Dr Oliver Mathematics
Applied Mathematics: Mechanics or Statistics
Section B
2014 Paper
1 hour

The total number of marks available is 32.
You must write down all the stages in your working.

1. Find the gradient of the tangent to the curve (4)

$$y = 2x\sqrt{x-1}$$

at the point where $x = 10$.

Solution

$$u = 2x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2$$
$$v = (x-1)^{\frac{1}{2}} \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \tfrac{1}{2}(x-1)^{-\frac{1}{2}}$$

Now,

$$\begin{aligned} y = 2x\sqrt{x-1} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= (2x)\left[\tfrac{1}{2}(x-1)^{-\frac{1}{2}}\right] + (2)\left[(x-1)^{\frac{1}{2}}\right] \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= (x-1)^{-\frac{1}{2}}[x + 2(x-1)] \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= (x-1)^{-\frac{1}{2}}(3x-2) \end{aligned}$$

and

$$x = 10 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \underline{\underline{\tfrac{28}{3}}}.$$

2. Matrices are given as

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 4 \\ k & 0 & -1 \\ 5 & 3 & 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 & -10 & 2 \\ -3 & 9 & 0 \\ 0 & -2 & 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 & 2 & -6 \\ 1 & 1 & -2 \\ 2 & 2 & -1 \end{pmatrix}.$$

(a) Calculate $\mathbf{A}+\mathbf{B}$. (1)

Solution

$$\begin{aligned}\mathbf{A}+\mathbf{B} &= \begin{pmatrix} 1 & 3 & 4 \\ k & 0 & -1 \\ 5 & 3 & 0 \end{pmatrix} + \begin{pmatrix} 3 & -10 & 2 \\ -3 & 9 & 0 \\ 0 & -2 & 1 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 4 & -7 & 6 \\ k-3 & 9 & -1 \\ 5 & 1 & 1 \end{pmatrix}}}.\end{aligned}$$

(b) Find the determinant of $\mathbf{A}$. (2)

Solution

$$\begin{aligned}\det \mathbf{A} &= 1(0+3) - 3(0+5) + 4(3k-0) \\ &= 3 - 15 + 12k \\ &= \underline{\underline{12k - 12}}.\end{aligned}$$

(c) Calculate $\mathbf{BC}$. (1)

Solution

$$\begin{aligned}\mathbf{BC} &= \begin{pmatrix} 3 & -10 & 2 \\ -3 & 9 & 0 \\ 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 & -6 \\ 1 & 1 & -2 \\ 2 & 2 & -1 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}}}.\end{aligned}$$

(d) Describe the relationship between $\mathbf{B}$ and $\mathbf{C}$. (2)

Solution

$$\mathbf{BC} = 3\mathbf{I} \Rightarrow \underline{\underline{\mathbf{B} = 3\mathbf{C}^{-1}}}.$$

3. Find the exact value of (5)

$$\int_0^{2\pi} x \sin 3x \, \mathrm{d}x.$$

Solution

$$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \sin 3x \Rightarrow v = -\tfrac{1}{3}\cos 3x$$

Now,

$$\begin{aligned}\int_0^{2\pi} x \sin 3x \, \mathrm{d}x &= \left[-\tfrac{1}{3}x\cos 3x\right]_{x=0}^{2\pi} + \tfrac{1}{3}\int_0^{2\pi} \cos 3x \, \mathrm{d}x \\ &= \left(-\tfrac{2}{3}\pi - 0\right) + \tfrac{1}{3}\left[\tfrac{1}{3}\sin 3x\right]_{x=0}^{2\pi} \\ &= -\tfrac{2}{3}\pi + \tfrac{1}{3}(0-0) \\ &= \underline{\underline{-\tfrac{2}{3}\pi}}.\end{aligned}$$

4. Evaluate (2)

$$\sum_{r=1}^{80} 3r^2.$$

Solution

$$\begin{aligned}\sum_{r=1}^{80} 3r^2 &= 3\sum_{r=1}^{80} r^2 \\ &= 3 \times \tfrac{1}{6}(80)(81)(161) \\ &= 3 \times 173\,880 \\ &= \underline{\underline{521\,640}}.\end{aligned}$$

5. (a) Write down and simplify the binomial expansion of (3)

$$(\mathrm{e}^x + 2)^4.$$

Solution

You ought to know the binomial sequence is 1 4 6 4 1:

$$\begin{aligned}(\mathrm{e}^x + 2)^4 &= (\mathrm{e}^x)^4 + 4(\mathrm{e}^x)^3(2) + 6(\mathrm{e}^x)^2(2)^2 + 4(\mathrm{e}^x)(2)^3 + (2)^4 \\ &= \underline{\underline{\mathrm{e}^{4x} + 8\mathrm{e}^{3x} + 24\mathrm{e}^{2x} + 32\mathrm{e}^x + 16}}.\end{aligned}$$

(b) Hence obtain (2)

$$\int (\mathrm{e}^x + 2)^4 \, \mathrm{d}x.$$

Solution

$$\begin{aligned}\int (\mathrm{e}^x + 2)^4 \, \mathrm{d}x &= \int \left(\mathrm{e}^{4x} + 8\mathrm{e}^{3x} + 24\mathrm{e}^{2x} + 32\mathrm{e}^x + 16\right) \, \mathrm{d}x \\ &= \underline{\underline{\tfrac{1}{4}\mathrm{e}^{4x} + \tfrac{8}{3}\mathrm{e}^{3x} + 12\mathrm{e}^{2x} + 32\mathrm{e}^x + 16x + c}}.\end{aligned}$$

6. A flu-like virus starts to spread through the 20 000 inhabitants of Dumbarton.
The situation can be modelled by the differential equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{N(20\,000 - N)}{10\,000},$$

where N is the number of people infected after t days and $0 < N < 20\,000$.

(a) How many people are infected when the infection is spreading most rapidly? (1)

Solution

$$\begin{aligned}\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{N(20\,000 - N)}{10\,000} &\Rightarrow \frac{\mathrm{d}N}{\mathrm{d}t} = \tfrac{1}{10\,000}(20\,000N - N^2) \\ &\Rightarrow \frac{\mathrm{d}^2N}{\mathrm{d}t^2} = \tfrac{1}{10\,000}(20\,000 - 2N) \\ &\Rightarrow \frac{\mathrm{d}^2N}{\mathrm{d}t^2} = \tfrac{1}{5\,000}(10\,000 - N)\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}^2N}{\mathrm{d}t^2} = 0 &\Rightarrow \tfrac{1}{5\,000}(10\,000 - N) \\ &\Rightarrow \underline{\underline{N = 10\,000}}.\end{aligned}$$

(b) Express (5)

$$\frac{10\,000}{N(20\,000-N)}$$

in partial fractions and show that

$$\ln\left(\frac{N}{20\,000-N}\right) = 2t + c,$$

for some constant c.

Solution

$$\begin{aligned}\frac{10\,000}{N(20\,000-N)} &\equiv \frac{A}{N} + \frac{B}{20\,000-N}\\ &\equiv \frac{A(20\,000-N)+BN}{N(20\,000-N)}\end{aligned}$$

which means

$$10\,000 \equiv A(20\,000-N) + BN.$$

$\underline{N=0}$: $10\,000 = 20\,000A \Rightarrow A = \frac{1}{2}$.
$\underline{N=20\,000}$: $10\,000 = 20\,000B \Rightarrow B = \frac{1}{2}$.
Hence,

$$\frac{10\,000}{N(20\,000-N)} \equiv \underline{\underline{\frac{\frac{1}{2}}{N} + \frac{\frac{1}{2}}{20\,000-N}}}$$

and

$$\begin{aligned}\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{N(20\,000-N)}{10\,000} &\Rightarrow \frac{10\,000}{N(20\,000-N)}\,\mathrm{d}N = \mathrm{d}t\\ &\Rightarrow \left(\frac{\frac{1}{2}}{N} + \frac{\frac{1}{2}}{20\,000-N}\right)\mathrm{d}N = \mathrm{d}t\\ &\Rightarrow \int\left(\frac{\frac{1}{2}}{N} + \frac{\frac{1}{2}}{20\,000-N}\right)\mathrm{d}N = \int \mathrm{d}t\\ &\Rightarrow \tfrac{1}{2}\ln N - \tfrac{1}{2}\ln(20\,000-N) = t + a\\ &\Rightarrow \tfrac{1}{2}\ln\left(\frac{N}{20\,000-N}\right) = t + a\\ &\Rightarrow \underline{\underline{\ln\left(\frac{N}{20\,000-N}\right) = 2t + c,}}\end{aligned}$$

where $c = 2a$.

Initially there were 100 people infected.

(c) Show that (4)

$$N = \frac{20\,000\,\mathrm{e}^{2t}}{199 + \mathrm{e}^{2t}}.$$

Solution

$$\begin{aligned} t = 0, N = 100 &\Rightarrow \ln\left(\frac{100}{20\,000 - 100}\right) = 0 + c \\ &\Rightarrow c = \ln \tfrac{1}{199} \end{aligned}$$

and

$$\begin{aligned} \ln\left(\frac{N}{20\,000 - N}\right) = 2t + \ln \tfrac{1}{199} &\Rightarrow \frac{N}{20\,000 - N} = \mathrm{e}^{2t + \ln \frac{1}{199}} \\ &\Rightarrow \frac{N}{20\,000 - N} = \mathrm{e}^{2t}\,\mathrm{e}^{\ln \frac{1}{199}} \\ &\Rightarrow \frac{N}{20\,000 - N} = \tfrac{1}{199}\mathrm{e}^{2t} \\ &\Rightarrow N = \tfrac{1}{199}\mathrm{e}^{2t}(20\,000 - N) \\ &\Rightarrow N = \tfrac{20\,000}{199}\mathrm{e}^{2t} - \tfrac{1}{199}\mathrm{e}^{2t}N \\ &\Rightarrow N + \tfrac{1}{199}\mathrm{e}^{2t}N = \tfrac{20\,000}{199}\mathrm{e}^{2t} \\ &\Rightarrow \tfrac{1}{199}N\left(199 + \mathrm{e}^{2t}\right) = \tfrac{20\,000}{199}\mathrm{e}^{2t} \\ &\Rightarrow N\left(199 + \mathrm{e}^{2t}\right) = 20\,000\mathrm{e}^{2t} \\ &\Rightarrow \underline{\underline{N = \frac{20\,000\mathrm{e}^{2t}}{199 + \mathrm{e}^{2t}}}}, \end{aligned}$$

as required.