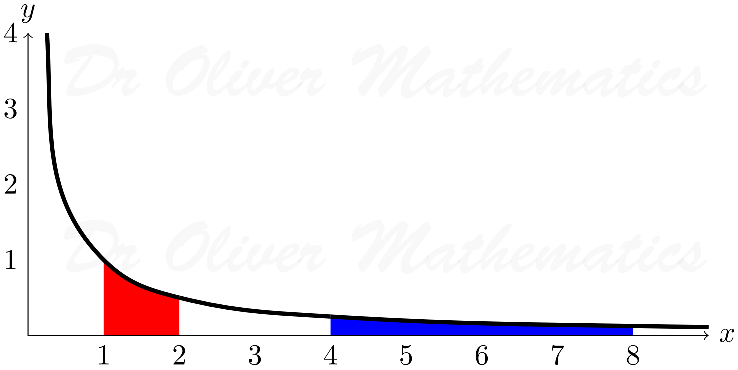
# Dr Oliver Mathematics

An Area Function

9 June 2014

# Two areas under the curve $y = x^{-1}$



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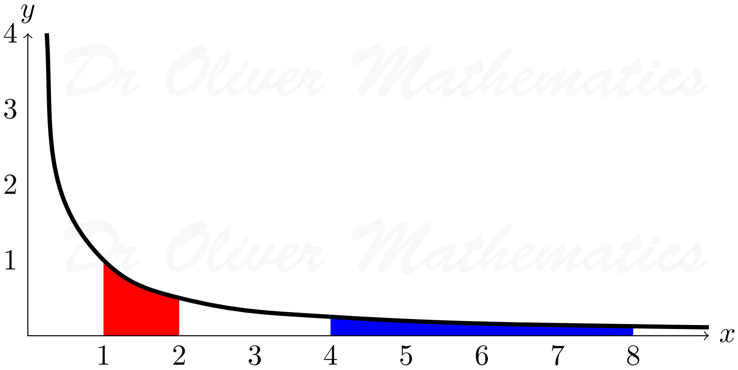


| | |
|---|---|
| A. The red area is biggest | B. The areas are equal |
| C. The blue area is biggest | D. Impossible to calculate |

| $x$ | 1 | 2 |
|---|---|---|
| $y=\frac{1}{x}$ | 1 | $\frac{1}{2}$ |

Hence

$$\int_1^2 \frac{1}{x}\,\mathrm{d}x \approx \frac{1}{2} \times 1 \times \left[1 + \frac{1}{2}\right] = \frac{3}{4}.$$

| $x$ | 4 | 8 |
|---|---|---|
| $y=\frac{1}{x}$ | $\frac{1}{4}$ | $\frac{1}{8}$ |

Hence

$$\int_4^8 \frac{1}{x}\,\mathrm{d}x \approx \frac{1}{2} \times 4 \times \left[\frac{1}{4} + \frac{1}{8}\right] = \frac{3}{4}.$$

| $x$ | $1$ | $1\frac{1}{2}$ | $2$ |
|---|---|---|---|
| $y=\frac{1}{x}$ | $1$ | $\frac{2}{3}$ | $\frac{1}{2}$ |

Hence

$$\int_1^2 \frac{1}{x}\,\mathrm{d}x \approx \frac{1}{2} \times \frac{1}{2} \times \left[1 + 2 \times \frac{2}{3} + \frac{1}{2}\right] = \frac{17}{24}.$$

| $x$ | $4$ | $6$ | $8$ |
|---|---|---|---|
| $y=\frac{1}{x}$ | $\frac{1}{4}$ | $\frac{1}{6}$ | $\frac{1}{8}$ |

Hence

$$\int_4^8 \frac{1}{x}\,\mathrm{d}x \approx \frac{1}{2} \times 2 \times \left[\frac{1}{4} + 2 \times \frac{1}{6} + \frac{1}{8}\right] = \frac{17}{24}.$$

# Trapezium Rule: 3 Strips

| $x$ | $4$ | $1\frac{1}{3}$ | $1\frac{2}{3}$ | $2$ |
|---|---|---|---|---|
| $y = \frac{1}{x}$ | $1$ | $\frac{3}{4}$ | $\frac{3}{5}$ | $\frac{1}{2}$ |

Hence

$$\int_1^2 \frac{1}{x}\,\mathrm{d}x \approx \frac{1}{2} \times \frac{1}{3} \times \left[1 + 2\left(\frac{3}{4} + \frac{3}{5}\right) + \frac{1}{2}\right] = \frac{7}{10}.$$

| $x$ | $4$ | $5\frac{1}{3}$ | $6\frac{2}{3}$ | $8$ |
|---|---|---|---|---|
| $y = \frac{1}{x}$ | $\frac{1}{4}$ | $\frac{3}{16}$ | $\frac{3}{20}$ | $\frac{1}{8}$ |

Hence

$$\int_4^8 \frac{1}{x}\,\mathrm{d}x \approx \frac{1}{2} \times \frac{4}{3} \times \left[\frac{1}{4} + 2\left(\frac{3}{16} + \frac{3}{20}\right) + \frac{1}{8}\right] = \frac{7}{10}.$$

| $x$ | $1$ | $1\frac{1}{4}$ | $1\frac{1}{2}$ | $1\frac{3}{4}$ | $2$ |
|---|---|---|---|---|---|
| $y = \frac{1}{x}$ | $1$ | $\frac{4}{5}$ | $\frac{2}{3}$ | $\frac{4}{7}$ | $\frac{1}{2}$ |

Hence

$$\int_1^2 \frac{1}{x}\,\mathrm{d}x \approx \frac{1}{2} \times \frac{1}{4} \times \left[1 + 2\left(\frac{4}{5} + \frac{2}{3} + \frac{4}{7}\right) + \frac{1}{2}\right] = \frac{1171}{1680}.$$

| $x$ | $4$ | $5$ | $6$ | $7$ | $8$ |
|---|---|---|---|---|---|
| $y = \frac{1}{x}$ | $\frac{1}{4}$ | $\frac{1}{5}$ | $\frac{1}{6}$ | $\frac{1}{7}$ | $\frac{1}{8}$ |

Hence

$$\int_4^8 \frac{1}{x}\,\mathrm{d}x \approx \frac{1}{2} \times 1 \times \left[\frac{1}{4} + 2\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \frac{1}{8}\right] = \frac{1171}{1680}.$$

# The Areas are Equal

Hence

$$\int_1^2 \frac{1}{x}\,\mathrm{d}x = \int_4^8 \frac{1}{x}\,\mathrm{d}x$$

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$$\int_9^{63} \frac{1}{x}\,\mathrm{d}x =$$

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Hence

$$\begin{aligned}
\int_1^2 \frac{1}{x}\,\mathrm{d}x &= \int_4^8 \frac{1}{x}\,\mathrm{d}x \\
\int_5^{10} \frac{1}{x}\,\mathrm{d}x &= \int_1^2 \frac{1}{x}\,\mathrm{d}x \\
\int_7^{21} \frac{1}{x}\,\mathrm{d}x &= \int_1^3 \frac{1}{x}\,\mathrm{d}x \\
\int_9^{63} \frac{1}{x}\,\mathrm{d}x &= \int_1^7 \frac{1}{x}\,\mathrm{d}x
\end{aligned}$$

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$$\int_{23}^{92} \frac{1}{x}\,\mathrm{d}x =$$

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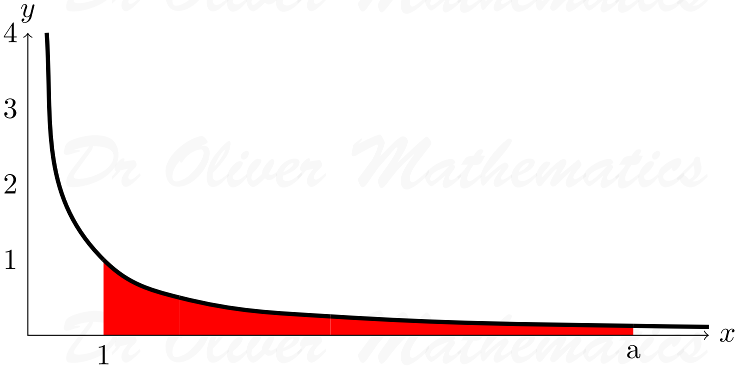
$$\int_{23}^{92} \frac{1}{x}\,\mathrm{d}x = \int_1^4 \frac{1}{x}\,\mathrm{d}x$$

Let

$$F(a) = \int_1^a \frac{1}{x}\,dx,\ a > 0.$$




Then the value of $F(a)$ is simply the area that is shaded red in the picture above.

$$\begin{aligned}
\mathrm{F}(2 \times 3) &= \mathrm{F}(6) \\
&= \int_1^6 \frac{1}{x}\,\mathrm{d}x \\
&= \int_1^2 \frac{1}{x}\,\mathrm{d}x + \int_2^6 \frac{1}{x}\,\mathrm{d}x \\
&= \int_1^2 \frac{1}{x}\,\mathrm{d}x + \int_1^3 \frac{1}{x}\,\mathrm{d}x \\
&= \mathrm{F}(2) + \mathrm{F}(3).
\end{aligned}$$

$$
\begin{aligned}
\mathrm{F}(24) &= \int_1^{24} \frac{1}{x}\,\mathrm{d}x \\
&= \int_1^{8} \frac{1}{x}\,\mathrm{d}x + \int_8^{24} \frac{1}{x}\,\mathrm{d}x && = \int_1^{4} \frac{1}{x}\,\mathrm{d}x + \int_4^{24} \frac{1}{x}\,\mathrm{d}x \\
&= \int_1^{8} \frac{1}{x}\,\mathrm{d}x + \int_1^{3} \frac{1}{x}\,\mathrm{d}x && = \int_1^{4} \frac{1}{x}\,\mathrm{d}x + \int_1^{6} \frac{1}{x}\,\mathrm{d}x \\
&= \mathrm{F}(3) + \mathrm{F}(8) && = \mathrm{F}(4) + \mathrm{F}(6)
\end{aligned}
$$

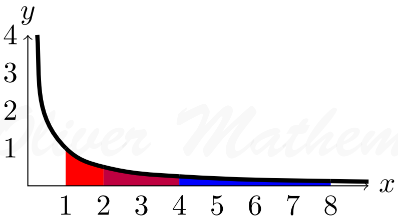
Hence

$$\mathrm{F}(24) = \mathrm{F}(2) + \mathrm{F}(12) = \mathrm{F}(3) + \mathrm{F}(8) = \mathrm{F}(4) + \mathrm{F}(6) = \ldots$$

What functions do we know that have the property

$$\mathrm{F}(ab) = \mathrm{F}(a) + \mathrm{F}(b)?$$

# What about powers?




$$\begin{aligned}\int_1^8 \frac{1}{x}\,\mathrm{d}x &= \int_1^2 \frac{1}{x}\,\mathrm{d}x + \int_2^4 \frac{1}{x}\,\mathrm{d}x + \int_4^8 \frac{1}{x}\,\mathrm{d}x \\ &= \int_1^2 \frac{1}{x}\,\mathrm{d}x + \int_1^2 \frac{1}{x}\,\mathrm{d}x + \int_1^2 \frac{1}{x}\,\mathrm{d}x \\ &= 3\int_1^2 \frac{1}{x}\,\mathrm{d}x\end{aligned}$$

and hence

$$\mathrm{F}(2^3) = \mathrm{F}(8) = 3\,\mathrm{F}(2).$$

I have £1 to invest and there are some exciting opportunities available on the high street this week.

| Bank | Interest Rate | Interest Paid | Total |
|---|---|---|---|
| 1 | 100% | Once a year | |
| | | | |
| | | | |
| | | | |

I have £1 to invest and there are some exciting opportunities available on the high street this week.

| Bank | Interest Rate | Interest Paid | Total |
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| 1 | 100% | Once a year | £2 |
| | | | |
| | | | |
| | | | |

# Some investment opportunities

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| 2 | 50% | Twice a year | |
| | | | |
| | | | |

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| | | | |
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If I go to bank after bank and get similar deals, is there a limit to how much I could get? Is there a bank where I could end up with £3? Or £100? Or £1000? If there is a limit, what do you think that limit is?

# The limit is $e = 2.718\,281\,828\ldots$

$$\left(1+\frac{1}{1}\right)^{1} = 2$$

$$\left(1+\frac{1}{2}\right)^{2} = 2.25$$

$$\left(1+\frac{1}{4}\right)^{4} = 2.441\,406\,25 \text{ (FCD)}$$

$$\left(1+\frac{1}{10}\right)^{10} = 2.593\,742\,46 \text{ (FCD)}$$

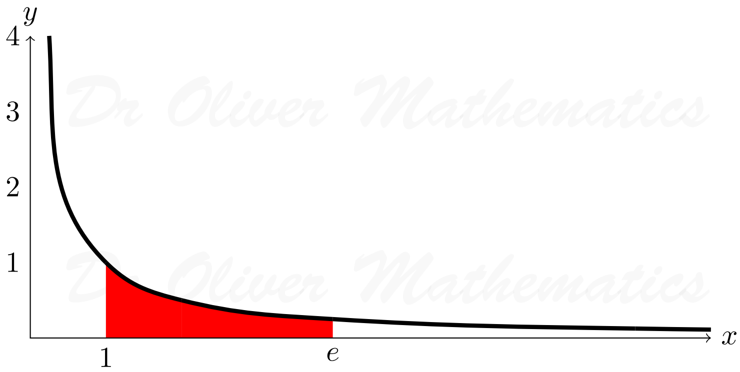
$$\left(1+\frac{1}{100}\right)^{100} = 2.704\,813\,829 \text{ (FCD)}$$

$$\vdots$$

$$\left(1+\frac{1}{n}\right)^{n} \to 2.718\,281\,828 \text{ (FCD) as } n \to \infty$$

This number $e$ is the value for which

$$\mathrm{F}(e) = \int_1^e \frac{1}{x}\,\mathrm{d}x = 1.$$



# So the area function is the logarithm to the base $e$

Recall the definition of logarithms:

$$a = b^c \Leftrightarrow \log_b a = c.$$

Now, for any $a > 0$, there exists some real number $c$ such that $a = e^c$. Then

$$\mathrm{F}(a) = \mathrm{F}(e^c) = c\,\mathrm{F}(e) = c \times 1 = c.$$

But

$$\log_e a = c$$

and so

$$\mathrm{F}(a) = \log_e a.$$

This logarithm has its own notation:

$$\ln x = \log_e x.$$

Although there are infinitely many exponential functions,

$$y = e^x$$

is the exponential function and although there are infinitely many logarithms,

$$y = \ln x$$

is the natural logarithm.

# And from here you can . . .

1. Since
$$\ln x = \int_1^x \frac{1}{t}\,\mathrm{d}t,\ x > 0$$
you can show that
$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x}.$$

2. By considering the integral
$$\int_{-3}^{-2} \frac{1}{x}\,\mathrm{d}x$$
you can show that
$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + c.$$

3. Or you can show that
$$\frac{\mathrm{d}}{\mathrm{d}x}(e^x) = e^x.$$

# And from here you can . . .

4. By using integration by substitution you can rigorously prove that

$$\int_a^{ab} \frac{1}{x}\,\mathrm{d}x = \int_1^a \frac{1}{x}\,\mathrm{d}x \text{ and } \int_1^{a^n} \frac{1}{x}\,\mathrm{d}x = n\int_1^a \frac{1}{x}\,\mathrm{d}x.$$

5. Or you can tell them something about the history and properties of $e$.