Dr Oliver Mathematics
Mathematics: Advanced Higher
2016 Paper
3 hours

The total number of marks available is 100.
You must write down all the stages in your working.

1. (a) Differentiate (2)

$$y = x \tan^{-1} 2x.$$

Solution

$$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$
$$v = \tan^{-1} 2x \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{2}{1 + (2x)^2}.$$

Now,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= x \cdot \frac{2}{1 + (2x)^2} + 1 \cdot \tan^{-1} 2x \\ &= \underline{\underline{\frac{2x}{1 + 4x^2} + \tan^{-1} 2x}}.\end{aligned}$$

(b) Given (3)

$$\mathrm{f}(x) = \frac{1 - x^2}{1 + 4x^2},$$

find $\mathrm{f}'(x)$, simplifying your answer.

Solution

$$u = 1 - x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -2x$$
$$v = 1 + 4x^2 \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 8x.$$

Now,

$$\begin{aligned}
f'(x) &= \frac{(1+4x^2)\cdot(-2x)-(1-x^2)\cdot(8x)}{(1+4x^2)^2} \\
&= \frac{(-2x-8x^3)-(8x-8x^3)}{(1+4x^2)^2} \\
&= \underline{\underline{\frac{-10x}{(1+4x^2)^2}}}.
\end{aligned}$$

A curve is given by the parametric equations

$$x = 6t \text{ and } y = 1 - \cos t.$$

(c) Find $\frac{dy}{dx}$ in terms of t. (3)

Solution

$$\begin{aligned}
x = 6t &\Rightarrow \frac{dx}{dt} = 6 \\
y = 1 - \cos t &\Rightarrow \frac{dy}{dt} = \sin t.
\end{aligned}$$

Now,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \\
&= \underline{\underline{\frac{\sin t}{6}}}.
\end{aligned}$$

2. A geometric sequence has second and fifth terms 108 and 4 respectively.

(a) Calculate the value of the common ratio. (3)

Solution

So,

$$\begin{aligned}
ar &= 108 \quad (1) \\
ar^4 &= 4 \quad (2).
\end{aligned}$$

Now, (2) ÷ (1):

$$r^3 = \tfrac{1}{27} \Rightarrow \underline{\underline{r = \tfrac{1}{3}}}.$$

(b) State why the associated geometric series has a sum to infinity. (1)

Solution

r is less than one in modulus, i.e., $\underline{\underline{|r| < 1}}$.

(c) Find the value of this sum to infinity. (2)

Solution

$$a = \frac{108}{\frac{1}{3}} = 324$$

and

$$\begin{aligned} S_\infty &= \frac{324}{1 - \frac{1}{3}} \\ &= \frac{324}{\frac{2}{3}} \\ &= \underline{\underline{486}}. \end{aligned}$$

3. (a) Write down and simplify the general term in the binomial expansion of (1)

$$\left(\frac{3}{x} - 2x\right)^{13}.$$

Solution

$$\text{General term} = \underline{\underline{\binom{13}{n}\left(\frac{3}{x}\right)^n (-2x)^{13-n}}}.$$

(b) Hence, or otherwise, find the term in x^9. (4)

Solution

$$\begin{aligned}
\text{General term} &= \binom{13}{n}\left(\frac{3}{x}\right)^n(-2x)^{13-n}\\
&= \binom{13}{n}\left(3x^{-1}\right)^n(-2x)^{13-n}\\
&= \binom{13}{n}3^n(-2)^{13-n}x^{-n}x^{13-n}\\
&= \binom{13}{n}3^n(-2)^{13-n}x^{13-2n}
\end{aligned}$$

and we want

$$9 = 13 - 2n \Rightarrow n = 2.$$

Hence, the term in x^9 is

$$\binom{13}{2}3^2(-2)^{11}x^9 = \underline{\underline{-1\,437\,696x^9}}.$$

4. Below is a system of equations: (4)

$$\begin{aligned}
x + 2y + 3z &= 3\\
2x - y + 4z &= 5\\
x - 3y + 2\lambda z &= 2.
\end{aligned}$$

Use Gaussian elimination to find the value of λ which leads to redundancy.

Solution

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 3\\ 2 & -1 & 4 & 5\\ 1 & -3 & 2\lambda & 2 \end{array}\right)$$

Do $R_2 - 2R_1$ and $rR_3 - R_1$:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 3\\ 0 & -5 & -2 & -1\\ 0 & -5 & 2\lambda - 3 & -1 \end{array}\right)$$

Now, the value of λ which leads to redundancy is

$$\begin{aligned}
2\lambda - 3 = -2 &\Rightarrow 2\lambda = 1\\
&\Rightarrow \underline{\underline{\lambda = \tfrac{1}{2}}}.
\end{aligned}$$

5. Prove by induction that (4)

$$\sum_{r=1}^{n} r(3r-1) = n^2(n+1),\ \forall n \in \mathbb{N}.$$

Solution

$\underline{n=1}$: LHS $= 1(3-1) = 2$ and RHS $= 1^2 \times 2 = 2$. So, it is true for $n = 1$.

Suppose that it is true for $n = k$, i.e.,

$$\sum_{r=1}^{k} r(3r-1) = k^2(k+1).$$

Then

$$\begin{aligned}\sum_{r=1}^{k+1} r(3r-1) &= \sum_{r=1}^{k} r(3r-1) + (k+1)[3(k+1)-1] \\ &= k^2(k+1) + (k+1)(3k+2) \text{ (by the inductive hypothesis)} \\ &= (k+1)(k^2+3k+2) \\ &= (k+1)(k+1)(k+2) \\ &= (k+1)^2[(k+1)+1],\end{aligned}$$

and so it is true for $n = k + 1$.

Hence, by mathematical induction, it is true for all $n \in \mathbb{N}$.

6. (a) Find Maclaurin expansions for $\sin 3x$ and e^{4x} up to and including the term in x^3. (4)

Solution

$$\begin{aligned}\sin 3x &= (3x) - \tfrac{1}{3!}(3x)^3 + \ldots \\ &= \underline{\underline{3x - \tfrac{9}{2}x^3 + \ldots}}\end{aligned}$$

and

$$\begin{aligned}e^{4x} &= 1 + (4x) + \tfrac{1}{2!}(4x)^2 + \tfrac{1}{3!}(4x)^3 + \ldots \\ &= \underline{\underline{1 + 4x + 8x^2 + \tfrac{32}{3}x^3 + \ldots}}.\end{aligned}$$

(b) Hence obtain an expansion for $e^{4x}\sin 3x$ up to and including the term in x^3. (2)

Solution

$\times$	1	$+4x$	$+8x^2$	$+\frac{32}{3}x^3$
$3x$	$3x$	$+12x^2$	$+24x^3$	$\ldots$
$-\frac{9}{2}x^3$	$-\frac{9}{2}x^3$	$\ldots$	$\ldots$	$\ldots$

Hence,

$$e^{4x}\sin 3x = \underline{\underline{3x + 12x^2 + \tfrac{39}{2}x^3 + \ldots}}.$$

7. $\mathbf{A}$ is the matrix

$$\begin{pmatrix} 2 & 0 \\ \lambda & -1 \end{pmatrix}.$$

(a) Find the determinant of matrix $\mathbf{A}$. (1)

Solution

$\underline{\underline{\det \mathbf{A} = -2}}$.

(b) Show that $\mathbf{A}^2$ can be expressed in the form $p\mathbf{A} + q\mathbf{I}$, stating the values of p and q. (3)

Solution

$$\begin{aligned}\mathbf{A}^2 &= \begin{pmatrix} 2 & 0 \\ \lambda & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ \lambda & -1 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 0 \\ \lambda & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 \\ \lambda & -1 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \\ &= \underline{\underline{\mathbf{A} + 2\mathbf{I}}};\end{aligned}$$

hence, $\underline{\underline{p = 1}}$ and $\underline{\underline{q = 2}}$.

(c) Obtain a similar expression for $\mathbf{A}^4$. (2)

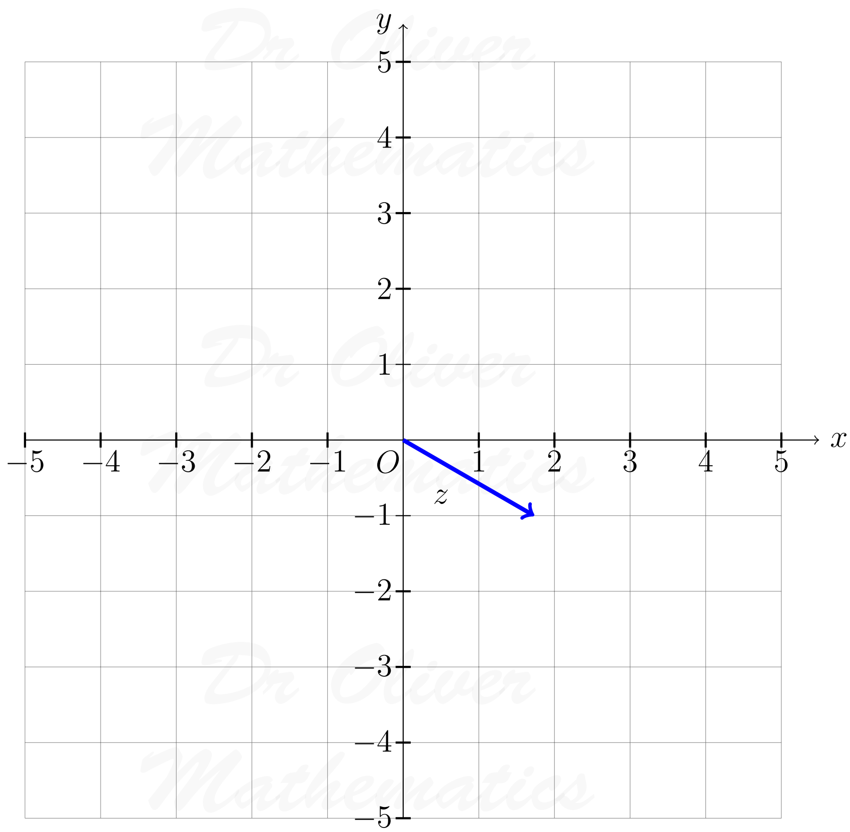
Solution

$$
\begin{aligned}
\mathbf{A}^4 &= (\mathbf{A}^2)^2 \\
&= (\mathbf{A} + 2\mathbf{I})^2 \\
&= \mathbf{A}^2 + 4\mathbf{A} + 4\mathbf{I} \\
&= (\mathbf{A} + 2\mathbf{I}) + 4\mathbf{A} + 4\mathbf{I} \\
&= \underline{\underline{5\mathbf{A} + 6\mathbf{I}}}.
\end{aligned}
$$

8. Let $z = \sqrt{3} - \mathrm{i}$.

(a) Plot z on an Argand diagram. (1)

Solution



Let $w = az$, where $a > 0$, $a \in \mathbb{R}$.

(b) Express w in polar form. (2)

Solution

$$\begin{aligned} w &= a(\sqrt{3} - \mathrm{i}) \\ &= 2a\left(\tfrac{\sqrt{3}}{2} - \tfrac{1}{2}\mathrm{i}\right) \\ &= 2a\left(\cos\tfrac{1}{6}\pi - \mathrm{i}\sin\tfrac{1}{6}\pi\right) \\ &= \underline{\underline{2a\left(\cos(-\tfrac{1}{6}\pi) + \mathrm{i}\sin(-\tfrac{1}{6}\pi)\right)}}. \end{aligned}$$

(c) Express w^8 in the form $ka^n(x + \mathrm{i}y)$ where $k, x, y \in \mathbb{Z}$. (3)

Solution

$$\begin{aligned} w &= \left[2a\left(\cos(-\tfrac{1}{6}\pi) + \mathrm{i}\sin(-\tfrac{1}{6}\pi)\right)\right]^8 \\ &= 64a^8\left(\cos(-\tfrac{4}{3}\pi) + \mathrm{i}\sin(-\tfrac{4}{3}\pi)\right) \\ &= 64a^8\left(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}\mathrm{i}\right) \\ &= \underline{\underline{32a^8\left(-1 + \sqrt{3}\mathrm{i}\right)}}; \end{aligned}$$

hence, for example, $\underline{\underline{k = 32}}$, $\underline{\underline{n = 8}}$, $\underline{\underline{x = -1}}$, and $\underline{\underline{y = \sqrt{3}}}$.

9. Obtain (6)

$$\int x^7 (\ln x)^2 \,\mathrm{d}x.$$

Solution

$$\begin{aligned} u = (\ln x)^2 &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2\ln x}{x} \\ \frac{\mathrm{d}v}{\mathrm{d}x} = x^7 &\Rightarrow v = \tfrac{1}{8}x^8 \end{aligned}$$

$$\int x^7 (\ln x)^2 \,\mathrm{d}x = \tfrac{1}{8}x^8(\ln x)^2 - \tfrac{1}{4}\int x^7 \ln x \,\mathrm{d}x$$

$$u = \ln x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = x^7 \Rightarrow v = \tfrac{1}{8}x^8$$

$$\begin{aligned} &= \tfrac{1}{8}x^8(\ln x)^2 - \tfrac{1}{4}\left[\tfrac{1}{8}x^8 \ln x - \tfrac{1}{8}\int x^7 \, \mathrm{d}x\right] \\ &= \tfrac{1}{8}x^8(\ln x)^2 - \tfrac{1}{32}x^8 \ln x + \tfrac{1}{32}\int x^7 \, \mathrm{d}x \\ &= \underline{\underline{\tfrac{1}{8}x^8(\ln x)^2 - \tfrac{1}{32}x^8 \ln x + \tfrac{1}{256}x^8 + c.}} \end{aligned}$$

10. For each of the following statements, decide whether it is true or false. If true, give a proof; if false, give a counterexample.

(a) If a positive integer p is prime, then so is $2p + 1$. (1)

Solution
False: $p = 7$ is prime but

$$\begin{aligned} 2p + 1 &= 2 \times 7 + 1 \\ &= 15 \\ &= 3 \times 5, \end{aligned}$$

which is composite.

(b) If a positive integer n has remainder 1 when divided by 3, then n^3 also has remainder 1 when divided by 3. (3)

Solution
True: Suppose n has remainder 1 when divided by 3: then $n = 3k + 1$ for some positive integer k. Then

$$\begin{aligned} n^3 &= (3k + 1)^3 \\ &= 27k^3 + 27k^2 + 9k + 1 \\ &= 3(9k^3 + 9k^2 + k) + 1 \\ &= 3 \times \text{some constant} + 1. \end{aligned}$$

11. The height of a cube is increasing at the rate of 5 $\text{cm}\,\text{s}^{-1}$. (4)
Find the rate of increase of the volume when the height of the cube is 3 cm.

Solution

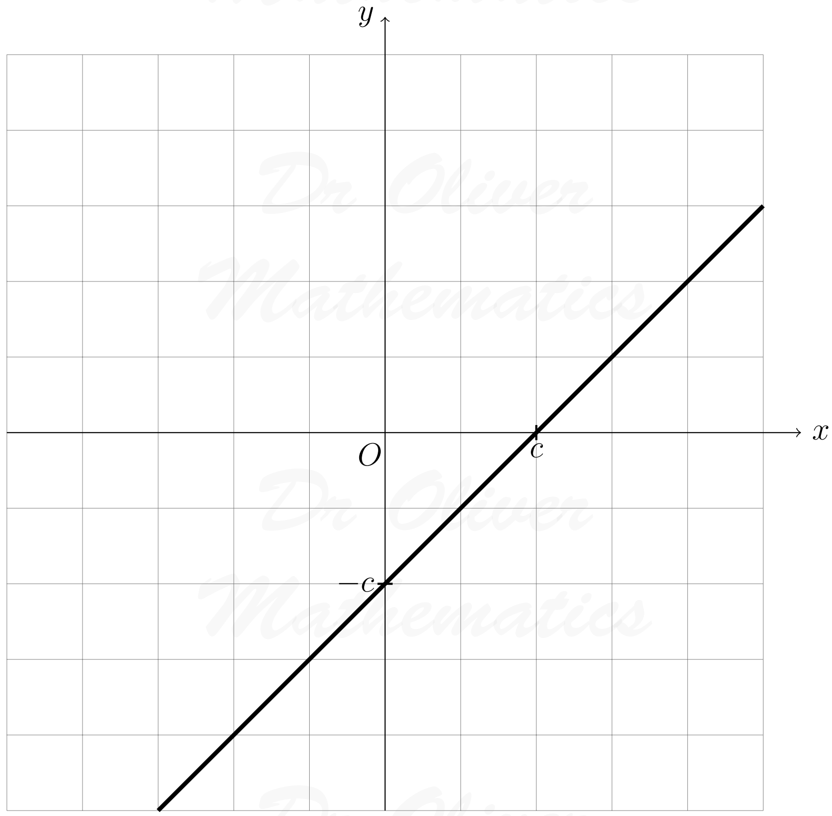
So,

$$\frac{\mathrm{d}h}{\mathrm{d}t} = 5 \text{ and } V = h^3 \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}h} = 3h^2.$$

Finally,

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &= \frac{\mathrm{d}V}{\mathrm{d}h} \times \frac{\mathrm{d}h}{\mathrm{d}t} \\ &= 3 \cdot 3^2 \times 5 \\ &= \underline{\underline{135 \text{ cm}^3\,\text{s}^{-1}}}.\end{aligned}$$

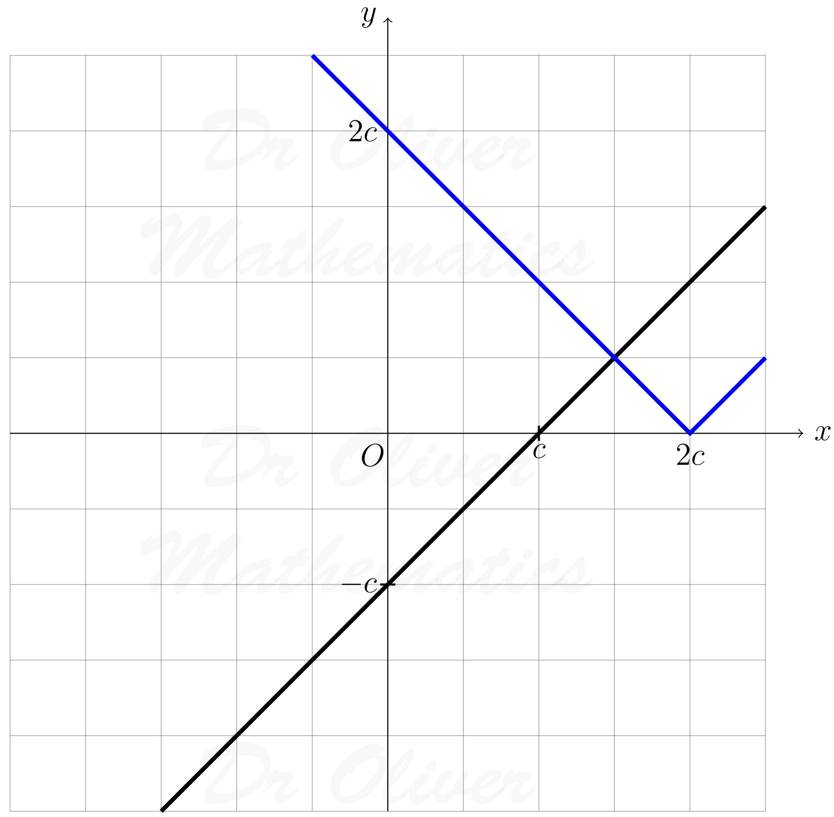
12. Below is a diagram showing the graph of a linear function, $y = \mathrm{f}(x)$.



On separate diagrams show:

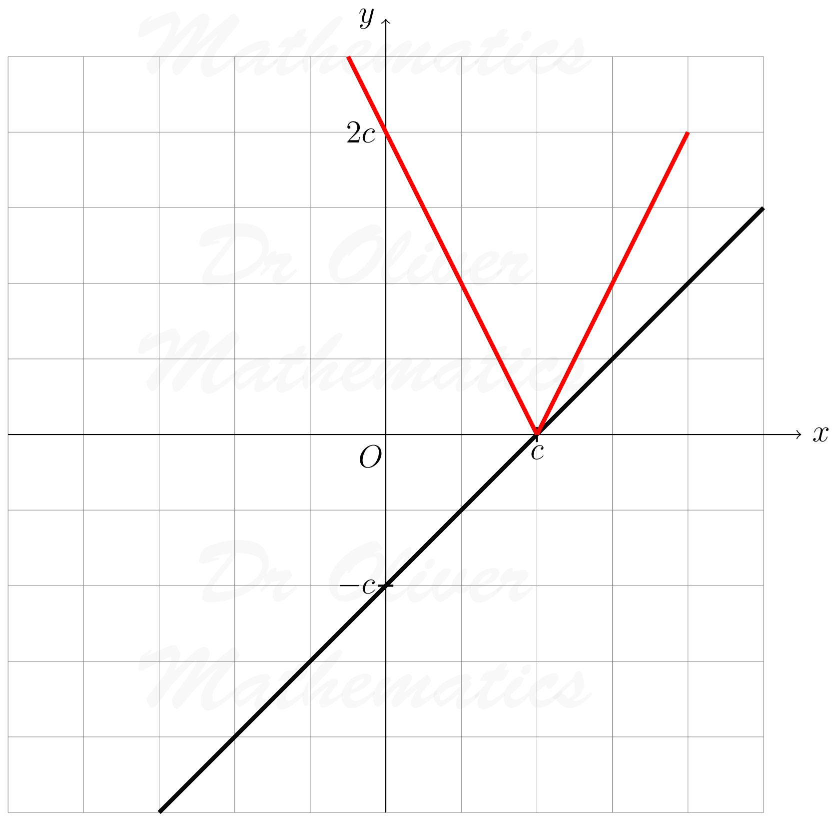
(a) $y = |\,\mathrm{f}(x) - c|$, (2)

Solution



(b) $y = |2\,\mathrm{f}(x)|$. (2)

Solution



13. (a) Express (4)

$$\frac{3x+32}{(x+4)(6-x)}$$

in partial fractions.

Solution

$$\frac{3x+32}{(x+4)(6-x)} \equiv \frac{A}{x+4} + \frac{B}{6-x}$$
$$\equiv \frac{A(6-x)+B(x+4)}{(x+4)(6-x)}$$

and so

$$3x+32 \equiv A(6-x)+B(x+4).$$

$\underline{x=6}$: $50 = 10B \Rightarrow B = 5$.
$\underline{x=-4}$: $20 = 10A \Rightarrow A = 2$.
Hence,

$$\frac{3x+32}{(x+4)(6-x)} = \underline{\underline{\frac{2}{x+4} + \frac{5}{6-x}}}.$$

(b) Hence evaluate (5)

$$\int_3^4 \frac{3x+32}{(x+4)(6-x)}\,\mathrm{d}x.$$

Give your answer in the form $\ln\left(\dfrac{p}{q}\right)$.

Solution

$$\begin{aligned}
\int_3^4 \frac{3x+32}{(x+4)(6-x)}\,\mathrm{d}x &= \int_3^4 \left(\frac{2}{x+4}+\frac{5}{6-x}\right)\,\mathrm{d}x \\
&= \left[2\ln|x+4| - 5\ln|6-x|\right]_{x=3}^{4} \\
&= (2\ln 8 - 5\ln 2) - (2\ln 7 - 5\ln 3) \\
&= (\ln 64 - \ln 32) - (\ln 49 - \ln 243) \\
&= \ln\left(\frac{64}{32}\right) - \ln\left(\frac{49}{243}\right) \\
&= \ln 2 - \ln\left(\frac{49}{243}\right) \\
&= \ln\left(\frac{2}{\frac{49}{243}}\right) \\
&= \underline{\underline{\ln\left(\frac{486}{49}\right)}};
\end{aligned}$$

hence, $\underline{\underline{p = 486}}$ and $\underline{\underline{q = 49}}$.

14. Two lines L_1 and L_2 are given by the equations:

$$L_1: \quad x = 4 + 3\lambda,\ y = 2 + 4\lambda,\ z = -7\lambda$$
$$L_2: \quad \frac{x-3}{-2} = \frac{y-8}{1} = \frac{z+1}{3}.$$

(a) Show that the lines L_1 and L_2 intersect and find the point of intersection. (5)

Solution

Let L_2 be

$$\frac{x-3}{-2} = \frac{y-8}{1} = \frac{z+1}{3} = \mu$$

and then

$$\begin{aligned} 4 + 3\lambda &= 3 - 2\mu \quad (1) \\ 2 + 4\lambda &= 8 + \mu \quad (2) \\ -7\lambda &= -1 + 3\mu \quad (3). \end{aligned}$$

Do $2 \times (2)$:

$$4 + 8\lambda = 16 + 2\mu \quad (4)$$

and do $(1) + (4)$:

$$\begin{aligned} 8 + 11\lambda = 19 &\Rightarrow 11\lambda = 11 \\ &\Rightarrow \lambda = 1 \\ &\Rightarrow 7 = 3 - 2\mu \\ &\Rightarrow 2\mu = -4 \\ &\Rightarrow \mu = -2. \end{aligned}$$

Check in (3):

$$-7\lambda = -7 \text{ and } -1 + 3\mu = -7. \quad \checkmark$$

So, the lines <u>intersect</u> and the point of intersection is <u>$(7, 6, -7)$</u>.

(b) Calculate the obtuse angle between the lines L_1 and L_2. (4)

Solution

Let θ° be the angle between the two lines. Now,

$$|3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}| = \sqrt{74}$$

and

$$|-2\mathbf{i} + \mathbf{j} + 3\mathbf{k}| = \sqrt{14}.$$

Finally,

$$\begin{aligned} & (3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}).(-2\mathbf{i} + \mathbf{j} + 3\mathbf{k} = |3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}||-2\mathbf{i} + \mathbf{j} + 3\mathbf{k}| \cos\theta^\circ \\ \Rightarrow \quad & -6 + 4 - 21 = \sqrt{74} \cdot \sqrt{14} \cdot \cos\theta^\circ \\ \Rightarrow \quad & \cos\theta^\circ = -\frac{23}{\sqrt{74} \cdot \sqrt{14}} \\ \Rightarrow \quad & \theta = 135.608\,398\,6 \text{ (FCD)} \\ \Rightarrow \quad & \underline{\underline{\theta = 136 \text{ (3 sf)}}}. \end{aligned}$$

15. Solve the differential equation (10)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 12x^2 + 2x - 5,$$

given $y = -6$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 3$ when $x = 0$.

Solution

Complementary function:

$$m^2 + 5m + 6 = 0 \Rightarrow (m+2)(m+3) = 0 \Rightarrow m = -3, -2$$

and hence the complementary function is

$$y = A\mathrm{e}^{-3x} + B\mathrm{e}^{-2x}.$$

Particular integral: try

$$\begin{aligned} y = Cx^2 + Dx + E \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= 2Cx + D \\ \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 2C. \end{aligned}$$

Now,

$$\begin{aligned} & \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 12x^2 + 2x - 5 \\ \Rightarrow \quad & 2C + 5(2Cx + D) + 6(Cx^2 + Dx + E) = 12x^2 + 2x - 5 \\ \Rightarrow \quad & 6Cx^2 + (10C + 6D)x + (2C + 5D + 6E) = 12x^2 + 2x - 5. \end{aligned}$$

Next,

$$\begin{aligned} 6C = 12 &\Rightarrow C = 2 \\ 10C + 6D = 2 &\Rightarrow 6D = -18 \\ &\Rightarrow D = -3 \\ 2C + 5D + 6E = -5 &\Rightarrow 6E = 6 \\ &\Rightarrow E = 1. \end{aligned}$$

The particular integral is $y = 2x^2 - 3x + 1$.

The general solution is

$$y = A\mathrm{e}^{-3x} + B\mathrm{e}^{-2x} + 2x^2 - 3x + 1.$$

Now,

$$x = 0, y = -6 \Rightarrow -6 = A + B + 1$$
$$\Rightarrow A + B = -7. \quad (1)$$

Next,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -3A\mathrm{e}^{-3x} - 2B\mathrm{e}^{-2x} + 4x - 3$$

and

$$x = 0, \frac{\mathrm{d}y}{\mathrm{d}x} = 3 \Rightarrow 3 = -3A - 2B - 3$$
$$\Rightarrow -3A - 2B = 6. \quad (2)$$

Do $2 \times (1)$:

$$2A + 2B = -14$$

and (2) + (3):

$$-A = -14 \Rightarrow A = 8$$
$$\Rightarrow B = -15.$$

Hence,

$$\underline{\underline{y = 8\mathrm{e}^{-3x} - 15\mathrm{e}^{-2x} + 2x^2 - 3x + 1}}.$$

16. A beaker of liquid was placed in a fridge. (9)
The rate of cooling is given by

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -k(T - T_F),\ k > 0,$$

where T_F is the constant temperature in the fridge and T is the temperature of the liquid at time t.

- The constant temperature in the fridge is 4°C.
- When first placed in the fridge, the temperature of the liquid was 25°C.
- At 12 noon, the temperature of the liquid was 9.8°C.
- At 12:15 pm, the temperature of the liquid had dropped to 6.5°C.

At what time, to the nearest minute, was the liquid placed in the fridge?

Solution

$$
\begin{aligned}
\frac{\mathrm{d}T}{\mathrm{d}t} = -k(T-4) &\Rightarrow \frac{1}{T-4}\,\mathrm{d}T = -k\,\mathrm{d}t \\
&\Rightarrow \int \frac{1}{T-4}\,\mathrm{d}T = -\int k\,\mathrm{d}t \\
&\Rightarrow \ln(T-4) = -kt + c \text{ (for some constant } c) \\
&\Rightarrow T - 4 = \mathrm{e}^{-kt+c} \\
&\Rightarrow T - 4 = \mathrm{e}^{-kt}\mathrm{e}^{c} \\
&\Rightarrow T = A\mathrm{e}^{-kt} + 4 \text{ (for some constant } A).
\end{aligned}
$$

Now,

$$
\begin{aligned}
9.8 = A + 4 &\Rightarrow A = 5.8 \\
6.5 = 5.8\mathrm{e}^{-0.25k} + 4 &\Rightarrow 5.8\mathrm{e}^{-0.25k} = 2.5 \\
&\Rightarrow \mathrm{e}^{-0.25k} = \tfrac{25}{58} \\
&\Rightarrow -0.25k = \ln\tfrac{25}{58} \\
&\Rightarrow k = -4\ln\tfrac{25}{58}.
\end{aligned}
$$

Hence,

$$T = 5.8\mathrm{e}^{\left(4\ln\frac{25}{58}\right)t} + 4.$$

Finally,

$$
\begin{aligned}
25 = 5.8\mathrm{e}^{\left(4\ln\frac{25}{58}\right)t} + 4 &\Rightarrow 21 = 5.8\mathrm{e}^{\left(4\ln\frac{25}{58}\right)t} \\
&\Rightarrow \tfrac{105}{29} = \mathrm{e}^{\left(4\ln\frac{25}{58}\right)t} \\
&\Rightarrow \ln\tfrac{105}{29} = \left(4\ln\tfrac{25}{58}\right)t \\
&\Rightarrow t = \frac{\ln\frac{105}{29}}{4\ln\frac{25}{58}} \\
&\Rightarrow t = -0.382\,222\,757\,2 \text{ hours (FCD)} \\
&\Rightarrow t = -22.933\,365\,43 \text{ minutes (FCD)};
\end{aligned}
$$

to the nearest minute, the liquid placed in the fridge at 11 : 37 am.