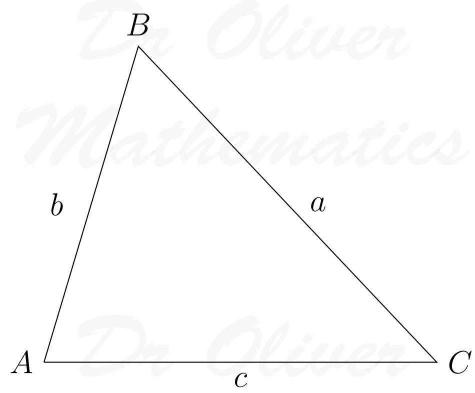
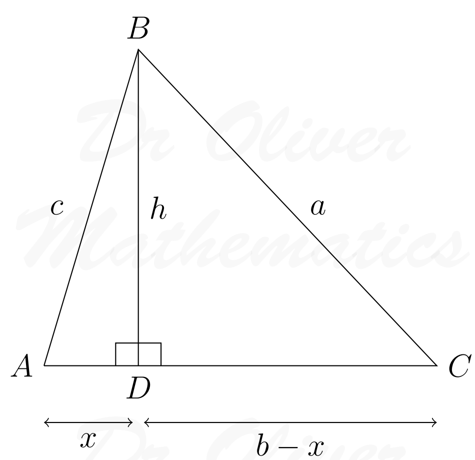
Dr Oliver Mathematics
The Cosine Rule

In this note, we will investigate the cosine rule.

Suppose we have the following triangle.



Split the triangle in two:



Then

$$c^2 = h^2 + x^2$$

and

$$\begin{aligned} a^2 = h^2 + (b - x)^2 &\Rightarrow a^2 = h^2 + (b^2 - 2bx + x^2) \\ &\Rightarrow a^2 = b^2 - 2bx + (h^2 + x^2) \\ &\Rightarrow a^2 = b^2 - 2bx + c^2. \end{aligned}$$

Now,

$$\frac{x}{c} = \cos A^\circ \Rightarrow x = c \cos A^\circ$$

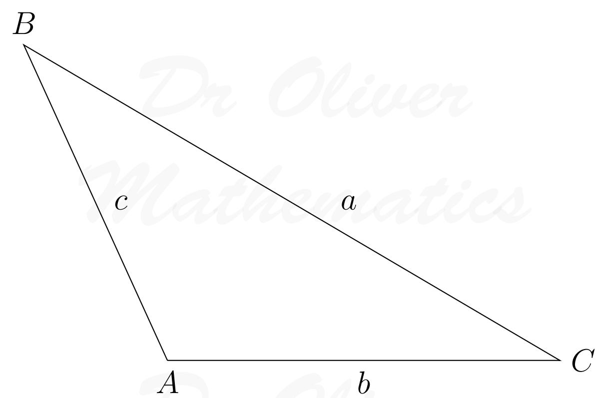
and

$$a^2 = b^2 + c^2 - 2bc \cos A^\circ.$$

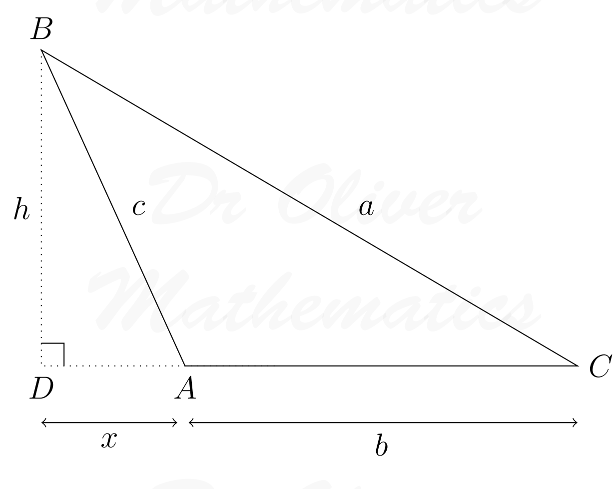
National Curriculum Mathematics Higher GCSE 10A (Bostock, Chandler, Shepherd, and Smith) says, "If were to draw a line from A perpendicular to BC, or from C perpendicular to AB, similar equations could be obtained, i.e.,

$$b^2 = a^2 + c^2 - 2ac \cos B^\circ$$
$$c^2 = a^2 + b^2 - 2ab \cos C^\circ."$$

How do you make work it in the following case:



Well. . .



Then

$$c^2 = h^2 + x^2$$

and

$$\begin{aligned} a^2 = h^2 + (b+x)^2 &\Rightarrow a^2 = h^2 + (b^2 + 2bx + x^2) \\ &\Rightarrow a^2 = b^2 + 2bx + (h^2 + x^2) \\ &\Rightarrow a^2 = b^2 + 2bx + c^2. \end{aligned}$$

Now,

$$\frac{x}{c} = \cos \angle BAD \Rightarrow x = c \cos BAD$$

and

$$a^2 = b^2 + c^2 + 2bc \cos BAD.$$

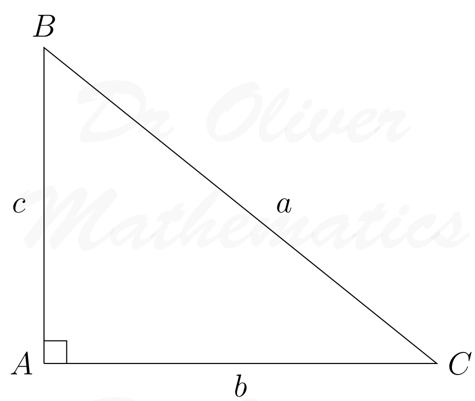
Next,

$$\begin{aligned} \cos \angle BAD &= \cos(180 - BAC)^\circ \\ &= \cos 180^\circ \cos BAC^\circ + \sin 180^\circ \sin BAC^\circ \\ &= -\cos BAC^\circ \end{aligned}$$

and we have

$$a^2 = b^2 + c^2 - 2bc \cos BAC^\circ.$$

And what about right-angled triangles?



Well,

$$a^2 = b^2 + c^2 = b^2 + c^2 - 2bc \cos A^\circ$$

as $\cos 90^\circ = 0$.

So, we have summary of what we have seen:

$$\boxed{\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A^\circ \\ b^2 &= a^2 + c^2 - 2ac \cos B^\circ \\ c^2 &= a^2 + b^2 - 2ab \cos C^\circ. \end{aligned}}$$

What about $\cos A^\circ$? Well,

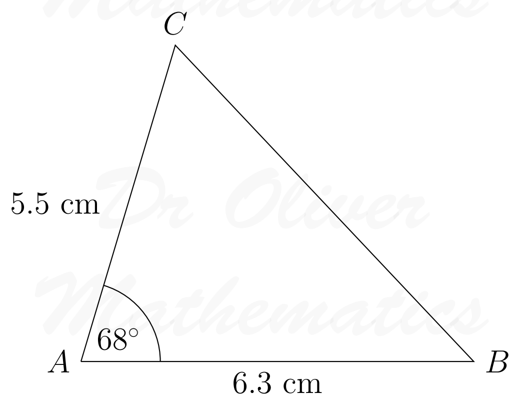
$$\begin{aligned} a^2 = b^2 + c^2 - 2bc\cos A^\circ \Rightarrow 2bc\cos A^\circ &= b^2 + c^2 - a^2 \\ \Rightarrow \cos A^\circ &= \frac{b^2 + c^2 - a^2}{2bc}. \end{aligned}$$

In fact, we have three equations with the cosine of an angle:

$$\boxed{\begin{aligned} \cos A^\circ &= \frac{b^2 + c^2 - a^2}{2bc} \\ \cos B^\circ &= \frac{a^2 + c^2 - b^2}{2ac} \\ \cos C^\circ &= \frac{a^2 + b^2 - ac^2}{2ab}. \end{aligned}}$$

Okay: a few examples. We will give our answers to 3 significant figures. Oh, the diagrams are *not* accurately drawn...

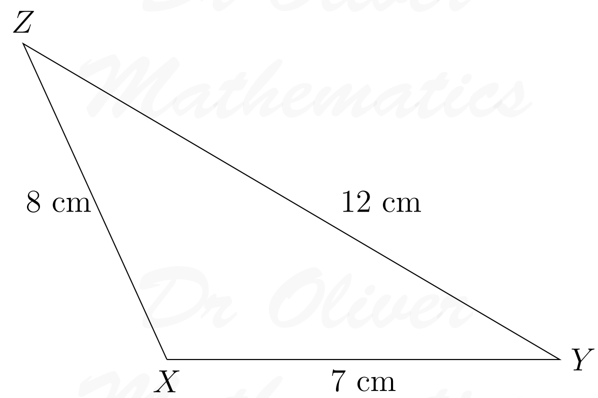
1. In $\triangle ABC$, find BC.



Solution

$$\begin{aligned} & a^2 = 5.5^2 + 6.3^2 - 2 \times 5.5 \times 6.3 \times \cos 68^\circ \\ \Rightarrow\ & a = 6.631\,723\,99 \text{ (FCD)} \\ \Rightarrow\ & \underline{\underline{a = 6.63 \text{ cm (3 sf)}}}. \end{aligned}$$

2. In $\triangle XYZ$, find X°.



Solution

$$\cos X^\circ = \frac{8^2 + 7^2 - 12^2}{2 \times 8 \times 7} \Rightarrow \cos X^\circ = -\tfrac{31}{112}$$
$$\Rightarrow X = 106.068\,459\,4 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{X = 106 \text{ (3 sf)}}}.$$

Note: exactly the same approach works if it is either acute or obtuse angles.