Dr Oliver Mathematics
Applied Mathematics: Mechanics or Statistics
Section B
2008 Paper
1 hour

The total number of marks available is 32.
You must write down all the stages in your working.

1. Given that **A**, **B**, **C**, and **D** are square matrices where:

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 & 6 \\ 0 & -3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} x & 2 \\ 0 & y \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 2 & 7 \\ 12 & -1 \end{pmatrix}.$$

(a) Find **AB**. (1)

Solution

$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 4 & 6 \\ 0 & -3 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 8 & 15 \\ 12 & 3 \end{pmatrix}}}. \end{aligned}$$

(b) Express $4\mathbf{C} + \mathbf{D}$ as a single matrix. (2)

Solution

$$\begin{aligned} 4\mathbf{C} + \mathbf{D} &= 4 \begin{pmatrix} x & 2 \\ 0 & y \end{pmatrix} + \begin{pmatrix} 2 & 7 \\ 12 & -1 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 4x+2 & 15 \\ 12 & 4y-1 \end{pmatrix}}}. \end{aligned}$$

(c) Given that (2)

$$\mathbf{AB} = 4\mathbf{C} + \mathbf{D},$$

find the values of x and y.

Solution

$$
\begin{aligned}
4x + 2 = 8 &\Rightarrow 4x = 6 \\
&\Rightarrow \underline{\underline{x = 1\tfrac{1}{2}}} \\
4y - 1 = 3 &\Rightarrow 4y = 4 \\
&\Rightarrow \underline{\underline{y = 1}}.
\end{aligned}
$$

2. Given that (3)

$$y = \mathrm{e}^{2x} \cos x,$$

find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

$$
\begin{aligned}
u = \mathrm{e}^{2x} &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2\mathrm{e}^{2x} \\
v = \cos x &\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = -\sin x
\end{aligned}
$$

$$
\begin{aligned}
y = \mathrm{e}^{2x} \cos x &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = (\mathrm{e}^{2x})(-\sin x) + (2\mathrm{e}^{2x})(\cos x) \\
&\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{2x}(2\cos x - \sin x)}}.
\end{aligned}
$$

3. Express (4)

$$y = \frac{4x - 3}{x(x^2 + 3)}, \; x \neq 0,$$

in partial fractions.

Solution

$$
\begin{aligned}
\frac{4x - 3}{x(x^2 + 3)} &\equiv \frac{A}{x} + \frac{B + Cx}{x^2 + 3} \\
&\equiv \frac{A(x^2 + 3) + (B + Cx)x}{x(x^2 + 3)}
\end{aligned}
$$

which means

$$4x - 3 \equiv A(x^2 + 3) + (B + Cx)x.$$

$\underline{x = 0}$: $-3 = 3A \Rightarrow A = -1$.
$\underline{x = 1}$: $1 = 4A + B + C \Rightarrow B + C = 5 \quad (2)$.
$\underline{x = -1}$: $-7 = 4A - B + C \Rightarrow -B + C = -3 \quad (3)$.
Now, (1) + (2):

$$\begin{aligned} 2C = 2 &\Rightarrow C = 1 \\ &\Rightarrow B = 4. \end{aligned}$$

Finally,

$$y = \underline{\underline{-\frac{1}{x} + \frac{4 + x}{x^2 + 3}}}.$$

4. (a) Use integration by parts to show that (2)

$$\int \ln x \, \mathrm{d}x = x \ln x - x + c.$$

Solution

$$u = \ln x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = 1 \Rightarrow v = x$$

$$\begin{aligned} \int \ln x \, \mathrm{d}x &= \int \ln x \cdot 1 \, \mathrm{d}x \\ &= x \ln x - \int 1 \, \mathrm{d}x \\ &= \underline{\underline{x \ln x - x + c}}, \end{aligned}$$

as required.

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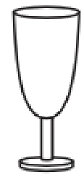
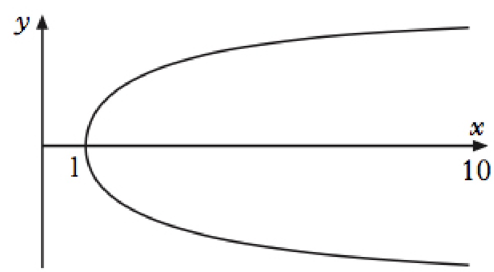
A goblet consists of a bowl and a short stem.



The diagram below shows the bowl section of the goblet (on its side).



The equation of the upper half of the curve is

$$y = 2\sqrt{\ln x}$$

for $1 \leqslant x \leqslant 10$.

(b) Given that the stem has length 1 and the overall height is 10, what is the capacity of the bowl? (4)

Solution

$$\begin{aligned} \text{Volume} &= \int_1^{10} \pi(2\sqrt{\ln x})^2 \, \mathrm{d}x \\ &= 4\pi \int_1^{10} \ln x \, \mathrm{d}x \\ &= 4\pi \left[x \ln x - x\right]_{x=1}^{10} \\ &= 4\pi \left[(10 \ln 10 - 10) - (0 - 1)\right] \\ &= \underline{\underline{4\pi\,(10 \ln 10 - 9)}}. \end{aligned}$$

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5. (a) Use the standard formulas for (3)

$$\sum_{r=1}^{n} r \text{ and } \sum_{r=1}^{n} r^2$$

to show that

$$\sum_{r=1}^{n}(6r^2 - r) = \tfrac{1}{2}n(n+1)(4n+1).$$

Solution

$$\begin{aligned}\sum_{r=1}^{n}(6r^2 - r) &= 6\sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r \\ &= n(n+1)(2n+1) - \tfrac{1}{2}n(n+1) \\ &= \tfrac{1}{2}n(n+1)[2(2n+1) - 1] \\ &= \underline{\underline{\tfrac{1}{2}n(n+1)(4n+1)}},\end{aligned}$$

as required.

(b) Hence evaluate (2)

$$\sum_{r=5}^{10}(6r^2 - r).$$

Solution

$$\begin{aligned}\sum_{r=5}^{10}(6r^2 - r) &= \sum_{r=1}^{10}(6r^2 - r) - \sum_{r=1}^{4}(6r^2 - r) \\ &= \tfrac{1}{2}(10)(11)(41) - \tfrac{1}{2}(4)(5)(17) \\ &= 2\,255 - 170 \\ &= \underline{\underline{2\,085}}.\end{aligned}$$

6. Newton's law of cooling states that a body loses heat at a rate which is proportional to the difference in temperature between itself and its surroundings. So, in a room with constant temperature 22°C, the temperature T°C of a body after a time t minutes satisfies

$$\frac{\mathrm{d}T}{\mathrm{d}t} = k(T - 22),$$

where k is a negative constant.

(a) Hence show that T can be expressed in the form (4)

$$T = A\mathrm{e}^{kt} + 22$$

for some arbitrary constant A.

Solution

$$\begin{aligned}
\frac{\mathrm{d}T}{\mathrm{d}t} = k(T-22) &\Rightarrow \frac{1}{(T-22)}\,\mathrm{d}T = k\,\mathrm{d}t \\
&\Rightarrow \int \frac{1}{(T-22)}\,\mathrm{d}T = \int k\,\mathrm{d}t \\
&\Rightarrow \ln(T-22) = kt + c \\
&\Rightarrow T - 22 = \mathrm{e}^{kt+c} \\
&\Rightarrow T - 22 = \mathrm{e}^{kt}\,\mathrm{e}^{c} \\
&\Rightarrow T - 22 = A\mathrm{e}^{kt} \text{ (for some constant } A) \\
&\Rightarrow \underline{\underline{T = A\mathrm{e}^{kt} + 22}},
\end{aligned}$$

as required.

In a restaurant, where the temperature remains constant at 22°C, a freshly baked roll, with temperature 82°C, is placed on a cooling tray. After 5 minutes, the temperature of the roll has fallen by 20°C.

(b) (i) Calculate the values of A and k. (2)

Solution

$$\begin{aligned}
t = 0, T = 82 &\Rightarrow 82 = A + 22 \\
&\Rightarrow \underline{\underline{A = 60}} \\
t = 5, T = 62 &\Rightarrow 62 = 60\mathrm{e}^{5k} + 22 \\
&\Rightarrow 40 = 60\mathrm{e}^{5k} \\
&\Rightarrow \mathrm{e}^{5k} = \tfrac{2}{3} \\
&\Rightarrow 5k = \ln\tfrac{2}{3} \\
&\Rightarrow \underline{\underline{k = \tfrac{1}{5}\ln\tfrac{2}{3}}}.
\end{aligned}$$

(ii) Write down an expression for the temperature of the roll after t minutes. (2)

Solution

$$\underline{\underline{T = 60e^{(\frac{1}{5}\ln\frac{2}{3})t} + 22}}.$$

(iii) Supposing the roll remains uneaten after a further 5 minutes, what will its temperature be? (1)

Solution

$$t = 10 \Rightarrow T = 60e^{2\ln\frac{2}{3}} + 22$$
$$\Rightarrow \underline{\underline{T = 48\tfrac{2}{3}{}^{\circ}\text{C}}}.$$

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