Dr Oliver Mathematics
Mathematics
Sequences
Past Examination Questions

This booklet consists of 19 questions across a variety of examination topics. The total number of marks available is 117.

1. The sequence of positive numbers $u_1, u_2, u_3, \ldots$ is given by

$$u_{n+1} = (u_n - 3)^2, \, u_1 = 1.$$

   (a) Find u_2, u_3, and u_4. (3)

Solution

$u_2 = (u_1 - 3)^2 = (1-3)^2 = \underline{\underline{4}}$.

$u_3 = (u_2 - 3)^2 = (4-3)^2 = \underline{\underline{1}}$.

$u_4 = (u_3 - 3)^2 = (1-3)^2 = \underline{\underline{4}}$.

   (b) Write down the value of u_{20}. (1)

Solution

The sequence is oscillating: 1, 4, 1, 4, … and so

$$u_{20} = (u_{19} - 3)^2 = (1-3)^2 = \underline{\underline{4}}.$$

2. A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = 3,$$
$$a_{n+1} = 3a_n - 5, \, n \geqslant 1.$$

   (a) Find the value of a_2 and the value of a_3. (2)

Solution

$a_2 = 3a_1 - 5 = 3 \times 3 - 5 = \underline{\underline{4}}$.

$a_3 = 3a_2 - 5 = 3 \times 4 - 5 = \underline{\underline{7}}$.

   (b) Calculate the value of $\displaystyle\sum_{r=1}^{5} a_r$. (3)

Solution

$a_4 = 3a_3 - 5 = 3 \times 7 - 5 = 16$.
$a_5 = 3a_4 - 5 = 3 \times 16 - 5 = 43$.
Now,

$$\sum_{r=1}^{5} a_r = 3 + 4 + 7 + 16 + 43 = \underline{\underline{73}}.$$

3. A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = k,$$
$$a_{n+1} = 3a_n + 5, \ n \geqslant 1,$$

where k is a positive integer.

(a) Write down an expression for a_2 in terms of k. (1)

Solution

$a_2 = 3a_1 + 5 = \underline{\underline{3k + 5}}$.

(b) Show that $a_3 = 9k + 20$. (2)

Solution

$a_3 = 3a_2 + 5 = 3(3k + 5) + 5 = 9k + 15 + 5 = \underline{\underline{9k + 20}}$.

(c) (i) Find $\sum_{r=1}^{4} a_r$ in terms of k. (4)

Solution

$a_4 = 3a_3 + 5 = 3(9k + 20) + 5 = 27k + 60 + 5 = 27k + 65$.
Now,

$$\sum_{r=1}^{4} a_r = k + (3k + 5) + (9k + 20) + (27k + 65) = \underline{\underline{40k + 90}}.$$

(ii) Show that $\sum_{r=1}^{4} a_r$ is divisible by 10.

Solution

$$\frac{\sum_{r=1}^{4} a_r}{10} = \frac{10(4k+9)}{10} = \underline{\underline{4k+9}}.$$

4. A sequence is given by:

$$x_1 = 1,$$
$$x_{n+1} = x_n(p + x_n),$$

where p is a constant ($p \neq 0$).

(a) Find x_2 in terms of p. (1)

Solution

$x_2 = x_1(p + x_1) = 1(p + 1) = \underline{\underline{p+1}}$.

(b) Show that $x_3 = 1 + 3p + 2p^2$. (2)

Solution

$$\begin{aligned} x_3 &= x_2(p + x_2) \\ &= (p+1)[p + (p+1)] \\ &= (p+1)(2p+1) \\ &= \underline{\underline{1 + 3p + 2p^2}}, \end{aligned}$$

as required.

Given that $x_3 = 1$,

(c) find the value of p, (3)

Solution

$$\begin{aligned} 1 + 3p + 2p^2 = 1 &\Rightarrow 3p + 2p^2 = 0 \\ &\Rightarrow p(3 + 2p) = 0 \\ &\Rightarrow \underline{\underline{p = -\tfrac{3}{2}}}, \end{aligned}$$

as $p \neq 0$.

(d) write down the value of x_{2008}. (2)

Solution

The sequence starts off with 1, $-\frac{1}{2}$, and 1 and we want to know about x_4:

$$x_4 = x_3(-\tfrac{3}{2} + x_3) = 1(-\tfrac{3}{2} + 1) = -\tfrac{1}{2};$$

clearly, $x_{2008} = \underline{\underline{-\frac{1}{2}}}$.

5. A sequence x_1, x_2, x_3, ... is defined by

$$x_1 = 1,$$
$$x_{n+1} = ax_n - 3,\ n \geqslant 1,$$

where a is a constant.

(a) Find an expression for x_2 in terms of a. (1)

Solution

$x_2 = ax_1 - 3 = a \times 1 - 3 = \underline{\underline{a - 3}}$.

(b) Show that $x_3 = a^2 - 3a - 3$. (2)

Solution

$x_3 = ax_2 - 3 = a(a - 3) - 3 = \underline{\underline{a^2 - 3a - 3}}$.

Given that $x_3 = 7$,

(c) find the possible values of a. (3)

Solution

$$\begin{aligned} x_3 = 7 &\Rightarrow a^2 - 3a - 3 = 7 \\ &\Rightarrow a^2 - 3a - 10 = 0 \\ &\Rightarrow (a - 5)(a + 2) = 0 \\ &\Rightarrow \underline{\underline{a = -2 \text{ or } a = 5}}. \end{aligned}$$

6. A sequence a_1, a_2, a_3, ... is defined by

$$a_1 = k,$$
$$a_{n+1} = 2a_n - 7,\ n \geqslant 1,$$

where k is a constant.

(a) Find an expression for a_2 in terms of a. (1)

Solution

$a_2 = 2a_1 - 7 = 2 \times k - 7 = \underline{\underline{2k - 7}}$.

(b) Show that $a_3 = 4k - 21$. (2)

Solution

$a_3 = 2a_2 - 7 = 2(2k - 7) - 7 = 4k - 14 - 7 = \underline{\underline{4k - 21}}$.

Given that $\sum_{r=1}^{4} a_r = 43$,

(c) find the value of k. (4)

Solution

$a_4 = 2a_3 - 7 = 2(4k - 21) - 7 = 8k - 42 - 7 = \underline{\underline{8k - 49}}$.

Now,

$$\begin{aligned}\sum_{r=1}^{4} a_r = 43 &\Rightarrow k + (2k - 7) + (4k - 21) + (8k - 49) = 43 \\ &\Rightarrow 15k - 77 = 43 \\ &\Rightarrow 15k = 120 \\ &\Rightarrow \underline{\underline{k = 8}}.\end{aligned}$$

7. A sequence of positive numbers is defined by

$$\begin{aligned}a_1 &= 2, \\ a_{n+1} &= \sqrt{a_n^2 + 3},\ n \geqslant 1.\end{aligned}$$

(a) Find a_2 and a_3, leaving your answers in surd form. (2)

Solution

$a_2 = \sqrt{a_1^2 + 3} = \sqrt{2^2 + 3} = \underline{\underline{\sqrt{7}}}$.

$a_3 = \sqrt{a_2^2 + 3} = \sqrt{\sqrt{7}^2 + 3} = \underline{\underline{\sqrt{10}}}$.

(b) Show that $a_5 = 4$. (2)

Solution

$a_4 = \sqrt{a_3^2 + 3} = \sqrt{\sqrt{10}^2 + 3} = \sqrt{13}$.

$a_5 = \sqrt{a_4^2 + 3} = \sqrt{\sqrt{13}^2 + 3} = \sqrt{16} = \underline{\underline{4}}$, as required.

8. A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = 2,$$
$$a_{n+1} = 3a_n - c,\ n \geqslant 1,$$

where c is a constant.

(a) Find an expression for a_2 in terms of c. (1)

Solution

$a_2 = 3a_1 - c = 3 \times 2 - c = \underline{\underline{6 - c}}$.

Given that $\sum_{i=1}^{3} a_i = 0$,

(b) find the value of c. (4)

Solution

$a_3 = 3a_2 - c = 3(6 - c) - c = 18 - 3c - c = 18 - 4c$.

Now,

$$\begin{aligned}\sum_{i=1}^{3} a_i = 0 &\Rightarrow 2 + (6 - c) + (18 - 4c) = 0\\ &\Rightarrow 26 = 5c\\ &\Rightarrow \underline{\underline{c = 5\tfrac{1}{5}}}.\end{aligned}$$

9. A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = k,$$
$$a_{n+1} = 5a_n + 3,\ n \geqslant 1,$$

where k is a positive integer.

(a) Write down an expression for a_2 in terms of k. (1)

Solution

$a_2 = 5a_1 + 3 = 5 \times k + 3 = \underline{\underline{5k + 3}}$.

(b) Show that $a_3 = 25k + 18$. (2)

Solution

$a_3 = 5a_2 + 3 = 5(5k + 3) + 3 = 25k + 15 + 3 = \underline{\underline{25k + 18}}$.

(c) (i) Find $\sum_{r=1}^{4} a_r$ in terms of k, in its simplest form. (4)

Solution

$a_4 = 5a_4 + 3 = 5(25k + 18) + 3 = 125k + 90 + 3 = 125k + 93$.

Now,

$$\begin{aligned}\sum_{r=1}^{4} a_r &= k + (5k + 3) + (25k + 18) + (125k + 93)\\ &= \underline{\underline{156k + 114}}.\end{aligned}$$

(ii) Show that $\sum_{r=1}^{4} a_r$ is divisible by 6.

Solution

$$\frac{\sum_{r=1}^{4} a_r}{6} = \frac{6(26k + 19)}{6} = \underline{\underline{26k + 19}}.$$

10. A sequence $x_1, x_2, x_3, \ldots$ is defined by

$$\begin{aligned}x_1 &= 1,\\ x_{n+1} &= ax_n + 5, \; n \geqslant 1,\end{aligned}$$

where a is a constant.

(a) Write down an expression for x_2 in terms of a. (1)

Solution

$x_2 = ax_1 + 5 = a \times 1 + 5 = \underline{\underline{a + 5}}$.

(b) Show that $x_3 = a^2 + 5a + 5$. (2)

Solution

$x_3 = ax_2 + 5 = a(a + 5) + 5 = \underline{\underline{a^2 + 5a + 5}}$.

Given that $x_3 = 41$,

(c) find the possible values of a. (3)

Solution

$$\begin{aligned} x_3 = 41 &\Rightarrow a^2 + 5a + 5 = 41 \\ &\Rightarrow a^2 + 5a - 36 = 0 \\ &\Rightarrow (a + 9)(a - 4) = 0 \\ &\Rightarrow \underline{\underline{a = -9 \text{ or } a = 4}}. \end{aligned}$$

11. A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$\begin{aligned} a_1 &= 3, \\ a_{n+1} &= 2a_n - c, \, n \geqslant 1, \end{aligned}$$

where c is a constant.

(a) Write down an expression, in terms of c, for a_2. (1)

Solution

$a_2 = 2a_1 - c = 2 \times 3 - c = \underline{\underline{6 - c}}$.

(b) Show that $a_3 = 12 - 3c$. (2)

Solution

$a_3 = 2a_2 - c = 2(6 - c) - c = 12 - 2c - c = \underline{\underline{12 - 3c}}$.

Given that $\displaystyle\sum_{r=1}^{4} a_r \geqslant 23$,

(c) find the range of values of c. (4)

Solution

$a_4 = 2a_3 - c = 2(12 - 3c) - c = 24 - 6c - c = 24 - 7c$.

Now,

$$\begin{aligned}\sum_{r=1}^{4} a_r \geqslant 23 &\Rightarrow 3 + (6 - c) + (12 - 3c) + (24 - 7c) \geqslant 23 \\ &\Rightarrow 45 - 11c \geqslant 23 \\ &\Rightarrow -11c \geqslant -22 \\ &\Rightarrow \underline{\underline{c \leqslant 2}}.\end{aligned}$$

12. A sequence u_1, u_2, u_3, ... satisfies

$$u_{n+1} = 2u_n - 1,\ n \geqslant 1.$$

Given that $u_2 = 9$,

(a) find the value of u_3 and the value of u_4, (2)

Solution

$u_3 = 2u_2 - c = 2 \times 9 - 1 = \underline{\underline{17}}$.

$u_4 = 2u_3 - c = 2 \times 17 - 1 = \underline{\underline{33}}$.

(b) evaluate $\sum_{r=1}^{4} u_r$. (3)

Solution

$$\begin{aligned}u_2 = 2u_1 - c &\Rightarrow 9 = 2u_1 - 1 \\ &\Rightarrow 10 = 2u_1 \\ &\Rightarrow u_1 = 5,\end{aligned}$$

and so

$$\sum_{r=1}^{4} u_r = 5 + 9 + 17 + 33 = \underline{\underline{64}}.$$

13. A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = 4,$$
$$a_{n+1} = k(a_n + 2),\ n \geqslant 1,$$

where k is a constant.

(a) Find an expression for a_2 in terms of k. (1)

Solution

$a_2 = k(a_1 + 2) = k(4 + 2) = \underline{\underline{6k}}$.

Given that $\sum_{i=1}^{3} a_i = 2$,

(b) find the two possible values of k. (6)

Solution

$a_3 = k(a_2 + 2) = k(6k + 2) = 6k^2 + 2k$.

Now,

$$\begin{aligned}\sum_{i=1}^{3} a_i = 2 &\Rightarrow 4 + 6k + (6k^2 + 2k) = 2 \\ &\Rightarrow 6k^2 + 8k + 2 = 0 \\ &\Rightarrow 2(3k^2 + 4k + 1) = 0 \\ &\Rightarrow 2(3k + 1)(k + 1) = 0 \\ &\Rightarrow \underline{\underline{k = -1 \text{ or } k = -\tfrac{1}{3}}}.\end{aligned}$$

14. A sequence $x_1, x_2, x_3, \ldots$ is defined by

$$x_1 = 1,$$
$$x_{n+1} = (x_n)^2 - kx_n,\ n \geqslant 1,$$

where k is a constant, $k \neq 0$.

(a) Find an expression for x_2 in terms of k. (1)

Solution

$x_2 = (x_1)^2 - kx_1 = 1^2 - k \times 1 = \underline{\underline{1 - k}}$.

(b) Show that $x_3 = 1 - 3k + 2k^2$. (2)

Solution

$$\begin{aligned} x_3 &= (x_2)^2 - kx_2 \\ &= (1-k)^2 - k(1-k) \\ &= (1 - 2k + k^2) - (k - k^2) \\ &= \underline{\underline{1 - 3k + 2k^2}}. \end{aligned}$$

Given also that $x_3 = 1$,

(c) calculate the value of k. (3)

Solution

$$\begin{aligned} x_3 = 1 &\Rightarrow 1 = 1 - 3k + 2k^2 \\ &\Rightarrow 2k^2 - 3k = 0 \\ &\Rightarrow k(2k-3) = 0 \\ &\Rightarrow \underline{\underline{k = \tfrac{3}{2}}}, \end{aligned}$$

as $k \neq 0$.

(d) Hence find the value of $\sum_{n=1}^{100} x_n$. (3)

Solution

The sequence starts off with $1, -\frac{1}{2}, 1, -\frac{1}{2}, \ldots$ and

$$\sum_{n=1}^{100} x_n = 50 \times 1 + 50 \times (-\tfrac{1}{2}) = \underline{\underline{25}}.$$

15. A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_{n+1} = 5a_n - 3, \; n \geqslant 1.$$

Given that $a_2 = 7$,

(a) find the value of a_1, (2)

Solution

$$\begin{aligned} a_2 = 5a_1 - 3 &\Rightarrow 7 = 5a_1 - 3 \\ &\Rightarrow 5a_1 = 10 \\ &\Rightarrow \underline{\underline{a_1 = 2}}. \end{aligned}$$

(b) Find the value of $\sum_{r=1}^{4} a_r$ (3)

Solution

$a_3 = 5a_2 - 3 = 5 \times 7 - 3 = 32.$
$a_4 = 5a_4 - 3 = 5 \times 32 - 3 = 157.$
Finally,

$$\sum_{r=1}^{4} a_r = 2 + 7 + 32 + 157 = \underline{\underline{198}}.$$

16. A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$\begin{aligned} a_1 &= k, \\ a_{n+1} &= 4a_n - 3, \, n \geqslant 1, \end{aligned}$$

where k is a positive integer.

(a) Write down an expression for a_2 in terms of k. (1)

Solution

$a_2 = 4a_1 - 3 = 4 \times k - 3 = \underline{\underline{4k - 3}}.$

Given that $\sum_{r=1}^{3} a_r = 66$,

(b) find the value of k. (4)

Solution

$a_3 = 4a_2 - 3 = 4(4k - 3) - 3 = 16k - 12 - 3 = 16k - 15.$

Now,

$$\sum_{r=1}^{3} a_r = 66 \Rightarrow k + (4k-3) + (16k-15) = 66$$
$$\Rightarrow 21k - 18 = 66$$
$$\Rightarrow 21k = 84$$
$$\Rightarrow \underline{\underline{k = 4}}.$$

17. (a) A sequence U_1, U_2, U_3, ... is defined by

$$U_1 = 4,$$
$$U_2 = 4,$$
$$U_{n+2} = 2U_{n+1} - U_n,\ n \geqslant 1.$$

Find the value of

(i) U_3, (1)

Solution

$U_3 = 2U_2 - U_1 = 2 \times 4 - 4 = \underline{\underline{4}}$.

(ii) $\sum_{n=1}^{20} U_n$. (2)

Solution

$$\sum_{n=1}^{20} U_n = 20 \times 4 = \underline{\underline{80}}.$$

(b) A sequence V_1, V_2, V_3, ... is defined by

$$V_1 = k,$$
$$V_2 = 2k,$$
$$V_{n+2} = 2V_{n+1} - V_n,\ n \geqslant 1,$$

where k is a constant.

(i) Find V_3 and V_4 in terms of k. (2)

Solution

$V_3 = 2V_2 - V_1 = 2(2k) - k = \underline{\underline{3k}}$.
$V_4 = 2V_3 - V_2 = 2(3k) - 2k = \underline{\underline{4k}}$.

Given that $\sum_{n=1}^{5} V_n = 165$,

(ii) find the value of k. (3)

Solution
$V_5 = 2V_4 - V_3 = 2(4k) - 3k = \underline{\underline{5k}}$.
Finally,

$$\begin{aligned}\sum_{n=1}^{5} V_n = 165 &\Rightarrow k + 2k + 3k + 4k + 5k = 165 \\ &\Rightarrow 15k = 165 \\ &\Rightarrow \underline{\underline{k = 11}}.\end{aligned}$$

18. A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$\begin{aligned} a_1 &= 4, \\ a_{n+1} &= 5 - ka_n,\ n \geqslant 1, \end{aligned}$$

where k is a constant.

(a) Write down expressions for a_2 and a_3 in terms of k. (2)

Solution
$a_2 = 5 - ka_2 = \underline{\underline{5 - 4k}}$.
$a_3 = 5 - ka_2 = 5 - k(5 - 4k) = \underline{\underline{5 - 5k + 4k^2}}$.

Find

(b) $\sum_{r=1}^{3}(1 + a_r)$ in terms of k, giving your answer in its simplest form, (3)

Solution

$$\sum_{r=1}^{3}(1+a_r) = (1+4)+(1+5-4k)+(1+5-5k+4k^2)$$
$$= \underline{\underline{4k^2-9k+17}}.$$

(c) $\sum_{r=1}^{100}(a_{r+1}+ka_r)$. (1)

Solution

$$\sum_{r=1}^{100}(a_{r+1}+ka_r) = \sum_{r=1}^{100} 5 = \underline{\underline{500}}.$$

19. A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = 1,$$
$$a_{n+1} = \frac{k(a_n+1)}{a_n}, \; n \geqslant 1,$$

where k is a positive constant.

(a) Write down expressions for a_2 and a_3 in terms of k, giving your answers in their simplest form. (3)

Solution

$a_2 = \dfrac{k(a_1+1)}{a_1} = \dfrac{k(1+1)}{1} = \underline{\underline{2k}}.$

$a_3 = \dfrac{k(a_2+1)}{a_2} = \underline{\underline{\dfrac{k(2k+1)}{2k}}}.$

Given that $\sum_{r=1}^{3} a_r = 10$,

(b) find an exact value for k. (3)

Solution

$$\begin{aligned}\sum_{r=1}^{3} a_r = 10 &\Rightarrow 1 + 2k + \frac{k(2k+1)}{2k} = 10\\ &\Rightarrow 2k + 4k^2 + (2k^2 + k) = 20k\\ &\Rightarrow 6k^2 - 17k = 0\\ &\Rightarrow k(6k - 17) = 0\\ &\Rightarrow \underline{\underline{k = \tfrac{17}{6}}},\end{aligned}$$

as k is a positive constant.