Dr Oliver Mathematics
Mathematics: Higher
2016 Paper 1: Non-Calculator
1 hour 10 minutes

The total number of marks available is 60.
You must write down all the stages in your working.

1. Find the equation of the line passing through the point $(-2, 3)$ which is parallel to the line with equation $y + 4x = 7$. (2)

Solution

Well, the equation is

$$y + 4x = c,$$

for some constant c. Using $(-2, 3)$,

$$y + 4x = 3 + 4(-2) \Rightarrow \underline{\underline{y + 4x = -5}}.$$

2. Given that (3)

$$y = 12x^3 + 8\sqrt{x}, \text{ where } x > 0,$$

find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

$$\begin{aligned} y = 12x^3 + 8\sqrt{x} &\Rightarrow y = 12x^3 + 8x^{\frac{1}{2}} \\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = 36x^2 + 4x^{-\frac{1}{2}}}}. \end{aligned}$$

3. A sequence is defined by the recurrence relation

$$u_{n+1} = \tfrac{1}{3}u_n + 10 \text{ with } u_3 = 6.$$

(a) Find the value of u_4. (1)

Solution

$$\begin{aligned} u_4 &= \tfrac{1}{3}u_3 + 10 \\ &= \tfrac{1}{3}(6) + 10 \\ &= 2 + 10 \\ &= \underline{\underline{12}}. \end{aligned}$$

(b) Explain why this sequence approaches a limit as $n \to \infty$. (1)

Solution

A limit exists as the recurrence relation is linear and $-1 < \frac{1}{3} < 1$.

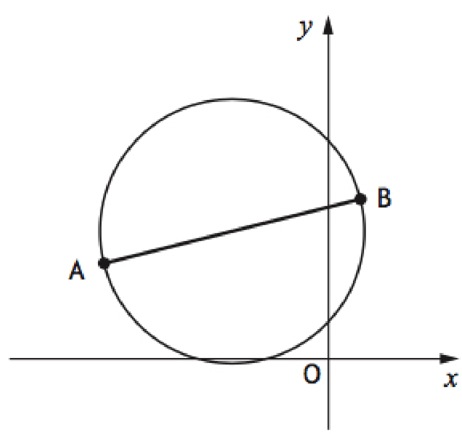
(c) Calculate this limit. (2)

Solution

Let the limit be u. Then

$$\begin{aligned} u = \tfrac{1}{3}u + 10 &\Rightarrow \tfrac{2}{3}u = 10 \\ &\Rightarrow \underline{\underline{u = 15}}. \end{aligned}$$

4. A and B are the points $(-7, 3)$ and $(1, 5)$. (3)
AB is a diameter of a circle.



Find the equation of this circle.

Solution

The centre is at

$$\left(\frac{-7+1}{2}, \frac{3+5}{2}\right) = (-3, 4)$$

and the radius is

$$\begin{aligned}\sqrt{[(1-(-3)]^2 + (5-4)^2} &= \sqrt{4^2+1^2} \\ &= \sqrt{17}.\end{aligned}$$

Hence, the equation of this circle is

$$\underline{\underline{(x+3)^2 + (y-4)^2 = 17}}.$$

5. Find (2)

$$\int 8\cos(4x+1)\,\mathrm{d}x.$$

Solution

$$\int 8\cos(4x+1)\,\mathrm{d}x = \underline{\underline{2\sin(4x+1)+c}}.$$

6. Functions f and g are defined on $\mathbb{R}$, the set of real numbers.
The inverse functions f^{-1} and g^{-1} both exist.

(a) Given (3)

$$\mathrm{f}(x) = 3x+5,$$

find $\mathrm{f}^{-1}(x)$.

Solution

$$\begin{aligned}y = 3x+5 &\Rightarrow 3x = y-5 \\ &\Rightarrow x = \tfrac{1}{3}(y-5);\end{aligned}$$

hence,

$$\mathrm{f}^{-1}(x) = \underline{\underline{\tfrac{1}{3}(x-5)}}.$$

(b) If (1)

$$g(2) = 7,$$

write down the value of $g^{-1}(7)$.

Solution

$g^{-1}(7) = \underline{\underline{2}}$.

7. Three vectors can be expressed as follows:

- $\overrightarrow{FG} = -2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$,
- $\overrightarrow{GH} = 3\mathbf{i} + 9\mathbf{j} - 7\mathbf{k}$, and
- $\overrightarrow{EH} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

(a) Find $\overrightarrow{FH}$. (2)

Solution

$$\begin{aligned}\overrightarrow{FH} &= \overrightarrow{FG} + \overrightarrow{GH} \\ &= (-2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}) + (3\mathbf{i} + 9\mathbf{j} - 7\mathbf{k}) \\ &= \underline{\underline{\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}}}.\end{aligned}$$

(b) Hence, or otherwise, find $\overrightarrow{FE}$. (2)

Solution

$$\begin{aligned}\overrightarrow{FE} &= \overrightarrow{FH} + \overrightarrow{HE} \\ &= (\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) - (2\mathbf{i} + 3\mathbf{j} + \mathbf{k}) \\ &= \underline{\underline{-\mathbf{i} - 5\mathbf{k}}}.\end{aligned}$$

8. Show that the line with equation (5)

$$y = 3x - 5$$

is a tangent to the circle with equation

$$x^2 + y^2 + 2x - 4y - 5 = 0$$

and find the coordinates of the point of contact.

Solution

$$\begin{aligned} & x^2 + y^2 + 2x - 4y - 5 = 0 \\ \Rightarrow \quad & x^2 + (3x-5)^2 + 2x - 4(3x-5) - 5 = 0 \end{aligned}$$

×	$3x$	-5
$3x$	$9x^2$	$-15x$
-5	$-15x$	$+25$

$$\begin{aligned} \Rightarrow \quad & x^2 + (9x^2 - 30x + 25) + 2x - 12x + 20 - 5 = 0 \\ \Rightarrow \quad & 10x^2 - 40x + 40 = 0 \\ \Rightarrow \quad & 10(x^2 - 4x + 4) = 0 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & -4 \\ \text{multiply to:} & +4 \end{array}\right\} -2, -2$$

$$\begin{aligned} \Rightarrow \quad & 10(x-2)^2 = 0 \\ \Rightarrow \quad & x = 2 \text{ (repeated, so a tangent);} \end{aligned}$$

hence, $y = 3x - 5$ is a tangent to the circle and the coordinates of the point of contact is $(2, 1)$.

9. (a) Find the x-coordinates of the stationary points on the graph with equation $y = \mathrm{f}(x)$, where (4)

$$\mathrm{f}(x) = x^3 + 3x^2 - 24x.$$

Solution

$$\mathrm{f}(x) = x^3 + 3x^2 - 24x \Rightarrow \mathrm{f}'(x) = 3x^2 + 6x - 24.$$

Now,

$$\begin{aligned} \mathrm{f}'(0) = 0 &\Rightarrow 3x^2 + 6x - 24 = 0 \\ &\Rightarrow 3(x^2 + 2x - 8) = 0 \end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & +2 \\ \text{multiply to:} & -8\end{array}\right\} +4,\ -2$$

$$\begin{aligned} &\Rightarrow 3(x+4)(x-2) = 0 \\ &\Rightarrow \underline{\underline{x = -4 \text{ or } x = 2}}. \end{aligned}$$

(b) Hence determine the range of values of x for which the function f is strictly increasing. (2)

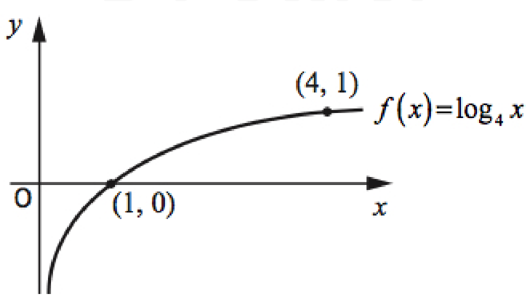
Solution

$$\begin{aligned} \mathrm{f}'(0) > 0 &\Rightarrow 3(x+4)(x-2) > 0 \\ &\Rightarrow \underline{\underline{x < -4 \text{ or } x > 2}}. \end{aligned}$$

10. The diagram below shows the graph of the function (2)

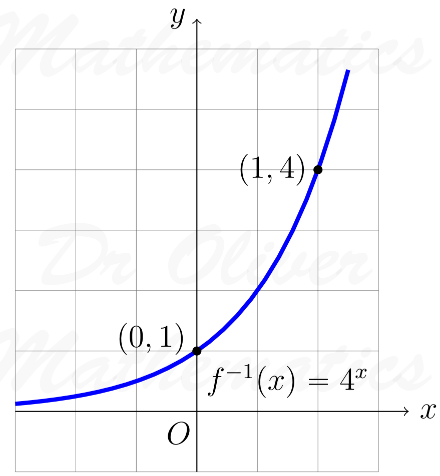
$$\mathrm{f}(x) = \log_4 x,$$

where $x > 0$.



The inverse function, f^{-1}, exists.
Sketch the graph of the inverse function.

Solution



11. A and C are the points $(1, 3, -2)$ and $(4, -3, 4)$ respectively.
Point B divides AC in the ratio $1 : 2$.

(a) Find the coordinates of B. (2)

Solution

$$\begin{aligned}\overrightarrow{OB} &= \overrightarrow{OA} + \tfrac{1}{3}\overrightarrow{AB}\\ &= (\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) + \tfrac{1}{3}[(4\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) - (\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})]\\ &= (\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) + \tfrac{1}{3}(3\mathbf{i} - 6\mathbf{j} + 6\mathbf{k})\\ &= (\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) + (\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})\\ &= 2\mathbf{i} + \mathbf{j};\end{aligned}$$

hence, $\underline{\underline{B(2, 1, 0)}}$.

$k\overrightarrow{AC}$ is a vector of magnitude 1, where $k > 0$.

(b) Determine the value of k. (3)

Solution

$$\begin{aligned}|\overrightarrow{AC}| &= \sqrt{3^2 + (-6)^2 + 6^2}\\ &= \sqrt{9 + 36 + 36}\\ &= \sqrt{81}\\ &= \pm\, 9;\end{aligned}$$

hence, as $k > 0$, $\underline{\underline{k = \frac{1}{9}}}$.

12. The functions f and g are defined on $\mathbb{R}$, the set of real numbers, by

$$\mathrm{f}(x) = 2x^2 - 4x + 5 \text{ and } \mathrm{g}(x) = 3 - x.$$

(a) Given (2)

$$\mathrm{h}(x) = \mathrm{f}(\mathrm{g}(x)),$$

show that

$$\mathrm{h}(x) = 2x^2 - 8x + 11.$$

Solution

$$\begin{aligned}\mathrm{h}(x) = \mathrm{f}(\mathrm{g}(x)) &\Rightarrow \mathrm{h}(x) = \mathrm{f}(3 - x) \\ &\Rightarrow \mathrm{h}(x) = 2(3 - x)x^2 - 4(3 - x) + 5\end{aligned}$$

$\times$	3	$-x$
3	9	$-3x$
$-x$	$-3x$	$+x^2$

$$\begin{aligned}&\Rightarrow \mathrm{h}(x) = 2(9 - 6x + x^2) - 12 + 4x + 5 \\ &\Rightarrow \mathrm{h}(x) = (18 - 12x + 2x^2) - 12 + 4x + 5 \\ &\Rightarrow \underline{\underline{\mathrm{h}(x) = 2x^2 - 8x + 11}},\end{aligned}$$

as required.

(b) Express $\mathrm{h}(x)$ in the form (3)

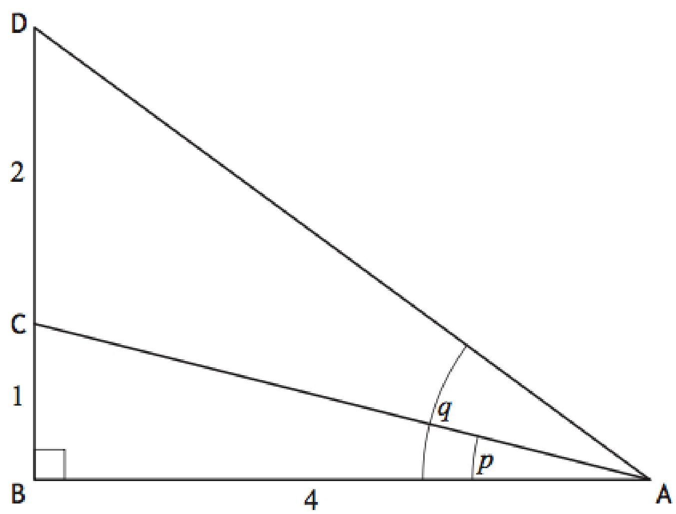
$$p(x + q)^2 + r.$$

Solution

$$\begin{aligned}
\mathrm{h}(x) &= 2x^2 - 8x + 11 \\
&= 2(x^2 - 4x) + 11 \\
&= 2[(x^2 - 4x + 4) - 4] + 11 \\
&= 2[(x-2)^2 - 4] + 11 \\
&= 2(x-2)^2 - 8 + 11 \\
&= \underline{\underline{2(x-2)^2 + 3}};
\end{aligned}$$

hence, $\underline{\underline{p = 2}}$, $\underline{\underline{q = -4}}$, and $\underline{\underline{r = 3}}$.

13. Triangle ABD is right-angled at B with angles $BAC = p$ and $BAD = q$ and lengths as shown in the diagram below. (5)



Show that the exact value of $\cos(q - p)$ is

$$\frac{19\sqrt{17}}{85}.$$

Solution

Now,

$$\begin{aligned} AD &= \sqrt{3^2 + 4^2} \\ &= \sqrt{9 + 16} \\ &= 25 \end{aligned}$$

and

$$\begin{aligned} AC &= \sqrt{1^2 + 4^2} \\ &= \sqrt{1 + 16} \\ &= \sqrt{17}. \end{aligned}$$

Finally,

$$\begin{aligned} \cos(q - p) &= \cos q \cos p + \sin q \sin p \\ &= (\tfrac{4}{5} \times \tfrac{4}{\sqrt{17}}) + (\tfrac{3}{5} \times \tfrac{1}{\sqrt{17}}) \\ &= \tfrac{16}{5\sqrt{17}} + \tfrac{3}{5\sqrt{17}} \\ &= \tfrac{19}{5\sqrt{17}} \\ &= \tfrac{19}{5\sqrt{17}} \times \tfrac{\sqrt{17}}{\sqrt{17}} \\ &= \underline{\underline{\tfrac{19\sqrt{17}}{85}}}. \end{aligned}$$

14. (a) Evaluate (1)

$$\log_5 25.$$

Solution

$$\begin{aligned} \log_5 25 &= \log_5 5^2 \\ &= 2 \log_5 5 \\ &= \underline{\underline{2}}. \end{aligned}$$

(b) Hence, solve (5)

$$\log_4 x + \log_4(x - 6) = \log_5 25,$$

where $x > 6$.

Solution

$$\begin{aligned}\log_4 x + \log_4(x-6) = \log_5 25 &\Rightarrow \log_4 x(x-6) = 2\\ &\Rightarrow x(x-6) = 4^2\\ &\Rightarrow x^2 - 6x = 16\\ &\Rightarrow x^2 - 6x - 16 = 0\end{aligned}$$

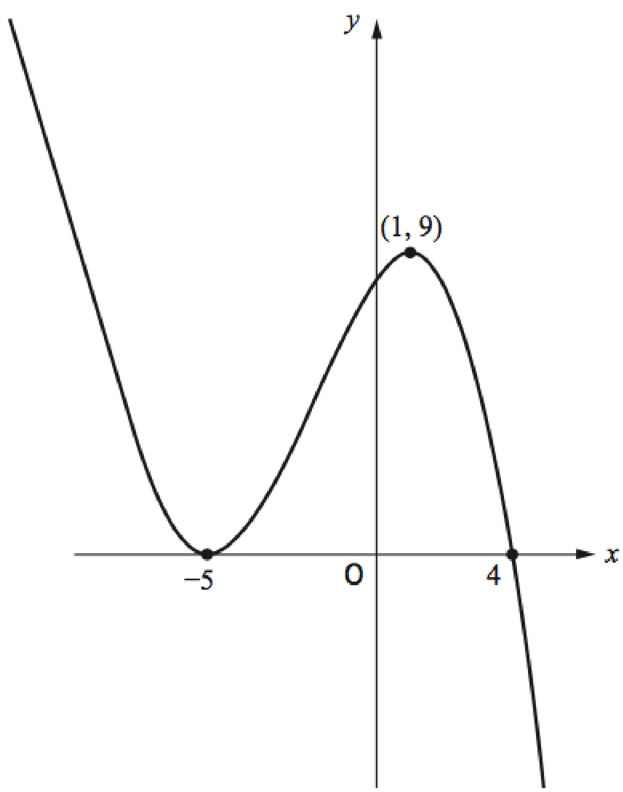
$$\left.\begin{array}{ll}\text{add to:} & -6\\ \text{multiply to:} & -16\end{array}\right\} -8,\ +2$$

$$\begin{aligned}&\Rightarrow (x-8)(x+2) = 0\\ &\Rightarrow x = -2 \text{ or } x = 8;\end{aligned}$$

hence, $x > 6$, $\underline{\underline{x = 8}}$.

15. The diagram below shows the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = k(x-a)(x-b)^2.$$



(a) Find the values of a, b, and k. (3)

Solution

$\underline{\underline{a = 4}}$, $\underline{\underline{b = -5}}$ (from the 'bounce'), and

$$\begin{aligned} 9 = k(1-4)[(1-(-5))]^2 &\Rightarrow 9 = k(-3)(6)^2 \\ &\Rightarrow 9 = -108k \\ &\Rightarrow \underline{\underline{k = -\tfrac{1}{12}}}. \end{aligned}$$

(b) For the function (1)

$$g(x) = f(x) - d,$$

where d is positive, determine the range of values of d for which $g(x)$ has exactly one real root.

Solution

Now, $d > 0$ and that means $\underline{\underline{d > 9}}$.