Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2006 November Paper 1: Calculator
2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

1. Express each of the following statements in appropriate set notation.

   (a) x is not an element of set A. (1)

   Solution
   $x \notin A$.

   (b) The number of elements not in set B is 16. (1)

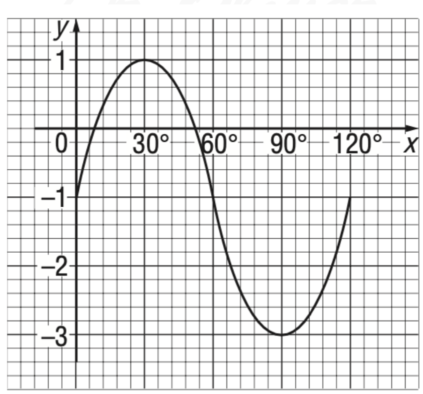
   Solution
   $\text{n}(B') = 16$.

   (c) Sets C and D have no common element. (1)

   Solution
   $C \cap D = \varnothing$.

2. The diagram shows part of the graph of

$$y = a\sin(bx) + c.$$



State the value of

(a) a, (1)

Solution

$\underline{\underline{a = 2}}$.

(b) b, (1)

Solution

$\underline{\underline{b = 3}}$.

(c) c. (1)

Solution

$\underline{\underline{c = -1}}$.

3. The equation of a curve is

$$y = \frac{8}{(3x-2)^2}.$$

(a) Find the gradient of the curve where $x = 2$. (3)

Solution

$$\begin{aligned} y = \frac{8}{(3x-2)^2} &\Rightarrow y = 8(3x-2)^{-2} \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 8 \times -2(3x-2)^{-3} \times 3 \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -48(3x-2)^{-3} \end{aligned}$$

and

$$\begin{aligned} x = 2 &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -48(2)^{-3} \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \underline{\underline{-6}}. \end{aligned}$$

(b) Find the approximate change in y when x increases from 2 to $2 + p$, where p is small. (2)

Solution
Well,

$$\delta y = \frac{\mathrm{d}y}{\mathrm{d}x} \times \delta x$$
$$\Rightarrow \delta y = \underline{\underline{-6p}}.$$

4. The vector $\overrightarrow{OP}$ has a magnitude of 10 units and is parallel to the vector $3\mathbf{i} - 4\mathbf{j}$.
The vector $\overrightarrow{OQ}$ has a magnitude of 15 units and is parallel to the vector $4\mathbf{i} + 3\mathbf{j}$.

(a) Express $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ in terms of $\mathbf{i}$ and $\mathbf{i}$. (3)

Solution
Well,

$$\sqrt{3^2 + 4^2} = 5.$$

Now,

$$\begin{aligned}\overrightarrow{OP} &= 10 \times \tfrac{1}{5}(3\mathbf{i} - 4\mathbf{j}) \\ &= 2(3\mathbf{i} - 4\mathbf{j}) \\ &= \underline{\underline{6\mathbf{i} - 8\mathbf{j}}}\end{aligned}$$

and

$$\begin{aligned}\overrightarrow{OQ} &= 15 \times \tfrac{1}{5}(4\mathbf{i} + 3\mathbf{j}) \\ &= 3(4\mathbf{i} + 3\mathbf{j}) \\ &= \underline{\underline{12\mathbf{i} + 9\mathbf{j}}}.\end{aligned}$$

(b) Given that the magnitude of $\overrightarrow{PQ}$ is $\lambda\sqrt{13}$, find the value of λ. (3)

Solution
Well,

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{PO} + \overrightarrow{OQ} \\ &= -\overrightarrow{OP} + \overrightarrow{OQ} \\ &= -(6\mathbf{i} - 8\mathbf{j}) + (12\mathbf{i} + 9\mathbf{j}) \\ &= -6\mathbf{i} + 8\mathbf{j} + 12\mathbf{i} + 9\mathbf{j} \\ &= 6\mathbf{i} + 17\mathbf{j}\end{aligned}$$

and

$$\begin{aligned} PQ &= \sqrt{6^2 + 17^2} \\ &= \sqrt{36 + 289} \\ &= \sqrt{325} \\ &= \underline{\underline{5\sqrt{13}}}; \end{aligned}$$

so, $\underline{\underline{\lambda = 5}}$.

5. A large airline has a fleet of aircraft consisting of 5 aircraft of type A, 8 of type B, 4 of type C and 10 of type D.
The aircraft have 3 classes of seat known as Economy, Business, and First.
The table below shows the number of these seats in each of the 4 types of aircraft.

	Economy	Business	First
A	300	60	40
B	150	50	20
C	120	40	0
D	100	0	0

(a) Write down two matrices whose product shows the total number of seats in each class. (1)

Solution

$$\underline{\underline{\begin{pmatrix} 5 & 8 & 4 & 10 \end{pmatrix} \begin{pmatrix} 300 & 60 & 40 \\ 150 & 50 & 20 \\ 120 & 40 & 0 \\ 100 & 0 & 0 \end{pmatrix}}}.$$

(b) Evaluate this product of matrices. (1)

Solution
Evaluating:

$$\underline{\underline{\begin{pmatrix} 4\,180 & 860 & 360 \end{pmatrix}}}.$$

On a particular day, each aircraft made one flight.
5% of the Economy seats were empty, 10% of the Business seats were empty, and 20% of the First seats were empty.

(c) Write down a matrix whose product with the matrix found in part (b) will give the total number of empty seats on that day. (1)

Solution

$$\underline{\underline{\begin{pmatrix} 0.05 \\ 0.1 \\ 0.2 \end{pmatrix}}}.$$

(d) Evaluate this total. (3)

Solution

$$\begin{pmatrix} 4\,180 & 860 & 360 \end{pmatrix} \begin{pmatrix} 0.05 \\ 0.1 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 367 \end{pmatrix}$$

so $\underline{\underline{367}}$.

6. Given that the coefficient of x^2 in the expansion of (6)

$$(k + x)(2 - \tfrac{1}{2}x)^6$$

is 84, find the value of the constant k.

Solution

Now,

$$\begin{aligned}(k + x)(2 - \tfrac{1}{2}x)^6 &= \ldots + k\binom{6}{2}(2)^4(-\tfrac{1}{2}x)^2 + x\binom{6}{1}(2)^5(-\tfrac{1}{2}x) + \ldots \\ &= \ldots + k(15)(16)(\tfrac{1}{4}x^2) + x(6)(32)(-\tfrac{1}{2}x) + \ldots \\ &= \ldots + 60kx^2 - 96x^2 + \ldots \\ &= \ldots + (60k - 96)x^2 + \ldots.\end{aligned}$$

Finally, the coefficient of x^2 is 84 which gives us

$$\begin{aligned}60k - 96 = 84 &\Rightarrow 60k = 180 \\ &\Rightarrow \underline{\underline{k = 3}}.\end{aligned}$$

7. The function f is defined for the domain $-3 \leqslant x \leqslant 3$ by

$$\mathrm{f}(x) = 9(x - \tfrac{1}{3})^2 - 11.$$

(a) Find the range of f. (3)

Solution

Well,

$$\begin{aligned} \mathrm{f}(-3) &= 89 \\ \mathrm{f}(3) &= 53 \\ \mathrm{f}(\tfrac{1}{3}) &= -11 \end{aligned}$$

and the range is

$$\underline{\underline{-11 \leqslant \mathrm{f}(x) \leqslant 89}}.$$

(b) State the coordinates and nature of the turning point of (4)

(i) the curve $y = \mathrm{f}(x)$,

Solution

The coordinate are $\underline{\underline{(\tfrac{1}{3}, -11)}}$ and this is a <u>minimum turning point</u>.

(ii) the curve $y = |\,\mathrm{f}(x)|$.

Solution

The coordinate are $\underline{\underline{(\tfrac{1}{3}, 11)}}$ and this is a <u>maximum turning point</u>.

8. (a) Solve the equation (3)

$$\log_{10}(x + 12) = 1 + \log_{10}(2 - x).$$

Solution

$$\begin{aligned}\log_{10}(x+12) = 1 + \log_{10}(2-x) &\Rightarrow \log_{10}(x+12) - \log_{10}(2-x) = 1\\ &\Rightarrow \log_{10}\left(\frac{x+12}{2-x}\right) = 1\\ &\Rightarrow \frac{x+12}{2-x} = 10\\ &\Rightarrow x+12 = 10(2-x)\\ &\Rightarrow x+12 = 20-10x\\ &\Rightarrow 11x = 8\\ &\Rightarrow \underline{\underline{x = \tfrac{8}{11}}}.\end{aligned}$$

(b) Given that (4)

$$\log_2 p = a, \log_8 q = b, \text{ and } \frac{p}{q} = 2^c,$$

express c in terms of a and b.

Solution

Now,

$$\begin{aligned}\log_8 q = b &\Rightarrow \frac{\log_2 q}{\log_2 8} = b\\ &\Rightarrow \frac{\log_2 q}{3} = b\\ &\Rightarrow \log_2 q = 3b\end{aligned}$$

and

$$\begin{aligned}\frac{p}{q} &= \frac{2^a}{2^{3b}}\\ &= \underline{\underline{2^{a-3b}}}.\end{aligned}$$

9. A curve has the equation

$$y = \frac{2x-4}{x+3}.$$

(a) Obtain an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ and hence explain why the curve has no turning points. (3)

Solution

Well,

$$u = 2x - 4 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2$$
$$v = x + 3 \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 1$$

and

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(x+3)(2) - (2x-4)(1)}{(x+3)^2} \\ &= \frac{2x + 6 - 2x + 4}{(x+3)^2} \\ &= \frac{10}{(x+3)^2}.\end{aligned}$$

Now, the numerator does not equal 0 for any value of x and so the curve has <u>no turning points</u>.

The curve intersects the x-axis at the point P.
The tangent to the curve at P meets the y-axis at the point Q.

(b) Find the area of the triangle POQ, where O is the origin. (5)

Solution

Well,

$$\begin{aligned}y = 0 &\Rightarrow 2x - 4 = 0 \\ &\Rightarrow 2x = 4 \\ &\Rightarrow x = 2\end{aligned}$$

and so $P(2, 0)$. Now,

$$x = 2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{2}{5}$$

and the equation of the tangent is

$$y = \tfrac{2}{5}(x - 2).$$

Next,

$$x = 0 \Rightarrow y = -\tfrac{4}{5}$$

so $Q(0, -\tfrac{4}{5})$.

Finally,

$$\begin{aligned}\text{area} &= \tfrac{1}{2} \times 2 \times \tfrac{4}{5} \\ &= \underline{\underline{\tfrac{4}{5}}}.\end{aligned}$$

10. The cubic polynomial $\mathrm{f}(x)$ is such that the coefficient of x^3 is 1 and the roots of $\mathrm{f}(x) = 0$ are 1, k, and k^2.
It is given that $\mathrm{f}(x)$ has a remainder of 7 when divided by $(x - 2)$.
   (a) Show that (3)
$$k^3 - 2k^2 - 2k - 3 = 0.$$

Solution
Well,

$$\mathrm{f}(x) = (x-1)(x-k)(x-k^2)$$

$\times$	x	$-k$
x	x^2	$-kx$
-1	$-x$	$+k$

$$= [x^2 - (k+1)x + k](x - k^2)$$

$\times$	x^2	$-(k+1)x$	$+k$
x	x^3	$-(k+1)x^2$	$+kx$
$-k^2$	$-k^2x^2$	$+k^2(k+1)x$	$-k^3$

$$= x^3 - (1 + k + k^2)x^2 - (k + k^2 + k^3)x - k^3.$$

Now, we use synthetic division:

2	1	$-1-k-k^2$	$k+k^2+k^3$	$-k^3$
	$\downarrow$	2	$2-2k-2k^2$	$4-2k-2k^2+2k^3$
	1	$1-k-k^2$	$2-k-k^2+k^3$	$4-2k-2k^2-k^3$

Next, $\mathrm{f}(x)$ has a remainder of 7 when divided by $(x-2)$ so

$$4 - 2k - 2k^2 + k^3 = 7 \Rightarrow \underline{\underline{k^3 - 2k^2 - 2k - 3 = 0}},$$

as required.

(b) Hence find a value for k and show that there are no other real values of k which satisfy this equation. (5)

Solution
Let

$$\mathrm{g}(k) = k^3 - 2k^2 - 2k - 3.$$

Then,

$$\begin{aligned} \mathrm{g}(1) &= 1 - 2 - 2 - 3 = -6 \\ \mathrm{g}(-1) &= -1 - 2 + 2 - 3 = -4 \\ \mathrm{g}(3) &= 27 - 18 - 6 - 3 = 0 \end{aligned}$$

and we know that $(k-3)$ divides $\mathrm{g}(k)$. So $\underline{\underline{k = 3}}$.

Now,

3	1	−2	−2	−3
	↓	3	3	3
	1	1	1	0

so

$$\mathrm{g}(k) = (k-3)(k^2 + k + 1).$$

Next,

$$\begin{aligned} k^2 + k + 1 &= (k^2 + k + \tfrac{1}{4}) + \tfrac{3}{4} \\ &= (k + \tfrac{1}{2})^2 + \tfrac{3}{4} \\ &> 0 \end{aligned}$$

so that there are $\underline{\underline{\text{no other real values}}}$ of k which satisfy this equation.

11. (a) Solve, for $0^\circ \leqslant x \leqslant 360^\circ$, the equation (5)

$$2 \cot x = 1 + \tan x.$$

Solution

$$\begin{aligned} 2\cot x = 1 + \tan x &\Rightarrow \frac{2}{\tan x} = 1 + \tan x \\ &\Rightarrow 2 = \tan x + \tan^2 x \\ &\Rightarrow \tan^2 x + \tan x - 2 = 0 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & +1 \\ \text{multiply to:} & -2 \end{array}\right\} +2,\ -1$$

$$\begin{aligned} &\Rightarrow (\tan x + 2)(\tan x - 1) = 0 \\ &\Rightarrow \tan x = -2 \text{ or } \tan x = 1. \end{aligned}$$

$\underline{\tan x = -2:}$

$$\begin{aligned} \tan x = -2 &\Rightarrow x = 116.565\,051\,2,\ 296.565\,051\,2 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 117,\ 297 \text{ (3 sf)}}}. \end{aligned}$$

$\underline{\tan x = 1:}$

$$\tan x = 1 \Rightarrow \underline{\underline{x = 45,\ 225}}.$$

(b) Given that y is measured in radians, find the two smallest positive values of y such that (5)

$$6\sin(2y + 1) + 5 = 0.$$

Solution

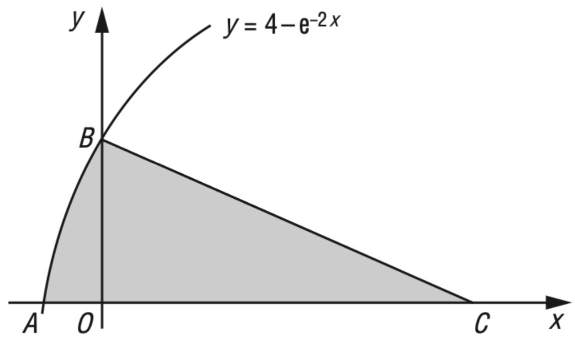
$$\begin{aligned} 6\sin(2y+1) + 5 = 0 &\Rightarrow 6\sin(2y+1) = -5 \\ &\Rightarrow \sin(2y+1) = -\tfrac{5}{6} \\ &\Rightarrow 2y + 1 = 4.126\,703\,437,\ 5.298\,074\,524 \text{ (FCD)} \\ &\Rightarrow 2y = 3.126\,703\,437,\ 4.298\,074\,524 \text{ (FCD)} \\ &\Rightarrow y = 1.563\,351\,718,\ 2.149\,037\,262 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{y = 1.56,\ 2.15 \text{ (3 sf)}}}. \end{aligned}$$

EITHER

12. The diagram shows part of the curve

$$y = 4 - \mathrm{e}^{-2x}$$

which crosses the axes at A and at B.



(a) Find the coordinates of A and of B. (2)

Solution
Well,

$$\begin{aligned} y = 0 \Rightarrow 0 &= 4 - \mathrm{e}^{-2x} \\ \Rightarrow \mathrm{e}^{-2x} &= 4 \\ \Rightarrow -2x &= \ln 4 \\ \Rightarrow x &= -\tfrac{1}{2}\ln 4 \end{aligned}$$

and

$$\begin{aligned} x = 0 \Rightarrow y &= 4 - 1 \\ \Rightarrow y &= 3; \end{aligned}$$

hence, $\underline{\underline{A(-\tfrac{1}{2}\ln 4, 0)}}$ and $\underline{\underline{B(0, 3)}}$.

The normal to the curve at B meets the x-axis at C.

(b) Find the coordinates of C. (4)

Solution
Now,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\mathrm{e}^{-2x}$$

and

$$x = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2.$$

Next, the normal to the curve has gradient $-\frac{1}{2}$ and the equation of the normal is

$$y - 3 = -\tfrac{1}{2}(x - 0) \Rightarrow y = -\tfrac{1}{2}x + 3.$$

Finally,

$$\begin{aligned} y = 0 \Rightarrow 0 &= -\tfrac{1}{2}x + 3 \\ \Rightarrow \tfrac{1}{2}x &= 3 \\ \Rightarrow x &= 6; \end{aligned}$$

hence, $\underline{\underline{C(6, 0)}}$.

(c) Show that the area of the shaded region is approximately 10.3 square units. (5)

Solution

Well,

$$\begin{aligned} \text{area} &= \int_{-\frac{1}{2}\ln 4}^{0} (4 - \mathrm{e}^{-2x})\,\mathrm{d}x + (\tfrac{1}{2} \times 6 \times 3) \\ &= \left[4x + \tfrac{1}{2}\mathrm{e}^{-2x}\right]_{x=-\frac{1}{2}\ln 4}^{0} + 9 \\ &= \left(0 + \tfrac{1}{2}\right) - \left(-2\ln 4 + \tfrac{1}{2}\mathrm{e}^{\ln 4}\right) + 9 \\ &= 10.272\,588\,72 \text{ (FCD)} \\ &= \underline{\underline{10.3 \text{ square units (3 sf)}}}. \end{aligned}$$

OR

13. The variables x and y are related by the equation

$$y = 10^{-A}b^x,$$

where A and b are constants.

The table below shows values of x and y.

x	15	20	25	30	35	40
y	0.15	0.38	0.95	2.32	5.90	14.80

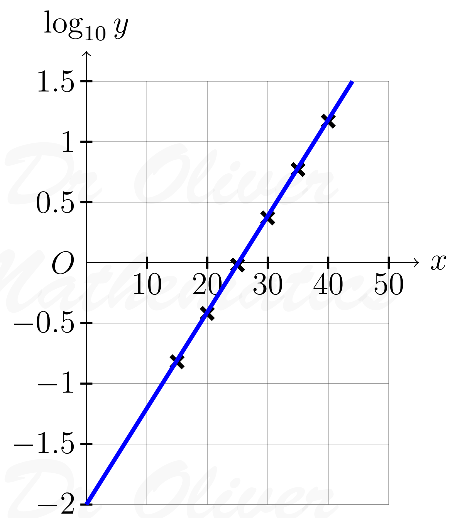
(a) Draw a straight line graph of $\log_{10} y$ against x, (2)

Solution

We will use 2 decimal places:

x	15	20	25	30	35	40
$\log_{10} y$	-0.82	-0.42	-0.02	0.37	0.77	1.17

We draw a plot the points together with a line of best fit:



(b) Use your graph to estimate the value of A and of b. (4)

Solution

The line of best fit goes through the points $(12, -1)$ and $(44, 1.5)$:

$$\begin{aligned} m &= \frac{1.5 - (-1)}{44 - 12} \\ &= \tfrac{5}{64} \end{aligned}$$

and

$$\begin{aligned}\log_{10} y - 1.5 = \tfrac{5}{64}(x-44) &\Rightarrow \log_{10} y - 1.5 = \tfrac{5}{64}x - \tfrac{55}{16}\\ &\Rightarrow \log_{10} y = \tfrac{5}{64}x - \tfrac{31}{16}\\ &\Rightarrow y = 10^{\frac{5}{64}x - \frac{31}{16}}\\ &\Rightarrow y = 10^{-\frac{31}{16}} \cdot (10^{\frac{5}{64}})^x\\ &\Rightarrow y = 10^{-1.9375} \cdot 1.19708503^x \text{ (FCD)}\\ &\Rightarrow \underline{\underline{y = 10^{-1.94} \cdot 1.20^x \text{ (3 sf)}}}.\end{aligned}$$

(c) Estimate the value of x when $y = 10$. (2)

Solution

$$\begin{aligned}y = 10 &\Rightarrow 10 = 10^{-1.94} \cdot 1.20^x\\ &\Rightarrow 1.20^x = 870.96359 \text{ (FCD)}\\ &\Rightarrow x = \log_{1.20} 870.96359 \text{ (FCD)}\\ &\Rightarrow x = 37.13000422 \text{ (FCD)}\\ &\Rightarrow \underline{\underline{x = 37.1 \text{ (3 sf)}}}.\end{aligned}$$

(d) On the same diagram, draw the line representing (3)

$$y^5 = 10^{-x}$$

and hence find the value of x for which

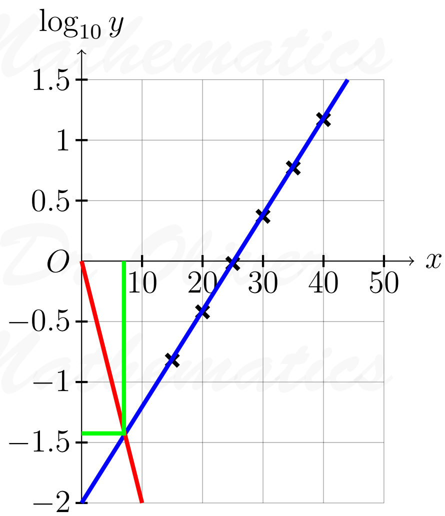
$$10^{A-\frac{1}{5}x} = b^x.$$

Solution

Well,

$$\begin{aligned}y^5 = 10^{-x} &\Rightarrow \log_{10} y^5 = \log_{10} 10^{-x}\\ &\Rightarrow 5\log_{10} y = -x\\ &\Rightarrow \log_{10} y = -\tfrac{1}{5}x\end{aligned}$$

and we draw the straight line on to the graph:



Correct read-off: approximately $\underline{\underline{x = 7}}$.