Dr Oliver Mathematics
OCR FMSQ Additional Mathematics
2012 Paper
2 hours

The total number of marks available is 100.
You must write down all the stages in your working.
You are permitted to use a scientific or graphical calculator in this paper.
Final answers should be given correct to three significant figures where appropriate.

Section A

1. (a) Find the range of values of x satisfying (3)

$$x^2 - 4x + 3 \leqslant 0.$$

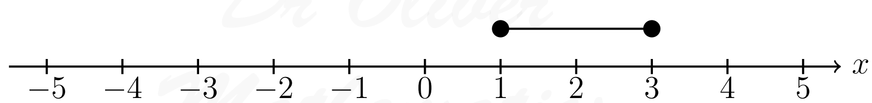
Solution

$$\left.\begin{array}{ll} \text{add to:} & -4 \\ \text{multiply to:} & +3 \end{array}\right\} -3,\ -1$$

$$\begin{aligned} x^2 - 4x + 3 \leqslant 0 &\Rightarrow (x-3)(x-1) \leqslant 0 \\ &\Rightarrow \underline{\underline{1 \leqslant x \leqslant 3}}. \end{aligned}$$

(b) Show this range on the number line. (1)

Solution



2. A die has 6 faces numbered one to six. The die is biased so that when it is thrown the probability of obtaining a six is $\frac{1}{5}$.

The die is thrown 5 times.

Find the probability of obtaining

(a) at least 1 six, (2)

Solution

$$\begin{aligned} \text{P(at least 1 six)} &= 1 - \text{P(no sixes)} \\ &= 1 - (\tfrac{4}{5})^5 \\ &= 1 - \tfrac{1\,024}{3\,125} \\ &= \underline{\underline{\tfrac{2\,101}{3\,125}}}. \end{aligned}$$

(b) exactly 3 sixes. (4)

Solution

$$\begin{aligned} \text{P(exactly 3 sixes)} &= \binom{5}{3}(\tfrac{1}{5})^3(\tfrac{4}{5})^2 \\ &= 10 \times \tfrac{1}{125} \times \tfrac{16}{25} \\ &= \underline{\underline{\tfrac{32}{625}}}. \end{aligned}$$

3. The function

$$f(x) = x^3 + ax + 6$$

is such that when $f(x)$ is divided by $(x - 3)$ the remainder is 12.

(a) Show that the value of a is -7. (2)

Solution

We use synthetic division:

3	1	0	a	6
	$\downarrow$	3	9	$3(a+9)$
	1	3	$a+9$	$6+3(a+9)$

Now,

$$\begin{aligned} 6 + 3(a+9) = 12 &\Rightarrow 3(a+9) = 6 \\ &\Rightarrow a + 9 = 2 \\ &\Rightarrow \underline{\underline{a = -7}}, \end{aligned}$$

as required.

(b) Factorise $f(x)$. (3)

Solution

Well,

$$f(1) = 1^3 - 7 + 6 = 0$$

so $x = 1$ is a root.

1	1	0	−7	6
	↓	1	1	−6
	1	1	−6	0

Now,

$$x^3 - 7x + 6 = (x-1)(x^2 + x - 6)$$

$$\left.\begin{array}{ll} \text{add to:} & +1 \\ \text{multiply to:} & -6 \end{array}\right\} -2, +3$$

$$= \underline{\underline{(x-1)(x-2)(x+3)}}.$$

4. A car moves from rest with constant acceleration on a straight road. When the car passes a point A it is travelling at 10 ms^{-1} and when it passes a point B further along the road it is travelling at 16 ms^{-1}.

The car takes 10 seconds to travel from A to B.

Find

(a) the distance AB, (2)

Solution

$s = ?$, $u = 10$, $v = 16$, $a = ?$, and $t = 10$: use $s = \frac{1}{2}(u+v)t$:

$$\begin{aligned} s &= \tfrac{1}{2}(10 + 16)(10) \\ &= \underline{\underline{130 \text{ m}}}. \end{aligned}$$

(b) the constant acceleration. (2)

Solution

Use $v = u + at$:

$$16 = 10 + 10a \Rightarrow 10a = 6$$
$$\Rightarrow \underline{\underline{a = 0.6 \text{ ms}^{-2}}}.$$

5. (a) Show that the equation (2)

$$3\cos^2\theta = \sin\theta + 1$$

can be written as

$$3\sin^2\theta + \sin\theta - 2 = 0.$$

Solution

$$3\cos^2\theta = \sin\theta + 1 \Rightarrow 3(1 - \sin^2\theta) = \sin\theta + 1$$
$$\Rightarrow 3 - 3\sin^2\theta = \sin\theta + 1$$
$$\Rightarrow \underline{\underline{3\sin^2\theta + \sin\theta - 2 = 0}},$$

as required.

(b) Solve this equation to find values of θ in the range $0^\circ < \theta < 360^\circ$ that satisfy (4)

$$3\cos^2\theta = \sin\theta + 1.$$

Solution

$$3\cos^2\theta = \sin\theta + 1 \Rightarrow 3\sin^2\theta + \sin\theta - 2 = 0$$

$$\left.\begin{array}{ll} \text{add to:} & +1 \\ \text{multiply to:} & (+3)\times(-2) = -6 \end{array}\right\} -2,\ +3$$

$$\Rightarrow 3\sin^2\theta + 3\sin\theta - 2\sin\theta - 2 = 0$$
$$\Rightarrow 3\sin\theta(\sin\theta + 1) - 2(\sin\theta + 1) = 0$$
$$\Rightarrow (3\sin\theta - 2)(\sin\theta + 1) = 0$$
$$\Rightarrow \sin\theta = \tfrac{2}{3} \text{ or } \sin\theta = -1.$$

Now,

$$\sin\theta = -1 \Rightarrow \underline{\underline{\theta = 270 \text{ (exact!)}}}$$

and

$$\begin{aligned}\sin\theta = \tfrac{2}{3} &\Rightarrow \theta = 41.810\,314\,9,\ 138.189\,685\,1 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{\theta = 41.8,\ 138 \text{ (3 sf)}}}.\end{aligned}$$

6. The equation of a curve is

$$y = 2x^3 - 9x^2 + 12x.$$

(a) Show that the curve has a stationary point where $x = 2$. (4)

Solution

$$y = 2x^3 - 9x^2 + 12x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 - 18x + 12$$

and

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 6x^2 - 18x + 12 = 0 \\ &\Rightarrow 6(x^2 - 3x + 2) = 0\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & -3 \\ \text{multiply to:} & +2\end{array}\right\} -2,\ -1$$

$$\begin{aligned}&\Rightarrow 6(x-2)(x-1) = 0 \\ &\Rightarrow x = 1 \text{ or } \underline{\underline{x = 2}},\end{aligned}$$

as required.

(b) Determine whether the stationary value where $x = 2$ is a maximum or minimum. (2)

Solution

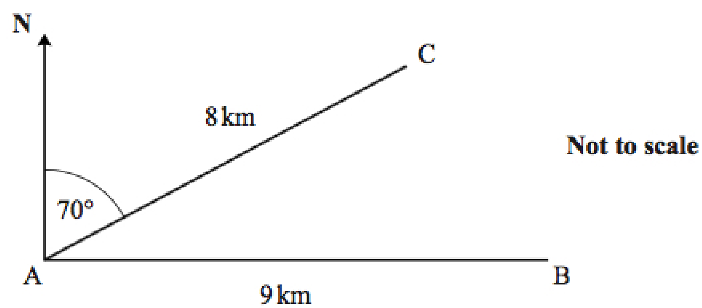
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 - 18x + 12 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x - 18.$$

Next,

$$x = 2 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6 > 0$$

which means $x = 2$ is a $\underline{\underline{\text{minimum}}}$.

7. A yachtsman wishes to sail from a port, A, to another port, B, which is 9 km due East of A.
Because of the wind he is unable to sail directly East and sails 8 km on a bearing of 070° to point C.



Calculate

(a) the distance he is now from port B, (3)

Solution

$$\angle CAB = 90 - 70 = 20^\circ$$

and we now apply the cosine rule:

$$\begin{aligned} BC &= \sqrt{9^2 + 8^2 - 2 \cdot 9 \cdot 8 \cdot \cos 20^\circ} \\ &= 3.111\,954\,789 \text{ (FCD)} \\ &= \underline{\underline{3.11 \text{ km (3 sf)}}}. \end{aligned}$$

(b) the angle ABC and hence the bearing on which he must sail to reach port B from point C, correct to the nearest degree. (4)

Solution

$$\begin{aligned} \frac{\sin ABC}{AC} = \frac{\sin CAB}{BC} \Rightarrow \frac{\sin ABC}{8} &= \frac{\sin 20^\circ}{3.111\ldots} \\ \Rightarrow \sin ABC &= \frac{8 \sin 20^\circ}{3.111\ldots} \\ \Rightarrow \angle ABC &= 61.551\,053\,15^\circ \text{ (FCD)} \\ \Rightarrow \underline{\underline{\angle ABC = 61.6^\circ \text{ (3 sf)}}}. \end{aligned}$$

Finally,

$$\begin{aligned}\text{bearing} &= 90 + 61.551\ldots \\ &= 151.551\,053\,15^\circ \text{ (FCD)} \\ &= \underline{\underline{152^\circ \text{ (nearest whole number)}}}.\end{aligned}$$

8. (a) Show that (3)

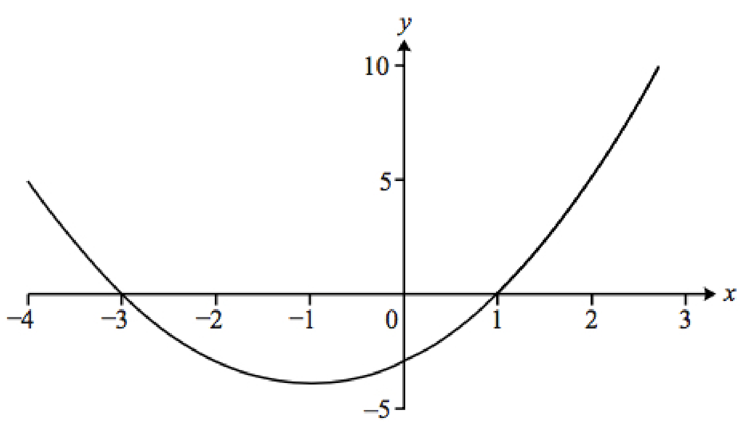
$$\int_0^2 (x^2 + 2x - 3)\mathrm{d}x = \tfrac{2}{3}.$$

Solution

$$\begin{aligned}\int_0^2 (x^2 + 2x - 3)\mathrm{d}x &= \left[\tfrac{1}{3}x^3 + x^2 - 3x\right]_{x=0}^2 \\ &= \left(2\tfrac{2}{3} + 4 - 6\right) - (0 + 0 - 0) \\ &= \underline{\underline{\tfrac{2}{3}}},\end{aligned}$$

as required.

The diagram shows part of the curve $y = x^2 + 2x - 3$.



Marc claims that the total area between the curve, the x-axis and the lines $x = 0$ and $x = 2$ is $\frac{2}{3}$.

(b) Explain why he is wrong. (1)

Solution

From $0 \leqslant x \leqslant 1$, the graph is *below* the x-axis. From $1 \leqslant x \leqslant 2$, the graph is *above* the x-axis. Hence, that is why the integral does not add up to $\frac{2}{3}$.

(c) Calculate the total area between the curve, the x-axis and the lines $x = 0$ and $x = 2$. (3)

Solution

$$\begin{aligned}\int_0^1 (x^2 + 2x - 3)\mathrm{d}x &= \left[\tfrac{1}{3}x^3 + x^2 - 3x\right]_{x=0}^1 \\ &= \left(\tfrac{1}{3} + 1 - 3\right) - (0 + 0 - 0) \\ &= -1\tfrac{2}{3}\end{aligned}$$

and

$$\begin{aligned}\int_1^2 (x^2 + 2x - 3)\mathrm{d}x &= \left[\tfrac{1}{3}x^3 + x^2 - 3x\right]_{x=1}^2 \\ &= \left(2\tfrac{2}{3} + 4 - 6\right) - \left(\tfrac{1}{3} + 1 - 3\right) \\ &= 2\tfrac{1}{3}.\end{aligned}$$

Finally,

$$\begin{aligned}\text{area} &= \left|\int_0^1 (x^2 + 2x - 3)\mathrm{d}x\right| + \left|\int_1^2 (x^2 + 2x - 3)\mathrm{d}x\right| \\ &= 1\tfrac{2}{3} + 2\tfrac{1}{3} \\ &= \underline{\underline{4}}.\end{aligned}$$

9. The height above the ground of a seat on a fairground big wheel is h metres. At time t **minutes** after the wheel starts, h is given by

$$h = 7 - 5\cos(480t)^\circ.$$

(a) Write down the initial height above the ground of the seat (when $t = 0$). (1)

Solution

$$t = 0 \Rightarrow h = 7 - 5 = \underline{\underline{2\text{ m}}}.$$

(b) Find the greatest height reached by the seat. (2)

Solution

$$\text{Greatest height} = 7 + 5 = \underline{\underline{12 \text{ m}}}.$$

(c) Calculate the time of the first occasion when the seat is 9 metres above the ground. Give your answer correct to the nearest second. (4)

Solution

$$\begin{aligned} 7 - 5\cos(480t)^\circ = 9 &\Rightarrow -5\cos(480t)^\circ = 2 \\ &\Rightarrow \cos(480t)^\circ = -\tfrac{2}{5} \\ &\Rightarrow 480t = 113.578\,178\,5 \text{ min (FCD)} \\ &\Rightarrow t = 0.236\,621\,205\,2 \text{ min (FCD)} \\ &\Rightarrow t = 14.197\,272\,31 \text{ s (FCD)} \\ &\Rightarrow \underline{\underline{t = 14 \text{ s (nearest second)}}}. \end{aligned}$$

Section B

10. $A(1, 10)$, $B(8, 9)$, and $C(7, 2)$ are three points.

(a) Find the coordinates of the midpoint, M, of AC. (1)

Solution

$$\left(\frac{1+7}{2}, \frac{10+2}{6}\right) = \underline{\underline{(4, 6)}}.$$

(b) Find the equation of the circle with AC as diameter. (4)

Solution

$$\begin{aligned} AM^2 &= (4-1)^2 + (6-10)^2 \\ &= 9 + 16 \\ &= 25 \end{aligned}$$

and the equation is

$$\underline{\underline{(x-4)^2 + (y-6)^2 = 25}}.$$

(c) Show that B lies on this circle. (1)

Solution

$$\begin{aligned}(8-4)^2+(9-6)^2 &= 4^2+3^2 \\ &= 16+9 \\ &= 25\end{aligned}$$

so, yes, B lies on this circle.

(d) Prove that AM and BM are perpendicular. (3)

Solution

$$\begin{aligned}\text{Grad}_{AM} &= \frac{10-6}{1-4} \\ &= -\tfrac{4}{3}\end{aligned}$$

and

$$\begin{aligned}\text{grad}_{BM} &= \frac{9-6}{8-4} \\ &= \tfrac{3}{4};\end{aligned}$$

as

$$\text{grad}_{AM} \times \text{grad}_{BM} = -1,$$

AM and BM are perpendicular.

BD is a diameter of this circle.

(e) Find the coordinates of D. (3)

Solution

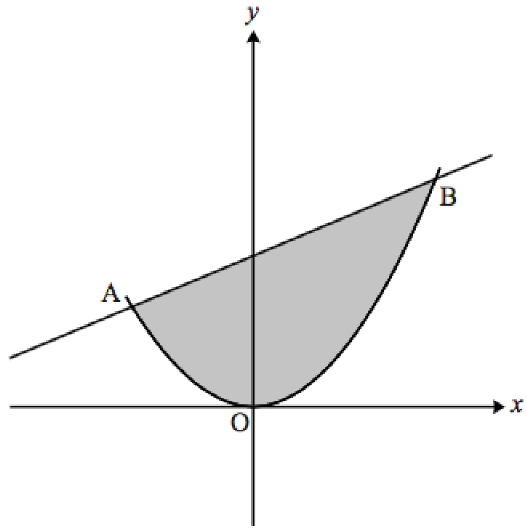
$$\begin{pmatrix} 4 \\ 6 \end{pmatrix} - \begin{pmatrix} 8 \\ 9 \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}$$

and we need to go along $(-4, -3)$ from M. So $D(0, 3)$.

11. The shaded region in the diagram shows a wooden shape. The curve has equation

$$y = \tfrac{1}{2}x^2$$

and the coordinates of A are $(-2, 2)$.



The line AB is the normal to the curve at the point A.

(a) Find the equation of the line AB. (5)

Solution

$$y = \tfrac{1}{2}x^2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = x.$$

Now,

$$x = -2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -2$$
$$\Rightarrow m_{\text{normal}} = \tfrac{1}{2}.$$

Hence, the equation of the line AB is

$$y - 2 = \tfrac{1}{2}(x+2) \Rightarrow y - 2 = \tfrac{1}{2}x + 1$$
$$\Rightarrow \underline{\underline{y = \tfrac{1}{2}x + 3}}.$$

(b) Find the coordinates of the point B where the line AB meets the curve again. (3)

Solution

$$\tfrac{1}{2}x^2 = \tfrac{1}{2}x + 3 \Rightarrow x^2 = x + 6$$
$$\Rightarrow x^2 - x - 6 = 0$$

$$\left.\begin{array}{ll}\text{add to:} & -1 \\ \text{multiply to:} & -6\end{array}\right\} -3,\ +2$$

$$\begin{aligned}&\Rightarrow (x-3)(x+2)=0\\ &\Rightarrow x=-2 \text{ or } x=3.\end{aligned}$$

Now,

$$x=3 \Rightarrow y=4\tfrac{1}{2}$$

and $\underline{\underline{B(3,4\tfrac{1}{2})}}$.

(c) Find the shaded area. (4)

Solution

$$\begin{aligned}\text{Area} &= \text{trapezium} - \text{area under the curve}\\ &= \tfrac{1}{2}(2+4\tfrac{1}{2})(2+3) - \int_{-2}^{3} \tfrac{1}{2}x^2 \, \mathrm{d}x\\ &= 16\tfrac{1}{4} - \left[\tfrac{1}{6}x^3\right]_{x=-2}^{3}\\ &= 16\tfrac{1}{4} - \left[4\tfrac{1}{2} - \left(-1\tfrac{1}{3}\right)\right]\\ &= 16\tfrac{1}{4} - 5\tfrac{5}{6}\\ &= \underline{\underline{10\tfrac{5}{12}}}.\end{aligned}$$

12. The Highway Code gives a table of shortest stopping distances (d feet) for a vehicle travelling at v miles per hour.

The formula used for this table is given by

$$d = av^2 + bv.$$

Two entries in the table are given below.

v mph	d feet
30	75
60	240

(a) By forming and solving a pair of simultaneous equations in a and b, show that the formula is (5)

$$d = \tfrac{1}{20}v^2 + v.$$

Solution

$$v = 30, d = 75 \Rightarrow 75 = 900a + 30b \quad (1)$$
$$v = 60, d = 240 \Rightarrow 240 = 3\,600a + 60b \quad (2)$$

Now, do $2 \times (1)$:

$$150 = 1\,800a + 60b \quad (3)$$

and $(2) - (3)$:

$$\begin{aligned} 90 = 1\,800a &\Rightarrow \underline{\underline{a = \tfrac{1}{20}}} \\ &\Rightarrow 75 = 900(\tfrac{1}{20}) + 30b \\ &\Rightarrow 75 = 45 + 30b \\ &\Rightarrow 30b = 30 \\ &\Rightarrow \underline{\underline{b = 1}}. \end{aligned}$$

(b) Find the difference between the stopping distances for a car travelling at 65 mph and a car travelling at 70 mph. (3)

Solution

Well,

$$v = 70 \Rightarrow d = \tfrac{1}{20}(70^2) + 70 = 315$$

and

$$v = 70 \Rightarrow d = \tfrac{1}{20}(65^2) + 65 = 276\tfrac{1}{4}.$$

Finally,

$$\begin{aligned} \text{difference} &= 315 - 276\tfrac{1}{4} \\ &= \underline{\underline{38\tfrac{3}{4} \text{ feet}}}. \end{aligned}$$

Many drivers maintain a distance of 50 feet or less when driving on a motorway.

(c) Use the formula in part (a) to find the speed at which the shortest stopping distance is 50 feet. (4)

Solution

$$\begin{aligned}\tfrac{1}{20}v^2 + v = 50 &\Rightarrow v^2 + 20v = 1\,000\\ &\Rightarrow v^2 + 20v + 100 = 1\,100\\ &\Rightarrow (v+10)^2 = 1\,100\\ &\Rightarrow v + 10 = \sqrt{1\,100}\\ &\Rightarrow v = \sqrt{1\,100} - 10\\ &\Rightarrow v = 23.166\,247\,9 \text{ (FCD)}\\ &\Rightarrow \underline{\underline{v = 23.2 \text{ mph (3 sf)}}}.\end{aligned}$$

13. (a) Find the coefficients a, b, and c in the expansion (3)

$$(2+h)^3 \equiv 8 + ah + bh^2 + ch^3.$$

Solution

$\times$	2	$+h$
2	4	$+2h$
$+h$	$+2h$	$+h^2$

$\times$	4	$+4h$	$+h^2$
2	8	$+8h$	$+2h^2$
$+h$	$+4h$	$+4h^2$	$+h^3$

So

$$(2+h)^3 \equiv \underline{\underline{8 + 12h + 6h^2 + h^3}};$$

$\underline{\underline{a = 12}}$, $\underline{\underline{b = 6}}$, and $\underline{\underline{c = 1}}$.

The graph of the equation $y = x^3$ passes through the points P and Q which have x-coordinates 2 and $2 + h$ respectively.

(b) Show that the gradient of the chord PQ is (3)

$$\frac{(2+h)^3 - 8}{h}.$$

Solution

$P(2, 8)$ and $Q(2 + h, (2 + h)^3)$. Now,

$$\begin{aligned}\text{Gradient} &= \frac{(2+h)^3 - 2^3}{(2+h) - 2} \\ &= \underline{\underline{\frac{(2+h)^3 - 8}{h}}},\end{aligned}$$

as required.

(c) Express (2)

$$\frac{(2+h)^3 - 8}{h}$$

as a quadratic function of h.

Solution

$$\begin{aligned}\frac{(2+h)^3 - 8}{h} &= \frac{(8 + 12h + 6h^2 + h^3) - 8}{h} \\ &= \frac{12h + 6h^2 + h^3}{h} \\ &= \underline{\underline{12 + 6h + h^2}}.\end{aligned}$$

As the value of h decreases, the point Q gets closer and closer to the point P on the curve. As h gets closer to 0 the chord PQ gets closer to being the tangent to the curve at P.

(d) Deduce the value of the gradient of the tangent at P. (1)

Solution

$\underline{\underline{12}}$.

Kareen uses the same method to deduce the value of the gradient of the tangent at the point $(2, 16)$ on the curve $y = x^4$.

The first three lines of her working are given below.

Take P to be the point $(2, 16)$.
Take Q to be the point $(2 + h, (2 + h)^4)$.
The gradient of the chord PQ is given by

$$\frac{(2+h)^4 - 16}{h} =$$

(e) Complete Kareen's working. (3)

Solution

$\times$	4	$+4h$	$+h^2$
4	16	$+16h$	$+4h^2$
$+4h$	$+16h$	$+16h^2$	$+4h^3$
$+h^2$	$+4h^2$	$+4h^3$	$+h^4$

$$\begin{aligned}\frac{(2+h)^4 - 16}{h} &= \frac{(16 + 32h + 24h^2 + 8h^3 + h^4) - 16}{h} \\ &= \frac{32h + 24h^2 + 8h^3 + h^4}{h} \\ &= 32 + 24h + 8h^2 + h^3\end{aligned}$$

and, as $h \to 0$, the gradient tends to $\underline{\underline{32}}$.