Dr Oliver Mathematics
OCR FMSQ Additional Mathematics
2011 Paper
2 hours

The total number of marks available is 100.
You must write down all the stages in your working.
You are permitted to use a scientific or graphical calculator in this paper.
Final answers should be given correct to three significant figures where appropriate.

Section A

1. Determine whether the point $(5, 2)$ lies inside or outside the circle whose equation is (3)

$$x^2 + y^2 = 30.$$

You must show your working.

Solution

$$\begin{aligned} \text{Length} &= \sqrt{5^2 + 2^2} \\ &= \sqrt{25 + 4} \\ &= \sqrt{29} \end{aligned}$$

and

$$\text{radius} = \sqrt{30};$$

hence, it lies inside the circle.

2. The equation of a curve is (5)

$$y = x^3 - x^2 - 2x - 3.$$

Find the equation of the tangent to this curve at the point $(3, 9)$.

Solution

$$y = x^3 - x^2 - 2x - 3 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 2x - 2.$$

Now,

$$x = 3 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 19$$

and the equation of the tangent is

$$\begin{aligned} y - 9 = 19(x-3) &\Rightarrow y - 9 = 19x - 57 \\ &\Rightarrow \underline{\underline{y = 19x - 46}}. \end{aligned}$$

3. In the triangle PQR, $PQ = 8$ cm, $RQ = 9$ cm, and $RP = 7$ cm.

   (a) Find the size of the largest angle. (4)

Solution

It is the angle QPR (why?):

$$\begin{aligned} \cos QPR = \frac{7^2 + 8^2 - 9^2}{2 \times 7 \times 8} &\Rightarrow \cos QPR = \tfrac{2}{7} \\ &\Rightarrow \angle QPR = 73.398\,450\,4 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{\angle QPR = 73.4^\circ \text{ (3 sf)}}}. \end{aligned}$$

   (b) Calculate the area of the triangle. (3)

Solution

We use Hero(n)'s Method:

$$s = \frac{7 + 8 + 9}{2} = 12$$

and

$$\begin{aligned} \text{area} &= \sqrt{12(12-7)(12-8)(12-9)} \\ &= \sqrt{12(5)(4)(3)} \\ &= \underline{\underline{12\sqrt{5} \text{ or } 26.8 \text{ cm}^2 \text{ (3 sf)}}}. \end{aligned}$$

4. Solve the equation (5)

$$5 \sin 2x = 2 \cos 2x$$

in the interval $0° \leqslant x \leqslant 360°$.
Give your answers correct to 1 decimal place.

Solution

$$\begin{aligned} & 5\sin 2x = 2\cos 2x \\ \Rightarrow\ & \tan 2x = \tfrac{2}{5} \\ \Rightarrow\ & 2x = 21.801\,409\,49,\ 201.801\,409\,49,\ 381.801\,409\,49,\ 561.801\,409\,49 \text{ (FCD)} \\ \Rightarrow\ & x = 10.900\,704\,74,\ 100.900\,704\,74,\ 190.900\,704\,74,\ 280.900\,704\,74 \text{ (FCD)} \\ \Rightarrow\ & \underline{\underline{x = 10.9,\ 100.9,\ 190.9,\ 280.9 \text{ (1 dp)}}}. \end{aligned}$$

5. The coordinates of the points A, B, and C are $(-2, 1)$, $(5, 2)$, and $(4, 9)$ respectively.

(a) Find the coordinates of the midpoint, M, of the line AC. (1)

Solution

$$\left(\frac{-2+4}{2}, \frac{1+9}{2}\right) = \underline{\underline{(1, 5)}}.$$

(b) Show that BM is perpendicular to AC. (3)

Solution

$$\begin{aligned} m_{BM} &= \frac{5-2}{1-5} \\ &= -\tfrac{3}{4} \end{aligned}$$

and

$$\begin{aligned} m_{AC} &= \frac{9-1}{4-(-2)} \\ &= \tfrac{4}{3}; \end{aligned}$$

hence,

$$m_{BM} \times m_{AC} = -1$$

which means BM is <u>perpendicular</u> to AC.

(c) (i) Use the result of part (b) to state the mathematical name of the triangle ABC. (1)

Solution
Triangle ABC is an $\underline{\underline{\text{isosceles triangle}}}$.

(ii) Prove this by another method. (2)

Solution

$$\begin{aligned} AB &= \sqrt{[5-(-2)]^2 + (2-1)^2} \\ &= \sqrt{49+1} \\ &= \sqrt{50} \end{aligned}$$

and

$$\begin{aligned} BC &= \sqrt{[(4-5)^2 + (9-2)^2} \\ &= \sqrt{1+49} \\ &= \sqrt{50}; \end{aligned}$$

hence, $\underline{\underline{AB = BC}}$.

6. Solve the inequality (4)

$$x^2 - 12x + 35 \leqslant 0.$$

Solution

$$\left.\begin{array}{ll} \text{add to:} & -12 \\ \text{multiply to:} & +35 \end{array}\right\} -7, -5$$

$$\begin{aligned} x^2 - 12x + 35 \leqslant 0 &\Rightarrow (x-5)(x-7) \leqslant 0 \\ &\Rightarrow \underline{\underline{5 \leqslant x < \leqslant 7}}. \end{aligned}$$

7. (a) Determine whether or not each of the following is a factor of the expression

$$x^3 - 7x + 6.$$

You must show your working.

(i) $(x-2)$, (2)

Solution
We use synthetic division:

$$\begin{array}{r|rrrr} 2 & 1 & 0 & -7 & 6 \\ & \downarrow & 2 & 4 & -6 \\ \hline & 1 & 2 & -3 & 0 \end{array}$$

Hence, the remainder is 0 which means that $\underline{\underline{(x-2)}}$ is a factor.

(ii) $(x+1)$. (1)

Solution

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -7 & 6 \\ & \downarrow & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & 12 \end{array}$$

Hence, the remainder is 12 which means that $(x+1)$ is $\underline{\underline{\text{not}}}$ a factor.

(b) (i) Factorise the function (3)

$$\mathrm{f}(x) = x^3 - 7x + 6.$$

Solution

$$x^3 - 7x + 6 = (x-2)(x^2 + 2x - 3)$$

$$\left.\begin{array}{ll} \text{add to:} & +2 \\ \text{multiply to:} & -3 \end{array}\right\} -1,\ +3$$

$$= \underline{\underline{(x-2)(x-1)(x+3)}}.$$

(ii) Solve the equationf$(x) = 0$. (1)

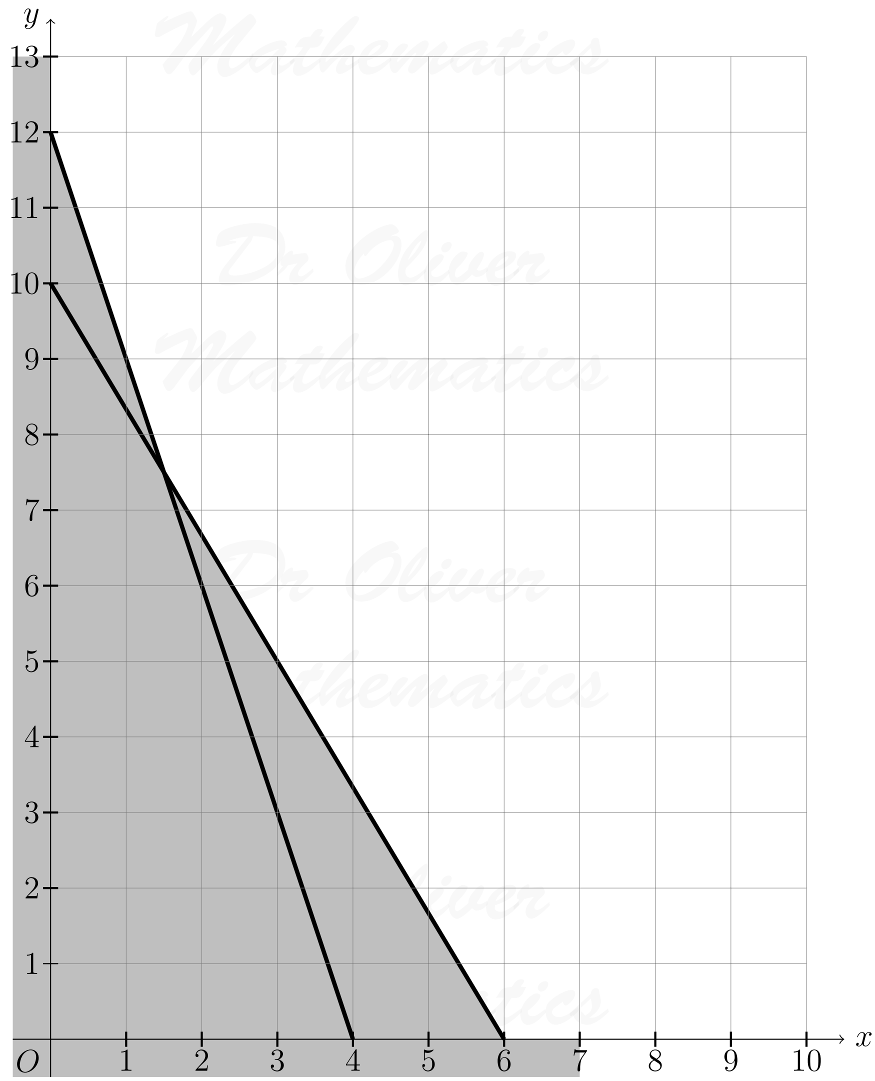
Solution

$$\begin{aligned} f(x) = 0 &\Rightarrow (x-2)(x-1)(x+3) = 0 \\ &\Rightarrow \underline{\underline{x = -3,\ x = 1,\ \text{or } x = 2}}. \end{aligned}$$

8. (a) On the axes given, indicate the region for which the following inequalities hold. You should shade the region which is **not** required. (5)

$$\begin{aligned} 5x + 3y &\geqslant 30 \\ 3x + y &\geqslant 12 \\ y &\geqslant 0 \\ x &\geqslant 0. \end{aligned}$$

Solution



(b) Find the minimum value of $6x + y$ subject to these conditions (2)

Solution

Two obvious methods. The first is to go around the polygon and determine $6x + y$ at its vertices. The second is to go $6x + y =?$ and translate this line out and upwards: $6x + y = 0$, $6x + y = 0.5$, $6x + y = 1$, $\ldots$, until we reach the very last point on its vertices. Hence, it is $x = 0$, $y = 12$ and the minimum is

$$6(0) + (12) = \underline{\underline{12}}.$$

9. The gradient function of a curve is given by (4)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 2x + 4.$$

Find the equation of the curve, given that it passes through the point $(2, 2)$.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 2x + 4 \Rightarrow y = x^3 - x^2 + 4x + c,$$

for some constant c. Now,

$$2 = 8 - 4 + 8 + c \Rightarrow c = -10.$$

Hence,

$$\underline{\underline{y = x^3 - x^2 + 4x - 10}}.$$

10. You are given that (3)

$$\sin\theta = \tfrac{2}{5}$$

with $0^\circ \leqslant \theta \leqslant 90^\circ$.

Using the identity

$$\sin^2\theta + \cos^2\theta = 1,$$

find an exact value for $\cos\theta$.

Solution

$$\begin{aligned}(\tfrac{2}{5})^2 + \cos^2\theta = 1 &\Rightarrow \tfrac{4}{25} + \cos^2\theta = 1\\ &\Rightarrow \cos^2\theta = \tfrac{21}{25}\\ &\Rightarrow \cos\theta = \pm\tfrac{\sqrt{21}}{5};\end{aligned}$$

we know that $0^\circ \leqslant \theta \leqslant 90^\circ$ which means

$$\underline{\underline{\cos\theta = \tfrac{\sqrt{21}}{5}}}.$$

Section B

11. Eggs are delivered to a supermarket in boxes of 6.
For each egg, the probability that it is cracked is 0.05 independently of other eggs.

Find the probability that

(a) in one box there are no cracked eggs, (2)

Solution

$$\begin{aligned} \text{P(no cracked eggs)} &= (0.95)^6 \\ &= 0.735\,091\,890\,6 \text{ (FCD)} \\ &= \underline{\underline{0.735 \text{ (3 sf)}}}. \end{aligned}$$

(b) in one box there is exactly 1 cracked egg. (4)

Solution

$$\begin{aligned} \text{P(1 cracked egg)} &= \binom{6}{1}(0.95)^5(0.05) \\ &= 0.232\,134\,281\,3 \text{ (FCD)} \\ &= \underline{\underline{0.232 \text{ (3 sf)}}}. \end{aligned}$$

The manager checks the eggs as follows.

- He takes a box at random from the delivery.
- He accepts the whole delivery if this box contains no cracked eggs.
- He rejects the whole delivery if the box contains 2 or more cracked eggs.
- If the box contains 1 cracked egg then he chooses another box at random.
- He accepts the delivery only if this second box contains no cracked eggs.

(c) Find the probability that the delivery is rejected. (6)

Solution

First,

$$\begin{aligned} \text{P(2 or more cracked)} &= 1 - \text{P(0 or 1 cracked)} \\ &= 1 - (0.735\ldots + 0.232\ldots) \text{ (FCD)} \\ &= 0.032\,773\,828\,12 \text{ (FCD)}. \end{aligned}$$

Second,

$$\begin{aligned} \text{P(2nd box has any cracked eggs)} &= 1 - \text{P(2nd box all intact)} \\ &= 1 - 0.735\ldots \\ &= 0.264\,908\,109\,4 \text{ (FCD)}. \end{aligned}$$

Finally,

$$\begin{aligned} \text{P(rejected)} &= 0.032\ldots + (0.264\ldots \times 0.232\ldots) \\ &= 0.094\,268\,081\,69 \text{ (FCD)} \\ &= \underline{\underline{0.0943 \text{ (3 sf)}}}. \end{aligned}$$

12. Two cars, A and B, move from rest away from a point O on a straight road starting at the same time.

(a) Car A moves with constant acceleration of 2 ms^{-2}. (2)
Express the displacement of car A after time t seconds as a function of t.

Solution

$$\begin{aligned} a = 2 &\Rightarrow v = 2t + c \\ &\Rightarrow s = t^2 + ct + d. \end{aligned}$$

Now,

$$t = 0, v = 0, s = 0 \Rightarrow c = 0, d = 0$$

and so

$$\underline{\underline{s = t^2}}.$$

(b) Car B moves with acceleration given by (4)

$$a = \tfrac{1}{2}t + 1.$$

Express the displacement of car B after t seconds as a function of t.

Solution

$$a = \tfrac{1}{2}t + 1 \Rightarrow v = \tfrac{1}{4}t^2 + t + c$$
$$\Rightarrow s = \tfrac{1}{12}t^3 + \tfrac{1}{2}t^2 + ct + d.$$

Now,

$$t = 0, v = 0, s = 0 \Rightarrow c = 0, d = 0$$

and so

$$\underline{\underline{s = \tfrac{1}{12}t^3 + \tfrac{1}{2}t^2}}.$$

(c) (i) Find the time at which the cars are the same distance from O. (2)

Solution

$$\tfrac{1}{12}t^3 + \tfrac{1}{2}t^2 = t^2 \Rightarrow \tfrac{1}{12}t^3 - \tfrac{1}{2}t^2 = 0$$
$$\Rightarrow \tfrac{1}{12}t^2(t-6) = 0$$
$$\Rightarrow t = 0 \text{ or } t = 6;$$

hence, when $\underline{\underline{t = 6 \text{ s}}}$.

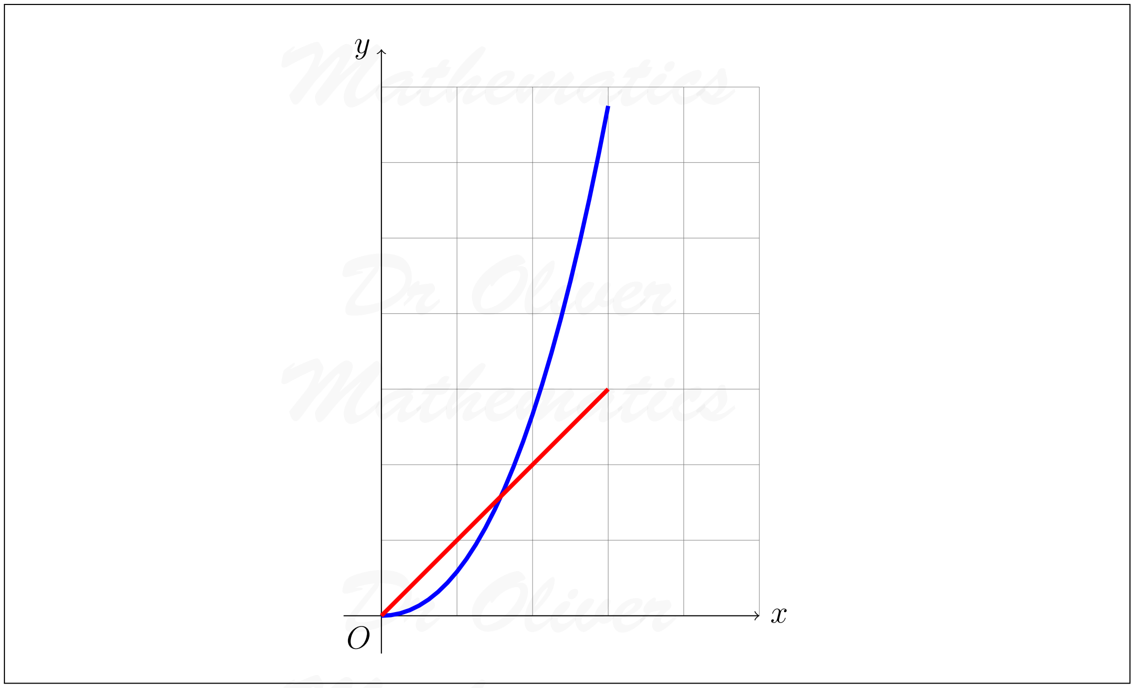
(ii) Find the distance they have travelled at that time. (2)

Solution

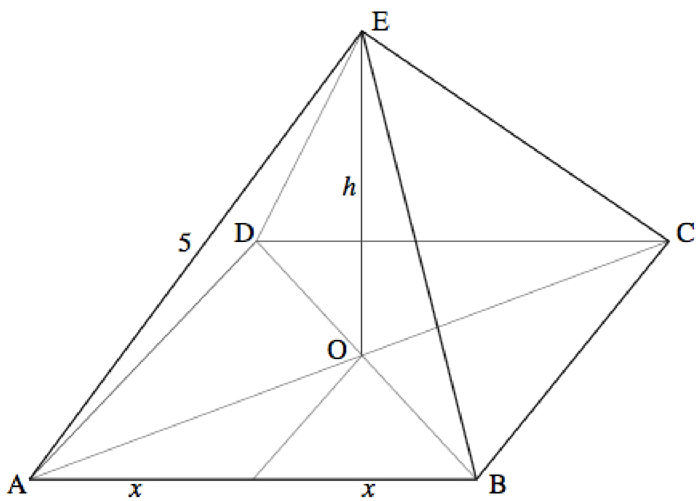
$$s = 6^2 = \underline{\underline{36 \text{ m}}}.$$

(d) Draw a sketch graph of the velocity of each car on the axes given. (2)

Solution



13. A pyramid has a square base, $ABCD$, with vertex E.
E is directly above the centre of the base, O, as shown in the figure.



The lengths of the sides of the base are each $2x$ metres and the height is h metres.
The lengths of the sloping edges, AE, BE, CE, and DE, are each 5 metres.

(a) Show that (2)

$$2x^2 = 25 - h^2.$$

Solution

Pythagoras' on $\triangle AEO$:

$$\begin{aligned} AE^2 = AO^2 + OE^2 &\Rightarrow 5^2 = (x^2 + x^2) + h^2 \\ &\Rightarrow 25 = 2x^2 + h^2 \\ &\Rightarrow \underline{\underline{2x^2 = 25 - h^2}}, \end{aligned}$$

as required.

(b) Show that the volume of the pyramid, V m^3, is given by (2)

$$V = \frac{50h - 2h^3}{3}.$$

Solution

$$\begin{aligned} V &= \tfrac{1}{3}Ah \\ &= \tfrac{1}{3}(2x)^2(h) \\ &= \tfrac{2}{3}(2x^2)(h) \\ &= \tfrac{2}{3}(25 - h^2)(h) \\ &= \underline{\underline{\frac{50h - 2h^3}{3}}}, \end{aligned}$$

as required.

(c) As h varies, find the value of h for which V has a stationary value. (4)

Solution

$$\begin{aligned} V = \tfrac{50}{3}h - \tfrac{2}{3}h^3 &\Rightarrow \frac{\mathrm{d}V}{\mathrm{d}h} = \tfrac{50}{3} - 2h^2 \\ &\Rightarrow \frac{\mathrm{d}^2V}{\mathrm{d}h^2} = -4h. \end{aligned}$$

Now,

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}h} = 0 &\Rightarrow \tfrac{50}{3} - 2h^2 = 0 \\ &\Rightarrow 2h^2 = \tfrac{50}{3} \\ &\Rightarrow h^2 = \tfrac{50}{6} \\ &\Rightarrow \underline{\underline{h = \tfrac{5}{3}\sqrt{3} \text{ or } 2.89 \text{ cm (3 sf)}}}.\end{aligned}$$

(d) Prove that this stationary value is a maximum. (2)

Solution

$$h = \tfrac{5}{3}\sqrt{3} \Rightarrow \frac{\mathrm{d}^2V}{\mathrm{d}h^2} = -\tfrac{20}{3}\sqrt{3} < 0$$

and this is a maximum.

(e) Calculate the angle between the edge AE and the base when h takes this value. (2)

Solution

$$\begin{aligned}h\tfrac{5}{3}\sqrt{3} &\Rightarrow 2x^2 = 16\tfrac{2}{3} \\ &\Rightarrow x^2 = 8\tfrac{1}{3} \\ &\Rightarrow x = \tfrac{5}{3}\sqrt{3} \\ &\Rightarrow \sqrt{2}x = \tfrac{5}{3}\sqrt{6}\end{aligned}$$

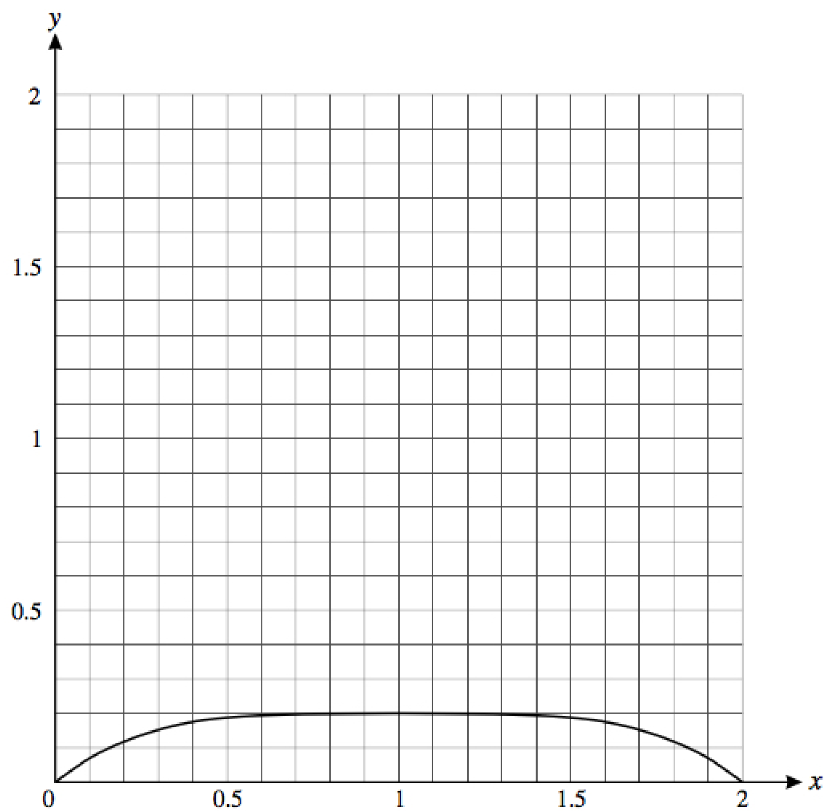
and

$$\begin{aligned}\tan(\text{angle}) = \frac{\tfrac{5}{3}\sqrt{3}}{\tfrac{5}{3}\sqrt{6}} &\Rightarrow \tan(\text{angle}) = \tfrac{1}{2}\sqrt{2} \\ &\Rightarrow \text{angle} = 35.264\,389\,68 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{\text{angle} = 35.3^\circ \text{ (3 sf)}}}.\end{aligned}$$

14. The cross-section of a speed hump is modelled by the region enclosed by the x-axis and the curve

$$y = \frac{1 - (x-1)^4}{5}.$$

The graph is shown below.



Units are metres.

(a) (i) Write down the maximum value of (1)

$$1 - (x - 1)^4.$$

Solution

$$x = 1 \Rightarrow \underline{\underline{\text{maximum value} = 1}}.$$

(ii) Hence write down the maximum height of the speed hump. (1)

Solution

$$x = 1 \Rightarrow \underline{\underline{y = 0.2 \text{ m}}}.$$

(b) Show that (3)

$$y = \tfrac{1}{5}(4x - 6x^2 + 4x^3 - x^4).$$

Solution

$$\begin{aligned} y &= \frac{1-(x-1)^4}{5} \\ &= \tfrac{1}{5}\left[1-(x^4-4x^3+6x^2-4x+1)\right] \\ &= \underline{\underline{\tfrac{1}{5}(4x-6x^2+4x^3-x^4)}}, \end{aligned}$$

as required.

(c) Find the area of the cross-section of the speed hump. (7)

Solution

$$\begin{aligned} \text{Cross-section} &= \int_0^2 \tfrac{1}{5}(4x-6x^2+4x^3-x^4)\,\mathrm{d}x \\ &= \tfrac{1}{5}\left[2x^2-2x^3+x^4-\tfrac{1}{5}x^5\right]_{x=0}^{2} \\ &= \tfrac{1}{5}\left[\left(8-16+16-6\tfrac{2}{5}\right)-(0-0+0-0)\right] \\ &= \underline{\underline{0.32\ \mathrm{m}^2}}. \end{aligned}$$