. An athlete prepares for a race by completing a practice run on each of 11 consecutive days. On each day after the first day, he runs further than he ran on the previous day. The lengths of his 11 practice runs form an arithmetic sequence with the first term a (7)

km and common difference d km. He runs 9 km on the 11th day, and runs a total of 77 km over the 11 day period. Find the value of a and find the value of d.

Solution

$a + 10d = 9$ and $\frac{11}{2}[2a + 10d] = 77$. Now, $a = 9 - 10d$ and so

$$\begin{aligned}\tfrac{11}{2}[2(9 - 10d) + 10d] = 77 &\Rightarrow \tfrac{11}{2}[(18 - 20d) + 10d] = 77\\ &\Rightarrow \tfrac{11}{2}[18 - 10d] = 77\\ &\Rightarrow 18 - 10d = 14\\ &\Rightarrow 4 = 10d\\ &\Rightarrow \underline{\underline{d = 0.4}},\end{aligned}$$

and hence $a = 9 - 10 \times 0.4 \Rightarrow \underline{\underline{a = 5}}$.

5. The first term of an arithmetic sequence is 30 the common difference is -1.5.

   (a) Find the value of the 25th term. (2)

   Solution

   $$30 + 24 \times (-1.5) = 30 - 36 = \underline{\underline{-6}}.$$

   The r term of the sequence is 0.

   (b) Find the value of r. (2)

   Solution

   $$30 + (n - 1) \times (-1.5) = 0 \Rightarrow n - 1 = 20 \Rightarrow \underline{\underline{n = 21}}.$$

   The of the first n terms of the sequence is S_n.

   (c) Find the largest positive value of S_n. (3)

   Solution

   $$S_n = \frac{21}{2}[2 \times 30 + 20 \times (-1.5)] = 10.5(60 - 30) = 10.5 \times 30 = \underline{\underline{315}}.$$

6. An arithmetic sequence has first term a and common difference d. The sum of the first 10 terms of the sequence is 162.

(a) Show that $10a + 45d = 162$. (2)

Solution

$$S_{10} = \frac{10}{2}(2a + 9d) = 162 \Rightarrow \underline{\underline{10a + 45d = 162}}.$$

Given also that the sixth term of the sequence is 17,

(b) write down a second equation in a and d, (1)

Solution

$\underline{\underline{a + 5d = 17}}$.

(c) find the value of a and find the value of d. (4)

Solution

$$a + 5d = 17 \Rightarrow 9a + 45d = 153$$

and so

$$\underline{\underline{a = 9}}$$

and

$$5d = 17 - 9 = 8 \Rightarrow \underline{\underline{d = 1.6}}.$$

7. A company, which is making 200 mobile phones each week, plans to increase its production. The number of mobile phones produced is to be increased by 20 each week from 200 week 1 to 220 in week 2, to 240 in week 3, and so on, until it is producing 600 in week N.

(a) Find the value of N. (2)

Solution

$a = 200$, $d = 20$, and $n = N$:

$$200 + 20(N - 1) = 600 \Rightarrow 20(N - 1) = 400 \Rightarrow N - 1 = 20 \Rightarrow \underline{\underline{N = 21}}.$$

The company then plans to continue to make 600 mobile phones each week.

(b) Find the total number of mobile phones that will be made in the first 52 weeks starting from and including week 2. (5)

Solution
$a = 200$, $d = 20$, and $n = 21$:

$$\frac{21}{2}[2 \times 200 + 20 \times 20] = \frac{21}{2} \times 800 = 8400.$$

We now add on to this $(52 - 21) \times 600 = 31 \times 600 = 18\,600$:

$$18\,600 + 8\,400 = \underline{\underline{27\,000 \text{ mobile phones}}}.$$

8. Jess started work 20 years ago. in year 1 her annual salary was £17 000. Her annual salary increased by £1500 each year, so that her annual salary in year 2 was £18 500, in year 3 was £20 000, and so on, forming an arithmetic sequence. This continued until she reached her maximum annual salary of £32 000 in year k. Her annual salary then remained at £32 000.

   (a) Find the value of the constant k. (2)

Solution
$17\,000 + 1500(k - 1) = 32\,000 \Rightarrow 170 + 15k - 15 = 320 \Rightarrow 15k = 165 \Rightarrow \underline{\underline{k = 11}}$.

   (b) Calculate the total amount that Jess earned in the 20 years. (5)

Solution
$a = 17\,000$, $d = 1500$, and $n = 11$:

$$\begin{aligned}\frac{11}{2}[2 \times 17\,000 + 10 \times 1500] &= \frac{11}{2}[34\,000 + 15\,000] \\ &= \frac{11}{2} \times 49\,000 \\ &= 11 \times 24\,500 \\ &= 245\,000 + 24\,500 \\ &= 269\,500;\end{aligned}$$

now, we add in 11 years that she was on the maximum:

$$9 \times 32\,000 + 269\,500 = 288\,000 + 269\,500 = \underline{\underline{£557\,500}}.$$

9. A 40-year building programme for new houses began in Oldtown in the year 1951 (Year 1)

and finished in 1990 (Year 40). The number of houses built each year form an arithmetic sequence with first term a and common difference d. Given that 2400 new houses were built in 1960 and 600 new houses were built in 1990, find

(a) the value of d, (3)

Solution

$a + 9d = 2400$ and $a + 39d = 600$:

$$30d = -1800 \Rightarrow \underline{\underline{d = -60}}.$$

(b) the value of a, (2)

Solution

$$a = 2400 - 9 \times (-60) = 2400 + 540 = \underline{\underline{2940}}.$$

(c) the total number of houses built in Oldtown over the 40-year period. (3)

Solution

$$S_{40} = \frac{40}{2}[2 \times 2940 + 39 \times (-60)] = 20(5880 - 2340) = 20 \times 3540 = \underline{\underline{70\,800}}.$$

10. A farmer has a pay scheme to keep fruit pickers working throughout the 30 day season. He pays £a for the first day, £$(a + d)$ for their second day, £$(a + 2d)$ for their third day, and so on, thus increasing the daily payment by £d for each extra day they work. A picker who works for all 30 days will earn £40.75 on the final day.

(a) Use this information to form an equation in a and d. (2)

Solution

$\underline{\underline{a + 29d = 40.75}}$.

A picker who works for all 30 days will earn a total of £1005.

(b) Show that $15(a + 40.75) = 1005$. (2)

Solution

$$\frac{30}{2}(a + 40.75) = 1005 \Rightarrow \underline{\underline{15(a + 40.75) = 1005}}.$$

(c) Hence find the value of a and find the value of d. (4)

Solution

$$15(a + 40.75) = 1005 \Rightarrow a + 40.75 = 67 \Rightarrow \underline{\underline{a = 26.25}}$$

and

$$29d = 40.75 - 26.25 = 14.50 \Rightarrow \underline{\underline{d = 0.5}}.$$

11. Lewis played a game of space invaders. He scored points for each spaceship that he captured. Lewis scored 140 points for capturing his first spaceship. He scored 160 points for capturing his second spaceship, 180 points for capturing his third spaceship, and so on. The number of points scored for capturing each successive spaceship form an arithmetic sequence.

(a) Find the number of points that Lewis scored for capturing his 20th spaceship. (2)

Solution

$a = 140$, $d = 20$, and $n = 20$:

$$140 + 20 \times 19 = 140 + 380 = \underline{\underline{520 \text{ points}}}.$$

(b) Find the total number of points Lewis scored for capturing his first 20 spaceships. (3)

Solution

$a = 140$, $d = 20$, and $n = 20$:

$$\frac{20}{2}[2 \times 140 + 20 \times 19] = 10(280 + 380) = 10 \times 660 = \underline{\underline{6600 \text{ points}}}.$$

Sian played an adventure game. She scored point for each dragon that she captured. The number of points that Sian scored for capturing each successive dragon form an arithmetic sequence. Sian captured n dragons and the total number of points that she scored for capturing all n dragons was 8500. Given than Sian scored 300 points for capturing her first dragon and then 700 points for capturing her nth dragon,

(c) find the value of n. (3)

Solution

$a = 300$ and $L = 700$:

$$\frac{n}{2}(300 + 700) = 8500 \Rightarrow 500n = 8500 \Rightarrow \underline{\underline{n = 17}}.$$

12. Xin has been given 14 day training schedule by her coach. Xin will run for A minutes on day 1, where A is a constant. She will then increase her running time by $(d+1)$ minutes each day, where d is a constant.

(a) Show that on day 14, Xin will run for (2)

$$(A + 13d + 13) \text{ minutes.}$$

Solution

$$u_{14} = A + 13(d+1) = \underline{\underline{(A + 13d + 13) \text{ minutes}}}.$$

Yi has also been given 14 day training schedule by her coach. Yi will run for $(A - 13)$ minutes on day 1. She will then increase her running time by $(2d - 1)$ minutes each day. Given that Xin and Yi will run for the same length of time day 14,

(b) find the value of d. (3)

Solution

$$\begin{aligned}(A - 13) + 13(2d - 1) = A + 13d + 13 &\Rightarrow -13 + 26d - 13 = 13d + 13 \\ &\Rightarrow 13d = 39 \\ &\Rightarrow \underline{\underline{d = 3}}.\end{aligned}$$

Given that Xin runs for a total time of 784 minutes over the 14 days,

(c) find the value of A. (3)

Solution

Now, $d = 3$ and we have

$$\begin{aligned}\frac{14}{2}[2A + 13 \times 4] = 784 &\Rightarrow 7(2A + 52) = 784 \\ &\Rightarrow 2A + 52 = 112 \\ &\Rightarrow 2A = 60 \\ &\Rightarrow \underline{\underline{A = 30}}.\end{aligned}$$

13. (a) Calculate the sum of all the even numbers from 2 to 100 inclusive, (3)

$$2 + 4 + 6 + \ldots + 100.$$

Solution

$a = 2$, $d = 2$, and $n = 50$:

$$\frac{50}{2}(2 \times 2 + 2 \times 49) = 25(4 + 98) = 25 \times 102 = \underline{\underline{2550}}.$$

(b) In the arithmetic series

$$k + 2k + 3k + \ldots + 100$$

k is a positive integer and k is a factor of 100.

(i) Find, in terms of k, an expression for the number of terms in this series. (1)

Solution

$\underline{\underline{\frac{100}{k}}}$.

(ii) Show that the sum of this series is (3)

$$50 + \frac{5000}{k}.$$

Solution

We use $a = k$ and $L = 100$:

$$\text{Sum} = \tfrac{1}{2} \times \frac{100}{k} \times (k + 100) = \frac{50}{k}(k + 100) = \underline{\underline{50 + \frac{5000}{k}}}.$$

(c) Find, in terms of k, the 50th term of the arithmetic sequence (2)

$$(2k + 1), (4k + 4), (6k + 7), \ldots,$$

giving your answer in its simplest form.

Solution

$a = 2k + 1$, $d = 2k + 3$, and $n = 50$:

$$2k + 1 + 49(2k + 3) = 2k + 1 + 98k + 147 = \underline{\underline{100k + 148}}.$$

14. A company offers two salary schemes for a 10-year period, Year 1 to Year 10 inclusive.

Scheme 1: Salary in Year 1 is £P, salary increases by £$(2T)$ each year, forming an arithmetic sequence.

Scheme 2: Salary in Year 1 is £$(P + 1800)$, salary increases by £T each year, forming an arithmetic sequence.

(a) Show that the **total** earned under Salary Scheme 1 for the 10-year period is (2)

$$£(10P + 90T).$$

Solution

$$\frac{10}{2}[2 \times P + 9 \times 2T] = 5(2P + 18T) = \underline{\underline{(10P + 90T)}}.$$

For the 10-year period, the **total** earned is the same for both salary schemes.

(b) Find the value of T. (4)

Solution

$$\begin{aligned}\frac{10}{2}[2(P + 1800) + 9T] = 10P + 90T &\Rightarrow 5[(2P + 3600) + 9T] = 10P + 90T \\ &\Rightarrow 10P + 18000 + 45T = 10P + 90T \\ &\Rightarrow 18000 + 45T = 90T \\ &\Rightarrow 45T = 18000 \\ &\Rightarrow \underline{\underline{T = 400}}.\end{aligned}$$

For this value of T, the salary in Year 10 under Salary Scheme 2 is £29 850.

(c) Find the value of P. (3)

Solution

$$(P + 1800) + 9 \times 400 = 29\,850 \Rightarrow P + 1800 + 3600 = 29\,850 \Rightarrow \underline{\underline{P = 24\,450}}.$$

15. Each year, Abbie pays into a savings scheme. In the first year she pays in £500. Her payments then increase by £200 each year so that she pays £700 in the second year, £900 in the third year, and so on.

(a) Find out how much Abbie pays into the savings scheme in the tenth year. (2)

Solution
$a = 500$, $d = 200$, and $n = 10$:

$$u_{50} = 500 + 200 \times 9 = 500 + 1800 = \underline{\underline{£2300}}.$$

Abbie pays into the scheme or n years until she has paid in a total of £67 200.

(b) Show that $n^2 + 4n - 24 \times 28 = 0$. (5)

Solution
$a = 500$ and $d = 200$:

$$\begin{aligned}\frac{n}{2}[2 \times 500 + 200 \times (n-1)] = 67\,200 &\Rightarrow \frac{n}{2}(1000 + 200n - 200) = 67\,200 \\ &\Rightarrow \frac{n}{2}(200n + 800) = 67\,200 \\ &\Rightarrow 100n^2 + 400n - 67\,200 = 0 \\ &\Rightarrow n^2 + 4n - 672 = 0 \\ &\Rightarrow \underline{\underline{n^2 + 4n - 24 \times 28 = 0}}.\end{aligned}$$

(c) Hence find the number of years that Abbie pays into the savings scheme. (2)

Solution

$$n^2 + 4n - 24 \times 28 = 0 \Rightarrow (n+28)(n-24) = 0 \Rightarrow n = -28 \text{ or } n = 24;$$

hence, $\underline{\underline{n = 24}}$.

16. In the year 2000 a shop sold 150 computers. Each year the shop sold 10 more computers than the year before, so that the shop sold 160 computers in 2001, 170 computers in 2002, and so on, forming an arithmetic sequence.

(a) Show that the shop sold 220 computers in 2007. (2)

Solution
$a = 150$, $d = 10$, and $n = 8$:

$$u_8 = 150 + 10 \times 7 = 150 + 70 = \underline{\underline{220 \text{ computers}}}.$$

(b) Calculate the total number of computers the shop sold from 2000 to 2013 inclusive. (3)

Solution

$a = 150$, $d = 10$, and $n = 14$:

$$S_{14} = \frac{14}{2}[2 \times 150 + 13 \times 10] = 7(300 + 130) = 7 \times 430 = \underline{\underline{3010 \text{ computers}}}.$$

In the year 2000, the selling price of each computer was £900. The selling price fell by £20 each year, so that in 2001 the selling price was £880, in 2002 the selling price was £860, and so on, forming an arithmetic sequence.

(c) In a particular year, the selling price of each computer in £s was equal to three times the number of computers the shop sold in that year. By forming and solving an equation, find the year in which this occurred. (4)

Solution

The selling price was $900 - 20(n - 1) = 900 - 20n + 20 = 920 - 20n$. Now,

$$\begin{aligned} 3[150 + 10(n - 1)] = 920 - 20n &\Rightarrow 3(150 + 10n - 10) = 920 - 20n \\ &\Rightarrow 3(140 + 10n) = 920 - 20n \\ &\Rightarrow 420 + 30n = 920 - 20n \\ &\Rightarrow 50n = 500 \\ &\Rightarrow n = 10, \end{aligned}$$

and this was $\underline{\underline{2009}}$.

17. A boy saves some money over a period of 60 weeks. He saves 10p in week 1, 15p in week 2, 20p in week 3, and so on until week 60. His weekly saving form an arithmetic sequence.

(a) Find how much he saves in week 15. (2)

Solution

$a = 10$, $d = 5$, and $n = 15$:

$$10 + 5 \times 14 = 10 + 70 = \underline{\underline{80\text{p}}}.$$

(b) Calculate the total amount he saves over the 60 week period. (3)

Solution

$a = 10$, $d = 5$, and $n = 60$:

$$\frac{60}{2}[2 \times 10 + 5 \times 59] = 30(20 + 295) = 30 \times 315 = \underline{\underline{9450\text{p or } £94.50}}.$$

The boy's sister also saves some money each week over a period of m weeks. He saves 10p in week 1, 20p in week 2, 30p in week 3, and so on so that her weekly saving form an arithmetic sequence. She saves a total of £63 in the m weeks.

(c) Show that (4)

$$m(m + 1) = 35 \times 36.$$

Solution

$a = 10$ and $d = 10$:

$$\begin{aligned}\frac{m}{2}[2 \times 10 + 10 \times (m - 1)] = 6300 &\Rightarrow \frac{m}{2}(20 + 10m - 10) = 6300\\ &\Rightarrow \frac{m}{2}(10m + 10) = 6300\\ &\Rightarrow 5m^2 + 5m = 6300\\ &\Rightarrow m^2 + m = 1260\\ &\Rightarrow \underline{\underline{m(m + 1) = 35 \times 36}}.\end{aligned}$$

(d) Hence write down the value of m. (1)

Solution

$$m^2 + m - 35 \times 36 = 0 \Rightarrow (m + 36)(m - 35) = 0 \Rightarrow m = -36 \text{ or } m = 35;$$

hence, $\underline{\underline{m = 35}}$.

18. On John's 10th birthday he received the first of an annual birthday gift of money from his uncle. The first gift was £60 and on each subsequent birthday the get was £15 more than the year before. The amounts of these gifts form an arithmetic sequence.

(a) Show that, immediately after his 12th birthday, the total of these gifts was £225. (1)

Solution

£60 + £75 + £90 = $\underline{\underline{£225}}$.

(b) Find the amount that John received from his uncle as a birthday on his 18th birthday. (2)

Solution

$a = 60$, $d = 15$, and $n = 9$:

$$60 + 8 \times 15 = \underline{\underline{£180}}.$$

(c) Find the total of these birthday gifts that John had received from his uncle up to and including his 21st birthday. (3)

Solution

$a = 60$, $d = 15$, and $n = 12$:

$$\tfrac{12}{2}[2 \times 60 + 11 \times 15] = 6 \times 285 = \underline{\underline{£1710}}.$$

When John had received n of these birthday gifts, the total money that he had received from these gifts was £3375.

(d) Show that $n^2 + 7n = 25 \times 18$. (3)

Solution

$$\begin{aligned} & \tfrac{n}{2}[2 \times 60 + 15(n-1)] = 3375 \\ \Rightarrow\quad & 60n + \frac{15n(n-1)}{2} = 3375 \\ \Rightarrow\quad & 120n + 15n(n-1) = 6750 \\ \Rightarrow\quad & 120n + 15n^2 - 15n = 6750 \\ \Rightarrow\quad & 15n^2 + 105n = 6750 \\ \Rightarrow\quad & n^2 + 7n = 450 \\ \Rightarrow\quad & \underline{\underline{n^2 + 7n = 25 \times 18}}, \end{aligned}$$

as required.

(e) Find the value of n, when he had received £3375 in total, and so determine John's age at this time. (2)

Solution

$$n^2 + 7n - 450 = 0 \Rightarrow (n + 25)(n - 18) = 0 \Rightarrow n = -25 \text{ or } n = 18.$$

Hence, $\underline{\underline{n = 18}}$ and John is $\underline{\underline{27}}$.

19. An arithmetic series has first term a and common difference d.

(a) Prove that the sum of the first n terms of this series is (4)

$$\tfrac{1}{2}n[2a + (n - 1)d].$$

Solution

$$\begin{aligned} S_n &= a + (a + d) + \ldots + [a + (n - 1)d] \\ S_n &= [a + (n - 1)d] + \ldots + (a + d) + a \\ \Rightarrow 2S_n &= \underbrace{[2a + (n - 1)d] + [2a + (n - 1)d] + \ldots + [2a + (n - 1)d]}_{n \text{ terms}} \\ \Rightarrow 2S_n &= n[2a + (n - 1)d] \\ \Rightarrow S_n &= \underline{\underline{\tfrac{1}{2}n[2a + (n - 1)d]}}. \end{aligned}$$

Sean repays a loan over a period of n months. His monthly repayments form an arithmetic sequence. He repays £149 in the first month, £147 in the second month, £145 in the third month, and so on. He makes his final repayment in the nth month, where $n > 21$.

(b) Find the amount Sean repays in the 21st month. (2)

Solution

$$£149 - 20 \times £2 = \underline{\underline{£109}}.$$

Over the n months, he repays a total of £5000.

(c) Form an equation in n, and show that your equation may be written as (3)

$$n^2 - 150n + 5000 = 0.$$

Solution

$$\begin{aligned}\tfrac{1}{2}n[2\times 149-2(n-1)]=5000 &\Rightarrow n(149-n+1)=5000\\ &\Rightarrow n(150-n)=5000\\ &\Rightarrow 150n-n^2=5000\\ &\Rightarrow \underline{\underline{n^2-150n+5000=0}}.\end{aligned}$$

(d) Solve the equation in part (c). (3)

Solution

$$n^2-150n+5000=0 \Rightarrow (n-50)(n-100)=0 \Rightarrow \underline{\underline{n=50}} \text{ or } \underline{\underline{n=100}}.$$

(e) State, with a reason, which of the solutions to the equation in part (c) is **not** a sensible solution to the repayment problem. (1)

Solution

Well,

$$£149-49\times £2=£51$$

and

$$£149-99\times £2=-£49.$$

I think that $\underline{\underline{\text{50 payments}}}$ are required.

20. Sue is training for a marathon. Her training includes a run every Saturday starting with a run of 5 km on the first Saturday. Each Saturday she increases the length of her run from the previous Saturday by 2 km.

(a) Show that on the 4th Saturday of the training she runs 11 km. (1)

Solution

$a=5$, $d=2$, and $n=4$:

$$5+2\times 3=5+6=\underline{\underline{11\text{ km}}}.$$

(b) Find an expression, in terms of n, for the length of her training run on the nth Saturday. (2)

Solution

$a = 5$ and $d = 2$,

$$5 + 2(n - 1) = 5 + 2n - 2 = \underline{\underline{(2n + 3)\text{ km}}}.$$

(c) Show that the total distance she runs on Saturdays in n weeks of training is $n(n+4)$ km. (3)

Solution

$$S_n = \frac{n}{2}[2 \times 5 + 2(n - 1)] = \frac{n}{2} \times (2n + 8) = \underline{\underline{n(n + 4)\text{ km}}}.$$

On the nth Saturday Sue runs 43 km.

(d) Find the value of n. (2)

Solution

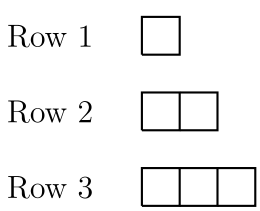
$$5 + 2(n - 1) = 43 \Rightarrow 2n + 3 = 43 \Rightarrow 2n = 40 \Rightarrow \underline{\underline{n = 20\text{ weeks}}}.$$

(e) Find the total distance, in km, Sue runs on Saturdays in n weeks of training. (2)

Solution

$$20 \times (20 + 4) = 20 \times 24 = \underline{\underline{480\text{ km}}}.$$

21. Ann has some sticks that are all of the same length. She arranges them in squares and has made the following 3 rows of patterns:



She notices that 4 sticks are required to make the the single square in the first row, 7 sticks make 2 squares in the second row and in third row she needs 10 sticks to make 3 squares.

(a) Find an expression, in terms of n, for the number of sticks required to make a similar arrangement of n squares in the nth row. (3)

Solution

$$4 + 3(n-1) = 4 + 3n - 3 = \underline{\underline{3n+1}}.$$

Ann continues to make squares following the same pattern. She makes 4 squares in the 4th row and so on until she has completed 10 rows.

(b) Find the total number of sticks Ann uses in making these 10 rows. (3)

Solution

$a = 4$, $d = 3$, and $n = 10$:

$$\frac{10}{2}[2 \times 4 + 9 \times 3] = 5 \times 35 = \underline{\underline{175 \text{ sticks}}}.$$

Ann started with 1750 sticks. Given that Ann continues the pattern to complete k rows but does not have sufficient sticks to complete the $(k+1)$th row,

(c) show that k satisfies $(3k-100)(k+35) < 0$. (4)

Solution

$$\begin{aligned}\frac{k}{2}[8 + 3(k-1)] < 1750 &\Rightarrow k[8 + 3k - 3] < 3500 \\ &\Rightarrow k(3k+5) < 3500 \\ &\Rightarrow 3k^2 + 5k - 3500 < 0 \\ &\Rightarrow \underline{\underline{(3k-100)(k+35) < 0}}.\end{aligned}$$

(d) Find the value of k. (2)

Solution

So $-35 < k < 33\frac{1}{3}$ in which case $\underline{\underline{k = 33}}$.

22. The first term of an arithmetic series is a and the common difference is d. The 18th term of this series is 25 and the 21st term of this series is $32\frac{1}{2}$.

(a) Use this information to write down two equations for a and d. (2)

Solution

$$\underline{\underline{a + 17d = 25}} \text{ and } \underline{\underline{a + 20d = 32\tfrac{1}{2}}}.$$

(b) Show that $a = -17.5$ and find the value of d. (2)

Solution

Subtract:

$$3d = 7\tfrac{1}{2} \Rightarrow \underline{\underline{d = 2\tfrac{1}{2}}}$$

and

$$a = 25 - 17 \times 2\tfrac{1}{2} = 25 - 42\tfrac{1}{2} = \underline{\underline{-17\tfrac{1}{2}}}.$$

The sum of the first n terms of the series is 2750.

(c) Show that n is given by (4)

$$n^2 - 15n = 55 \times 40.$$

Solution

$$\begin{aligned}\frac{n}{2}[2 \times (-17.5) + 2.5(n-1)] = 2750 &\Rightarrow \frac{n}{2}(-35 + 2.5n - 2.5) = 2750 \\ &\Rightarrow \frac{n}{2}(2.5n - 37.5) = 2750 \\ &\Rightarrow 1.25n^2 - 18.75n = 2750 \\ &\Rightarrow n^2 - 15n = 2200 \\ &\Rightarrow \underline{\underline{n^2 - 15n = 55 \times 40}}.\end{aligned}$$

(d) Hence find the value of n. (3)

Solution

$$n^2 - 15n - 2200 = 0 \Rightarrow (n-55)(n+40) = 0 \Rightarrow n = -40 \text{ or } n = 55;$$

on the whole, $\underline{\underline{n = 55}}$.

23. Jill gave money to a charity over a 20-year period, from Year 1 to Year 20 inclusive. She gave £150 inYear 1, £160 in Year 2, £170, and so on, so that the amounts of money she gave each year formed an arithmetic sequence.

(a) Find the amount of money she gave in Year 10. (2)

Solution
$a = 150$ and $d = 10$:
$$150 + 9 \times 10 = \underline{\underline{240}}.$$

(b) Calculate the total amount of money she gave over the 20-year period. (3)

Solution
$a = 150$, $d = 10$ and $n = 20$:
$$\frac{20}{2}[2 \times 150 + 19 \times 10] = 10(300 + 190) = 10 \times 490 = \underline{\underline{£4900}}.$$

Kevin also gave money to the charity over the 20-year period. He gave £A in Year 1 and the amounts of money he gave each year increased, forming an arithmetic sequence with common difference £30. The total amount of money that Kevin gave over the 20-year period was **twice** the total amount of money that Jill gave.

(c) Calculate the value of A. (4)

Solution
Kevin gave £9800:
$$\begin{aligned}\frac{20}{2}[2A + 19 \times 30] = 9800 &\Rightarrow 10(2A + 570) = 9800 \\ &\Rightarrow 2A + 570 = 980 \\ &\Rightarrow 2A = 410 \\ &\Rightarrow \underline{\underline{A = 205}}.\end{aligned}$$

24. On Alice's 11th birthday she started to receive an annual allowance. The first annual allowance was £500 and on each following birthday the allowance was increased by £200.

(a) Show that, immediately after her 12th birthday, the total of the allowances that Alice had received was £1200. (1)

Solution
$$£500 + £700 = \underline{\underline{£1200}}.$$

(b) Find the amount of Alice's annual allowance on her 18th birthday. (2)

Solution
$a = 500$ and $d = 200$:

$$\pounds 500 + 7 \times \pounds 200 = \underline{\underline{\pounds 1900}}.$$

(c) Find the total of the allowances that Alice had received up to and including her 18th birthday. (3)

Solution
$a = 500$, $d = 200$ and $n = 8$:

$$\frac{8}{2}[2 \times 500 + 7 \times 200] = 4 \times 2400 = \underline{\underline{\pounds 9600}}.$$

When the total of the allowances that Alice had received reached £32 000 the allowance stopped.

(d) Find how old Alice was when she received her last allowance. (7)

Solution

$$\begin{aligned}\frac{n}{2}[2 \times 500 + 200(n-1)] > 32\,000 &\Rightarrow 500n + 100n(n-1) > 32\,000 \\ &\Rightarrow 5n + n(n-1) > 320 \\ &\Rightarrow 5n + n^2 - n - 320 > 0 \\ &\Rightarrow n^2 + 4n - 320 > 0 \\ &\Rightarrow (n+20)(n-16) > 0 \\ &\Rightarrow n = -20 \text{ or } n = 16.\end{aligned}$$

Alice's age was $10 + 16 = \underline{\underline{26 \text{ years of age}}}$.

25. A company, which is making 140 bicycles each week, plans to increase its production. The number of bicycles produced is to be increased by d each week, starting from 140 in week 1, to $140 + d$ in week 2, to $140 + 2d$ in week 3 and so on, until the company is producing 206 in week 12.

(a) Find the value of d. (2)

Solution

$$\begin{aligned} S_n = a + (n-1)d &\Rightarrow 206 = 140 + 11d \\ &\Rightarrow 11d = 66 \\ &\Rightarrow \underline{\underline{d = 6}}. \end{aligned}$$

After week 12 the company plans to continue making 206 bicycles each week.

(b) Find the total number of bicycles that would be made in the first 52 weeks starting from and including week 1. (5)

Solution

$$\begin{aligned} \text{Sum} &= \tfrac{1}{2} \times 12 \times (2 \times 140 + 11 \times 6) + 40 \times 206 \\ &= 2076 + 8240 \\ &= \underline{\underline{10\,1316}}. \end{aligned}$$