Dr Oliver Mathematics
Further Mathematics: Further Pure Mathematics 2 (Paper 4A)
June 2022: Calculator
1 hour 30 minutes

The total number of marks available is 75.
You must write down all the stages in your working.
Inexact answers should be given to three significant figures unless otherwise stated.

1. The group S_4 is the set of all possible permutations that can be performed on the four numbers 1, 2, 3, and 4, under the operation of composition. For the group S_4,

   (a) write down the identity element, (1)

   (b) write down the inverse of the element a, where (1)

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}.$$

   (c) demonstrate that the operation of composition is associative using the following elements (2)

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}, c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}.$$

   (d) Explain why it is possible for the group S_4 to have a subgroup of order 4. (2)
   You do not need to find such a subgroup.

2. Matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & a \\ -3 & b & 1 \\ 0 & 1 & a \end{pmatrix},$$

   where a and b are integers, such that $a < b$.

   Given that the characteristic equation for $\mathbf{M}$ is

$$\lambda^3 - 7\lambda^2 + 13\lambda + c = 0,$$

   where c is a constant,

   (a) determine the values of a, b, and c. (5)

   (b) Hence, using the Cayley-Hamilton theorem, determine the matrix $\mathbf{M}^{-1}$. (3)

3. There are three lily pads on a pond.

A frog hops repeatedly from one lily pad to another.

The frog starts on lily pad A, as shown in Figure 1.

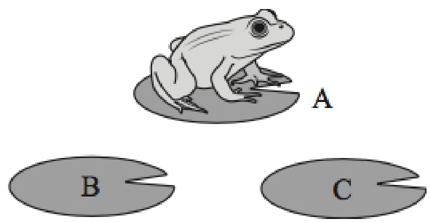

Figure 1: a frog hops repeatedly from one lily pad to another

In a model, the frog hops from its position on one lily pad to either of the other two lily pads with equal probability.

Let p_n be the probability that the frog is on lily pad A after n hops.

(a) Explain, with reference to the model, why $p_1 = 0$. (1)

The probability p_n satisfies the recurrence relation

$$p_{n+1} = \tfrac{1}{2}(1 - p_n),\ n \geqslant 1 \text{ where } p_1 = 0.$$

(b) Prove by induction that, for $n \geqslant 1$, (6)

$$p_n = \tfrac{2}{3}(-\tfrac{1}{2})^n + \tfrac{1}{3}.$$

(c) Use the result in part (b) to explain why, in the long term, the probability that the frog is on lily pad A is $\frac{1}{3}$. (1)

4. (a) Use the Euclidean algorithm to show that 124 and 17 are relatively prime (coprime) (2)

(b) Hence solve the equation (3)

$$124x + 17y = 10.$$

(c) Solve the congruence equation (2)

$$124x \equiv 6 (\text{mod } 17).$$

5. The locus of points z satisfies

$$|z + a\mathrm{i}| = 3|z - a|,$$

where a is an integer.

The locus is a circle with its centre in the third quadrant and radius $\frac{3}{2}\sqrt{2}$.

Determine

(a) the value of a, (4)

(b) the coordinates of the centre of the circle. (2)

6. (a) Determine the general solution of the recurrence relation (4)

$$u_n = 2u_{n-1} - u_{n-2} + 2^n, \geqslant 2.$$

(b) Hence solve this recurrence relation given that $u_0 = 2u_1$ and $u_4 = 3u_2$. (2)

7. The polynomial $\mathrm{F}(x)$ is a quartic such that

$$\mathrm{F}(x) = px^4 + qx^3 + 2x^2 + rx + s,$$

where p, q, r, and s are distinct constants.

Determine the number of possible quartics given that

(a) (i) the constants p, q, r, and s belong to the set $\{-4, -2, 1, 3, 5\}$, (1)

(ii) the constants p, q, r, and s belong to the set $\{-4, -2, 0, 1, 3, 5\}$. (1)

A 3-digit positive integer $N = abc$ has the following properties

- N is divisible by 11,
- the sum of the digits of N is even, and
- $N \equiv 8 (\mathrm{mod}\ 9)$.

(b) (i) Use the first two properties to show that (2)

$$a - b + c = 0.$$

(ii) Hence determine all possible integers N, showing all your working and reasoning. (4)

8. The locus of points $z = x + \mathrm{i}y$ that satisfy

$$\arg\left(\frac{z - 8 - 5\mathrm{i}}{z - 2 - 5\mathrm{i}}\right)$$

is an arc of a circle C.

(a) On an Argand diagram sketch the locus of z. (2)

(b) Explain why the centre of C has x-coordinate 5. (1)

(c) Determine the radius of C. (2)

(d) Determine the y-coordinate of the centre of C. (2)

9.

$$I_n = \int_0^{\frac{1}{2}\pi} \sin^n 2x \, \mathrm{d}x.$$

(a) Prove that, for $n \geqslant 2$, (4)

$$I_n = \frac{(n-1)}{n} I_{n-2}.$$

(b) Hence determine the exact value of (3)

$$\int_0^{\frac{1}{2}\pi} 64 \sin^5 x \cos^5 x \, \mathrm{d}x.$$

10. Figure 2 shows a picture of a plant pot.

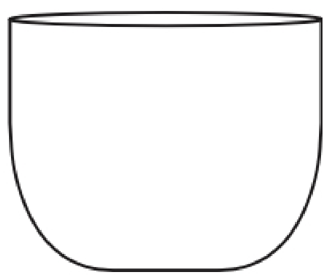

Figure 2: a plant pot

The plant pot has

- a flat circular base of radius 10 cm and
- a height of 15 cm

Figure 3 shows a sketch of the curve C with parametric equations

$$x = 10 + 15t - 5t^3,\ y = 15t^2,\ 0 \leqslant t \leqslant 1.$$

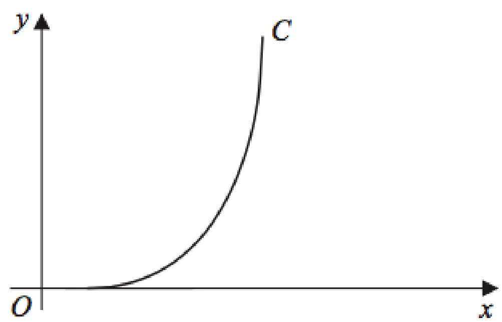

Figure 3: $x = 10 + 15t - 5t^3$, $y = 15t^2$

The curved inner surface of the plant pot is modelled by the surface of revolution formed by rotating curve C through 2π radians about the y-axis.

(a) Show that, according to the model, the area of the curved inner surface of the plant pot is given by (5)

$$150\pi \int_0^1 (2 + 3t + 2t^2 + 2t^3 - t^5)\, \mathrm{d}t.$$

(b) Determine, according to the model, the total area of the inner surface of the plant pot. (4)

Each plant pot will be painted with one coat of paint, both inside and outside.
The paint in one tin will cover an area of 12 m^2.

(c) Use the answer to part (b) to estimate how many plant pots can be painted using one tin of paint. (2)

(d) Give a reason why the model might not give an accurate answer to part (c). (1)