Dr Oliver Mathematics
If And Only If

In this note, we will examine "if and only if".

What it means: **either** both statements are true **or** both are false. The result is that the truth of either one of the connected statements requires the truth of the other. We can denote it as follows:

(i) $P \Leftrightarrow Q$,

(ii) P is necessary and sufficient for Q,

(iii) P is equivalent to Q.

Essentially, the term "if and only if" is really a code word for equivalence. To prove a theorem of this form, you must prove that A and B are equivalent; that is, not only is B true whenever A is true, but A is true whenever B is true.

Example 1
Let $a \in \mathbb{Z}$. Show that a is odd if and only if $3a + 8$ is odd.

Solution
<u>a odd $\Rightarrow 3a + 8$ odd</u>:
Now, a is odd so $a = 2n + 1$ for some integer $n \in \mathbb{Z}$. Next,

$$\begin{aligned} 3a + 8 &= 3(2n + 1) + 8 \\ &= 6n + 11 \\ &= 2(3n + 5) + 1, \end{aligned}$$

which means that $3a + 8$ is odd.

<u>$3a + 8$ odd $\Rightarrow a$ odd</u>:
Now, we want to use the contrapositive:

$$a \text{ even} \Rightarrow 3a + 8 \text{ even.}$$

So a is even so $a = 2m$ for some integer $m \in \mathbb{Z}$. Next,

$$\begin{aligned} 3a + 8 &= 3(2m) + 8 \\ &= 6m + 8 \\ &= 2(3m + 4), \end{aligned}$$

which means that $3a + 8$ is even.

Hence, a is odd if and only if $3a + 8$ is odd. ■

Example 2
Let $b \in \mathbb{Z}$. Show that $35|b$ if and only if $5|b$ and $7|b$.

Solution
$35|b \Rightarrow 5|b$ and $7|b$:
Now, $35|b$ and so $b = 35n$ for some integer $n \in \mathbb{Z}$. Next,

$$b = 5(7n) \Rightarrow 5|b$$

and

$$b = 7(5n) \Rightarrow 7|b.$$

Hence, $35|b \Rightarrow 5|b$ and $7|b$.

$5|b, 7|b \Rightarrow 35|b$:
Now, $b = 5k$ and $b = 7l$ for some constants $k, l \in \mathbb{Z}$. Next,

$$\begin{aligned} l &= \frac{b}{7} \\ &= \frac{5k}{7} \end{aligned}$$

and so we have

$$7|k \Rightarrow k = 7m$$

for some constant $m \in \mathbb{Z}$. Finally,

$$\begin{aligned} b &= 5k \\ &= 5(7m) \\ &= 35m \end{aligned}$$

and so $35|b$.

Hence, $35|b$ if and only if $5|b$ and $7|b$. ■

Here are some examples for you to try.

1. Prove that a whole number is divisible by 9 if and only if the sum of the digits is divisible by 9.
2. Suppose $x, y \geqslant 0$. Then $x = y$ if and only if $\dfrac{x+y}{2} = \sqrt{xy}$.
3. Let n be a positive integer. Then n is even if and only if n^2 is even.
4. Let x and y be two natural numbers. Then xy is odd if and only if x is odd and y is odd.