Dr Oliver Mathematics
Mathematics: Advanced Higher
2015 Paper
3 hours

The total number of marks available is 100.
You must write down all the stages in your working.

1. Use the binomial theorem to expand and simplify (4)

$$\left(\frac{x^2}{3} - \frac{2}{x}\right)^5.$$

Solution

$$\begin{aligned}
&\left(\frac{x^2}{3} - \frac{2}{x}\right)^5 \\
= \;& \left(\frac{x^2}{3}\right)^5 + \binom{5}{1}\left(\frac{x^2}{3}\right)^4\left(-\frac{2}{x}\right) + \binom{5}{2}\left(\frac{x^2}{3}\right)^3\left(-\frac{2}{x}\right)^2 + \binom{5}{3}\left(\frac{x^2}{3}\right)^2\left(-\frac{2}{x}\right)^3 \\
& + \binom{5}{4}\left(\frac{x^2}{3}\right)\left(-\frac{2}{x}\right)^4 + \left(-\frac{2}{x}\right)^5 \\
= \;& \underline{\underline{\tfrac{1}{243}x^{10} - \tfrac{10}{81}x^7 + \tfrac{40}{27}x^4 - \tfrac{80}{9}x + \tfrac{80}{3}x^{-2} - 32x^{-5}}}.
\end{aligned}$$

2. (a) For (3)

$$y = \frac{5x+1}{x^2+2},$$

find $\frac{\mathrm{d}y}{\mathrm{d}x}$.
Express your answer as a single, simplified fraction.

Solution

$$\begin{aligned}
u = 5x + 1 &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 5 \\
v = x^2 + 2 &\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 2x.
\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x^2+2)\cdot 5 - (5x+1)\cdot 2x}{(x^2+2)^2} \\ &= \frac{(5x^2+10)-(10x^2+2x)}{(x^2+2)^2} \\ &= \underline{\underline{\frac{10-2x-5x^2}{(x^2+2)^2}}}.\end{aligned}$$

(b) Given (3)

$$f(x) = e^{2x}\sin^2 3x,$$

obtain $f'(x)$.

Solution

$$u = e^{2x} \Rightarrow \frac{du}{dx} = 2e^{2x}$$
$$v = \sin^2 3x \Rightarrow \frac{dv}{dx} = 6\sin 3x\cos 3x.$$

$$\begin{aligned}f'(x) &= e^{2x}\cdot 6\sin 3x\cos 3x + 2e^{2x}\cdot\sin^2 3x \\ &= \underline{\underline{2e^{2x}\sin 3x(3\cos 3x + \sin 3x)}}.\end{aligned}$$

3. The sum of the first twenty terms of an arithmetic sequence is 320. (5)
The twenty-first term is 37.
What is the sum of the first ten terms?

Solution

Well,

$$u_{21} = 37 \Rightarrow a + 20d = 37 \quad (1)$$

and

$$\begin{aligned}S_{20} = 320 &\Rightarrow \tfrac{20}{2}[2a+19d] = 320 \\ &\Rightarrow 2a + 19d = 32 \quad (2).\end{aligned}$$

Now,

$$2a + 40d = 74 \quad (3)$$

and (3) − (2):

$$\begin{aligned} 21d = 42 &\Rightarrow d = 2 \\ &\Rightarrow a + 40 = 37 \\ &\Rightarrow a = -3. \end{aligned}$$

Finally,

$$\begin{aligned} S_{10} &= \tfrac{10}{2}[2a + 9d] \\ &= \underline{\underline{60}}. \end{aligned}$$

4. The equation (4)

$$x^4 + y^4 + 9x - 6y = 14$$

defines a curve passing through the point $A(1, 2)$.

Obtain the equation of the tangent to the curve at A.

Solution

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(x^4 + y^4 + 9x - 6y\right) = \frac{\mathrm{d}}{\mathrm{d}x}(14) &\Rightarrow 4x^3 + 4y^3\frac{\mathrm{d}y}{\mathrm{d}x} + 9 - 6\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}(4y^3 - 6) = -4x^3 - 9 \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-4x^3 - 9}{4y^3 - 6}. \end{aligned}$$

Now,

$$\begin{aligned} x = 1, y = 2 &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-4 - 9}{32 - 6} \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\tfrac{1}{2}. \end{aligned}$$

Finally, the equation of the tangent to the curve is

$$\begin{aligned} y - 2 = -\tfrac{1}{2}(x - 1) &\Rightarrow y - 2 = -\tfrac{1}{2}x + \tfrac{1}{2} \\ &\Rightarrow \underline{\underline{y = -\tfrac{1}{2}x + \tfrac{5}{2}}}. \end{aligned}$$

5. Obtain the value(s) of p for which the matrix (4)

$$\mathbf{A} = \begin{pmatrix} p & 2 & 0 \\ 3 & p & 1 \\ 0 & -1 & -1 \end{pmatrix}$$

is singular.

Solution

$$\begin{aligned} \det \mathbf{A} = 0 &\Rightarrow p(-p+1) - 2(-3-0) + 0 = 0 \\ &\Rightarrow -p^2 + p + 6 = 0 \\ &\Rightarrow p^2 - p - 6 = 0 \end{aligned}$$

$$\left.\begin{matrix} \text{add to:} & -1 \\ \text{multiply to:} & -6 \end{matrix}\right\} -3,\ +2$$

$$\begin{aligned} &\Rightarrow (p-3)(p+2) = 0 \\ &\Rightarrow \underline{\underline{p = -2 \text{ or } p = 3}}. \end{aligned}$$

6. For (3)

$$y = 3^{x^2},$$

obtain $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

$$\begin{aligned} y = 3^{x^2} &\Rightarrow \ln y = \ln 3^{x^2} \\ &\Rightarrow \ln y = x^2 \ln 3 \end{aligned}$$

now take $\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\begin{aligned} &\Rightarrow \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 2x \ln 3 \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2xy \ln 3 \\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = 2x \ln 3 \cdot 3^{x^2}}}. \end{aligned}$$

7. Use the Euclidean algorithm to find integers p and q such that (4)

$$3\,066p + 713q = 1.$$

Solution

$$\begin{aligned} 3\,066 &= 4 \times 713 + 214 \\ 713 &= 3 \times 214 + 71 \\ 214 &= 3 \times 71 + 1 \end{aligned}$$

and

$$\begin{aligned} 1 &= 214 - 3 \times 71 \\ &= 214 - 3(713 - 3 \times 214) \\ &= 10 \times 214 - 3 \times 713 \\ &= 10(3\,066 - 4 \times 713) - 3 \times 713 \\ &= \underline{\underline{10 \times 3\,066 - 43 \times 713}}; \end{aligned}$$

hence, $\underline{\underline{p = 10}}$ and $\underline{\underline{q = -43}}$.

8. Given (3)

$$x = \sqrt{t+1} \text{ and } y = \cot t,\ 0 < t < \pi,$$

obtain $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t.

Solution

$$\begin{aligned}
x = \sqrt{t+1} \Rightarrow x &= (t+1)^{\frac{1}{2}} \\
\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} &= \tfrac{1}{2}(t+1)^{-\frac{1}{2}}, \\
y = \cot t \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} &= -\operatorname{cosec}^2 t, \text{ and} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\
&= \frac{-\operatorname{cosec}^2 t}{\frac{1}{2}(t+1)^{-\frac{1}{2}}} \\
&= \underline{\underline{-2(t+1)^{\frac{1}{2}} \operatorname{cosec}^2 t}}.
\end{aligned}$$

9. Show that (4)

$$\binom{n+2}{3} - \binom{n}{3} = n^2,$$

for all integers, n, where $n \geqslant 3$.

Solution

$$
\begin{aligned}
\binom{n+2}{3} - \binom{n}{3} &= \frac{(n+2)!}{3![(n+2)-3]!} - \frac{n!}{3!(n-3)!} \\
&= \frac{(n+2)!}{3!(n-1)!} - \frac{n!}{3!(n-3)!} \\
&= \frac{n!}{3!(n-3)!}\left[\frac{(n+1)(n+2)}{(n-1)(n-2)} - 1\right] \\
&= \frac{n!}{3!(n-3)!}\left[\frac{n^2+3n+2}{n^2-3n+2} - 1\right] \\
&= \frac{n!}{3!(n-3)!}\left[\frac{(n^2+3n+2)-(n^2-3n+2)}{n^2-3n+2}\right] \\
&= \frac{n!}{3!(n-3)!} \cdot \frac{6n}{(n-1)(n-2)} \\
&= \frac{n!}{(n-3)!} \cdot \frac{n}{(n-1)(n-2)} \\
&= \frac{n!}{(n-1)!} \cdot n \\
&= n \cdot n \\
&= \underline{\underline{n^2}},
\end{aligned}
$$

as required.

10. Obtain the exact value of (5)

$$\int_0^2 x^2\, e^{4x}\, dx.$$

Solution

$$u = x^2 \Rightarrow \frac{du}{dx} = 2x,$$

$$\frac{dv}{dx} = e^{4x} \Rightarrow v = \tfrac{1}{4}e^{4x}.$$

Now,

$$\int_0^2 x^2\, e^{4x}\, dx = \left[\tfrac{1}{4}x^2 e^{4x}\right]_{x=0}^2 - \tfrac{1}{2}\int_0^2 x\, e^{4x}\, dx$$

$$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1,$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{4x} \Rightarrow v = \tfrac{1}{4}\mathrm{e}^{4x}$$

$$\begin{aligned}
&= \left(\mathrm{e}^8 - 0\right) - \tfrac{1}{2}\left(\left[\tfrac{1}{4}x\mathrm{e}^{4x}\right]_{x=0}^{2} - \tfrac{1}{4}\int_0^2 \mathrm{e}^{4x}\,\mathrm{d}x\right)\\
&= \mathrm{e}^8 - \tfrac{1}{2}\left(\tfrac{1}{2}\mathrm{e}^8 - 0\right) + \tfrac{1}{8}\int_0^2 \mathrm{e}^{4x}\,\mathrm{d}x\\
&= \tfrac{3}{4}\mathrm{e}^8 + \tfrac{1}{8}\left[\tfrac{1}{4}\mathrm{e}^{4x}\right]_{x=0}^{2}\\
&= \tfrac{3}{4}\mathrm{e}^8 + \tfrac{1}{8}\left(\tfrac{1}{4}\mathrm{e}^8 - \tfrac{1}{4}\right)\\
&= \tfrac{25}{32}\mathrm{e}^8 - \tfrac{1}{32}\\
&= \underline{\underline{\tfrac{1}{32}(25\mathrm{e}^8 - 1)}}.
\end{aligned}$$

11. (a) Write down the 2×2 matrix, $\mathbf{M}_1$, associated with a reflection in the y-axis. (1)

Solution

$$\underline{\underline{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}}.$$

(b) Write down a second 2×2 matrix, $\mathbf{M}_2$, associated with an anticlockwise rotation through an angle of $\frac{1}{2}\pi$ radians about the origin. (1)

Solution

$$\underline{\underline{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}}.$$

(c) Find the 2×2 matrix, $\mathbf{M}_3$, associated with an anticlockwise rotation through $\frac{1}{2}\pi$ radians about the origin followed by a reflection in the y-axis. (1)

Solution

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}}.$$

(d) What single transformation is associated with $\mathbf{M}_3$? (1)

Solution

A reflection in the line $y = x$.

12. Prove that the difference between the squares of any two consecutive odd numbers is divisible by 8. (3)

Solution

Let the numbers be $(2k + 1)$ and $(2k + 3)$ where $k \in \mathbb{Z}$. Then

$$\begin{aligned}(2k+3)^2 - (2k+1)^2 &= (4k^2 + 12k + 9) - (4k^2 + 4k + 1)\\ &= 8k + 8\\ &= 8(k+1);\end{aligned}$$

hence, two consecutive odd numbers is divisible by 8.

13. By writing z in the form $x + \mathrm{i}y$:

(a) solve the equation (3)

$$z^2 = |z|^2 - 4;$$

Solution

$$\begin{aligned}z^2 = |z|^2 - 4 &\Rightarrow (x + \mathrm{i}y)^2 = |x + \mathrm{i}y|^2 - 4\\ &\Rightarrow (x^2 - y^2) + 2\mathrm{i}xy = (x^2 + y^2) - 4\\ &\Rightarrow -y^2 + 2\mathrm{i}xy = y^2 - 4\\ &\Rightarrow \mathrm{i}(2xy) = 2y^2 - 4.\end{aligned}$$

We have purely imaginary on the left and purely real on the right which means that they are zero. Now,

$$\begin{aligned}2y^2 - 4 = 0 &\Rightarrow y^2 = 2\\ &\Rightarrow y = \pm\sqrt{2}\\ &\Rightarrow z = \pm\sqrt{2}\,\mathrm{i}.\end{aligned}$$

(b) find the solutions to the equation (4)

$$z^2 = \mathrm{i}(|z|^2 - 4).$$

Solution

$$\begin{aligned} z^2 = \mathrm{i}(|z|^2 - 4) &\Rightarrow (x + \mathrm{i}y)^2 = \mathrm{i}(|x + \mathrm{i}y|^2 - 4) \\ &\Rightarrow (x^2 - y^2) + 2\mathrm{i}xy = \mathrm{i}(x^2 + y^2) - 4\mathrm{i} \\ &\Rightarrow x^2 - y^2 = \mathrm{i}(x^2 - 2xy + y^2 - 4). \end{aligned}$$

Again, we have purely imaginary on the left and purely real on the right which means that they are zero. So

$$\begin{aligned} x^2 - y^2 = 0 &\Rightarrow y^2 = x^2 \\ &\Rightarrow y = \pm x \end{aligned}$$

and

$$\begin{aligned} x^2 - 2xy + y^2 - 4 = 0 &\Rightarrow (x - y)^2 = 4 \\ &\Rightarrow x - y = \pm 2 \\ &\Rightarrow y = x \mp 2. \end{aligned}$$

Combining:

$$\underline{\underline{z = 1 - \mathrm{i}}} \text{ and } \underline{\underline{z = -1 + \mathrm{i}}}.$$

14. For some function, f, define

$$\mathrm{g}(x) = \mathrm{f}(x) + \mathrm{f}(-x) \text{ and } \mathrm{h}(x) = \mathrm{f}(x) - f(-x).$$

(a) Show that $\mathrm{g}(x)$ is an even function and that $\mathrm{h}(x)$ is an odd function. (2)

Solution

$$\begin{aligned} \mathrm{g}(-x) &= \mathrm{f}(-x) + \mathrm{f}(-(-x)) \\ &= \mathrm{f}(-x) + \mathrm{f}(x) \\ &= \mathrm{g}(x) \end{aligned}$$

and

$$\begin{aligned}\mathrm{h}(-x) &= \mathrm{f}(-x) - \mathrm{f}(-(-x)) \\ &= \mathrm{f}(-x) - \mathrm{f}(x) \\ &= -(\mathrm{f}(x) - \mathrm{f}(-x)) \\ &= -\mathrm{h}(x);\end{aligned}$$

so, $\mathrm{g}(x)$ is an $\underline{\underline{\text{even function}}}$ and that $\mathrm{h}(x)$ is an $\underline{\underline{\text{odd function}}}$.

(b) Hence show that $\mathrm{f}(x)$ can be expressed as the sum of an even and an odd function. (2)

Solution

$$\begin{aligned}\mathrm{f}(x) &= \tfrac{1}{2}(2\,\mathrm{f}(x)) \\ &= \tfrac{1}{2}\big[(\mathrm{f}(x) + \mathrm{f}(-x)) + (\mathrm{f}(x) - \mathrm{f}(-x))\big] \\ &= \tfrac{1}{2}(\mathrm{f}(x) + \mathrm{f}(-x)) + \tfrac{1}{2}(\mathrm{f}(x) - \mathrm{f}(-x)) \\ &= \tfrac{1}{2}\,\mathrm{g}(x) + \tfrac{1}{2}\,\mathrm{h}(x) \\ &= \underline{\underline{\text{even function} + \text{odd function}}}.\end{aligned}$$

15. A line, L_1, passes through the point $P(2, 4, 1)$ and is parallel to

$$\mathbf{u}_1 = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

and a second line, L_2, passes through $Q(-5, 2, 5)$ and is parallel to

$$\mathbf{u}_2 = -4\mathbf{i} + 4\mathbf{j} + \mathbf{k}.$$

(a) Write down the vector equations for L_1 and L_2. (2)

Solution

For L_1,

$$\underline{\underline{\mathbf{r} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} - \mathbf{k})}},$$

for L_2,

$$\underline{\underline{\mathbf{r} = -5\mathbf{i} + 2\mathbf{j} + 5\mathbf{k} + \mu(-4\mathbf{i} + 4\mathbf{j} + \mathbf{k})}},$$

for some λ and μ.

(b) Show that the lines L_1 and L_2 intersect and find the point of intersection. (4)

Solution

$$\begin{aligned} x: &\quad 2+\lambda = -5-4\mu \quad (1)\\ y: &\quad 4+2\lambda = 2+4\mu \quad (2)\\ z: &\quad 1-\lambda = 5+\mu \quad (3). \end{aligned}$$

Do (1) + (2):

$$\begin{aligned} 6+3\lambda = -3 &\Rightarrow 3\lambda = -9\\ &\Rightarrow \lambda = -3\\ &\Rightarrow 2-3 = -5-4\mu\\ &\Rightarrow 4 = -4\mu\\ &\Rightarrow \mu = -1. \end{aligned}$$

Check in (3):

$$1-\lambda = 4 \text{ and } 5+\mu = 4. \quad \checkmark$$

So they intersect and the point of intersection is

$$2\mathbf{i}+4\mathbf{j}+\mathbf{k}-3(\mathbf{i}+2\mathbf{j}-\mathbf{k}),$$

i.e., $\underline{\underline{(-1,-2,4)}}$.

(c) Determine the equation of the plane containing L_1 and L_2. (4)

Solution

$$\begin{aligned} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2 & -1\\ -4 & 4 & 1 \end{vmatrix} &= (2+4)\mathbf{i}-(1-4)\mathbf{j}+(4+8)\mathbf{k}\\ &= 6\mathbf{i}+3\mathbf{j}+12\mathbf{k}; \end{aligned}$$

so, a normal is $2\mathbf{i}+\mathbf{j}+4\mathbf{k}$. Hence, the equation of the plane is

$$\begin{aligned} 2x+y+4z &= (2\mathbf{i}+\mathbf{j}+4\mathbf{k}).(-\mathbf{i}-2\mathbf{j}+4\mathbf{k})\\ &= -2-2+16\\ &= \underline{\underline{12}}. \end{aligned}$$

16. Solve the second order differential equation (10)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + 10y = 3\mathrm{e}^{2x},$$

given that when $x = 0$, $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

Solution

Complementary function:

$$m^2 + 2m + 10 = 0 \Rightarrow (m+1)^2 = -9 \Rightarrow m = -1 \pm 3\mathrm{i}$$

and hence the complementary function is

$$y = \mathrm{e}^{-x}(A\cos 3x + B\sin 3x).$$

Particular integral: try

$$\begin{aligned} y = C\mathrm{e}^{2x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= 2C\mathrm{e}^{2x} \\ \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 4Ce^{2x}. \end{aligned}$$

Now,

$$\begin{aligned} & \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + 10y = 3\mathrm{e}^{2x} \\ \Rightarrow \quad & 4C\mathrm{e}^{2x} + 4C\mathrm{e}^{2x} + 10C\mathrm{e}^{2x} = 3\mathrm{e}^{2x} \\ \Rightarrow \quad & 18C = 3 \\ \Rightarrow \quad & C = \tfrac{1}{6}. \end{aligned}$$

Hence the particular integral is $y = \frac{3}{25}\mathrm{e}^{3x}$.

The general solution is

$$y = \mathrm{e}^{-x}(A\cos 3x + B\sin 3x) + \tfrac{1}{6}\mathrm{e}^{2x}.$$

Now,

$$\begin{aligned} x = 0, y = 1 \Rightarrow 1 &= A + \tfrac{1}{6} \\ \Rightarrow A &= \tfrac{5}{6}. \end{aligned}$$

Next,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{-x}(-A\sin 3x + 3B\cos 3x) - \mathrm{e}^{-x}(A\cos 3x + B\sin 3x) + \tfrac{1}{3}\mathrm{e}^{2x}$$

and

$$x = 0, \frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow 0 = 3B - \tfrac{5}{6} + \tfrac{1}{3}$$
$$\Rightarrow 3B = \tfrac{1}{2}$$
$$\Rightarrow B = \tfrac{1}{6}.$$

Hence,

$$\underline{\underline{y = \mathrm{e}^{-x}(\tfrac{5}{6}\cos 3x + \tfrac{1}{6}\sin 3x) + \tfrac{1}{6}\mathrm{e}^{2x}}}.$$

17. Find (9)

$$\int \frac{2x^3 - x - 1}{(x-3)(x^2+1)}\,\mathrm{d}x\, x > 3.$$

Solution

$$\begin{aligned}\frac{2x^3 - x - 1}{(x-3)(x^2+1)} &= \frac{2x^3 - x - 1}{x^3 - 3x^2 + x - 3}\\ &= \frac{2(x^3 - 3x^2 + x - 3) + 6x^2 - 3x + 5}{x^3 - 3x^2 + x - 3}\\ &= 2 + \frac{6x^2 - 3x + 5}{(x-3)(x^2+1)}.\end{aligned}$$

Now,

$$\begin{aligned}\frac{6x^2 - 3x + 5}{(x-3)(x^2+1)} &\equiv \frac{A}{x-3} + \frac{Bx + C}{x^2+1}\\ &\equiv \frac{A(x^2+1) + (Bx + C)(x-3)}{(x-3)(x^2+1)}\end{aligned}$$

and this means that

$$6x^2 - 3x + 5 \equiv A(x^2+1) + (Bx + C)(x-3).$$

$\underline{x = 3}$: $50 = 10A \Rightarrow A = 5$.
$\underline{x = 0}$: $5 = A - (-3C) \Rightarrow 3C = 0 \Rightarrow C = 0$.
$\underline{x = 1}$: $8 = 2A - 2B \Rightarrow 2B = 2 \Rightarrow B = 1$.

Hence,

$$\frac{2x^3 - x - 1}{(x-3)(x^2+1)} = \frac{5}{x-3} + \frac{x}{x^2+1}$$

and

$$\int \frac{2x^3 - x - 1}{(x-3)(x^2+1)}\,\mathrm{d}x = \int \left(2 + \frac{5}{x-3} + \frac{x}{x^2+1}\right)\,\mathrm{d}x$$
$$= \underline{\underline{2x + 5\ln|x-3| + \tfrac{1}{2}\ln|x^2+1| + c}}.$$

18. Vegetation can be irrigated by putting a small hole in the bottom of a cylindrical tank, so that the water leaks out slowly. Torricelli's Law states that the rate of change of volume, V, of water in the tank is proportional to the square root of the height, h, of the water above the hole.

This is given by the differential equation

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -k\sqrt{h},\ k > 0.$$

(a) For a cylindrical tank with constant cross-sectional area, A, show that the rate of change of the height of the water in the tank is given by (2)

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{k}{A}\sqrt{h}.$$

Solution

Well,

$$V = Ah \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(Ah)$$
$$\Rightarrow -k\sqrt{h} = A\frac{\mathrm{d}h}{\mathrm{d}t}$$
$$\Rightarrow \underline{\underline{\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{k}{A}\sqrt{h}}},$$

as required.

Initially, when the height of the water is 144 cm, the rate at which the height is changing is -0.3 cm/hr.

(b) By solving the differential equation in part (a), show that (4)

$$h = (12 - \tfrac{1}{80}t)^2.$$

Solution

$$\begin{aligned}\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{k}{A}\sqrt{h} &\Rightarrow -0.3 = -\frac{k}{A}\sqrt{144} \\ &\Rightarrow \frac{k}{A} = \tfrac{1}{40} \\ &\Rightarrow A = 40k.\end{aligned}$$

Now,

$$\begin{aligned}\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{k}{40k}\sqrt{h} &\Rightarrow h^{-\frac{1}{2}}\,\mathrm{d}h = -\tfrac{1}{40}\,\mathrm{d}t \\ &\Rightarrow \int h^{-\frac{1}{2}}\,\mathrm{d}h = -\int \tfrac{1}{40}\,\mathrm{d}t \\ &\Rightarrow 2h^{\frac{1}{2}} = -\tfrac{1}{40}t + c.\end{aligned}$$

Next,

$$t = 0, h = 144 \Rightarrow 2 \cdot 12 = 0 + c \Rightarrow c = 24.$$

Finally,

$$\begin{aligned}2h^{\frac{1}{2}} = -\tfrac{1}{40}t + 24 &\Rightarrow h^{\frac{1}{2}} = 12 - \tfrac{1}{80}t \\ &\Rightarrow \underline{\underline{h = (12 - \tfrac{1}{80}t)^2}},\end{aligned}$$

as required.

(c) How many days will it take for the tank to empty? (2)

Solution

$$\begin{aligned}(12 - \tfrac{1}{80}t)^2 = 0 &\Rightarrow 12 - \tfrac{1}{80}t = 0 \\ &\Rightarrow \tfrac{1}{80}t = 12 \\ &\Rightarrow t = 960 \text{ hours} \\ &\Rightarrow \underline{\underline{t = 40 \text{ days}}}.\end{aligned}$$

(d) Given that the tank has radius 20 cm, find the rate at which the water was being delivered to the vegetation (in cm^3/hr) at the end of the fourth day. (3)

Solution

Well,

$$A = \pi \times 20^2 = 400\pi$$

and

$$\begin{aligned} \frac{k}{A} = \tfrac{1}{40} &\Rightarrow k = \frac{400\pi}{40} \\ &\Rightarrow k = 10\pi. \end{aligned}$$

Now, at the start of the fourth day, $t = 96$ hours and

$$h = (12 - \tfrac{96}{80})^2 = 116.64.$$

Finally,

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} &= -10\pi \times \sqrt{116.64} \\ &= -108\pi; \end{aligned}$$

hence, the rate at which the water was being delivered to the vegetation is $\underline{\underline{108\pi \text{ cm}^3/\text{hr}}}$.