Dr Oliver Mathematics
Mathematics
Vectors
Past Examination Questions

This booklet consists of 27 questions across a variety of examination topics.
The total number of marks available is 315.

1. The line l_1 has vector equation

$$\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$$

and line l_2 has vector equation

$$\mathbf{r} = \begin{pmatrix} 0 \\ 4 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$$

where λ and μ are parameters.

The lines l_1 and l_2 intersect at the point B and the acute angle between l_1 and l_2 is θ.

(a) Find the coordinates of B. (4)

Solution

Well,

$$3 + \lambda = \mu \quad (1)$$
$$1 - \lambda = 4 - \mu \quad (2)$$
$$2 + 4\lambda = -2 \quad (3).$$

In (3),

$$\begin{aligned} 2 + 4\lambda = -2 &\Rightarrow 4\lambda = -4 \\ &\Rightarrow \lambda = -1, \end{aligned}$$

and, from (1),

$$\mu = 2.$$

What about (2)? Now,

$$1 - (-1) = 2 \text{ and } 4 - 2 = 2 \checkmark.$$

Finally,

$$\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix},$$

and the coordinates of $\underline{\underline{B(2, 2, -2)}}$.

(b) Find the value of $\cos\theta$, giving your answer as a simplified fraction. (4)

Solution

Well,

$$\begin{aligned} \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} &= 1 + 1 + 0 \\ &= 2, \end{aligned}$$

$$\left| \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} \right| = \sqrt{1^2 + (-1)^2 + 4^2} = 3\sqrt{2},$$

and

$$\left| \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right| = \sqrt{1^2 + (-1)^2 + 0} = \sqrt{2}.$$

Finally,

$$2 = 3\sqrt{2} \times \sqrt{2} \times \cos\theta \Rightarrow \underline{\underline{\cos\theta = \tfrac{1}{3}}}.$$

- The point A, which lies on l_1, has position vector $\mathbf{a} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.
- The point C, which lies on l_2, has position vector $\mathbf{c} = 5\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.
- The point D is such that $ABCD$ is a parallelogram.

(c) Show that (3)

$$|\overrightarrow{AB}| = |\overrightarrow{BC}|.$$

Solution

Well,

$$\begin{aligned}|\overrightarrow{AB}| &= |(2\mathbf{i}+2\mathbf{j}-2\mathbf{k})-(3\mathbf{i}+\mathbf{j}+2\mathbf{k})| \\ &= |-\mathbf{i}+\mathbf{j}-4\mathbf{k}| \\ &= \sqrt{(-1)^2+1^2+(-4)^2} \\ &= 3\sqrt{2}\end{aligned}$$

and

$$\begin{aligned}|\overrightarrow{BC}| &= |(5\mathbf{i}-\mathbf{j}-2\mathbf{k})-(2\mathbf{i}+2\mathbf{j}-2\mathbf{k})| \\ &= |3\mathbf{i}-3\mathbf{j}| \\ &= \sqrt{3^2+(-3)^2} \\ &= 3\sqrt{2};\end{aligned}$$

hence, $\underline{\underline{|\overrightarrow{AB}| = |\overrightarrow{BC}|}}$, as required.

(d) Find the position vector of the point D. (2)

Solution
Well,

$$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$$

because $ABCD$ is a parallelogram:

$$\begin{aligned}&= \overrightarrow{OA} + \overrightarrow{BC} \\ &= (3\mathbf{i}+\mathbf{j}+2\mathbf{k})+(3\mathbf{i}-3\mathbf{j}) \\ &= 6\mathbf{i}-2\mathbf{j}+2\mathbf{k};\end{aligned}$$

hence, $\underline{\underline{D(6,-2,2)}}$.

2. The line l_1 has vector equation

$$\mathbf{r} = 8\mathbf{i}+12\mathbf{j}+14\mathbf{k}+\lambda(\mathbf{i}+\mathbf{j}-\mathbf{k}),$$

where λ is a parameter.

- The point A has coordinates $(4, 8, a)$, where a is a constant.
- The point B has coordinates $(b, 13, 13)$, where b is a constant.

- Points A and B lie on the line l_1.

(a) Find the values of a and b. (3)

Solution

For the **i**th component of A,

$$8 + \lambda = 4 \Rightarrow \lambda = -4,$$

and

$$\begin{aligned} a &= 14 - \lambda \\ &= 14 - (-4) \\ &= \underline{\underline{18}}. \end{aligned}$$

For the **j**th component of B,

$$12 + \lambda = 13 \Rightarrow \lambda = 1,$$

and

$$\begin{aligned} b &= 8 + \lambda \\ &= 8 + 1 \\ &= \underline{\underline{9}}. \end{aligned}$$

Given that the point O is the origin, and that the point P lies on l_1 such that OP is perpendicular to l_1,

(b) find the coordinates of P. (5)

Solution

Well,

$$\begin{aligned} \mathbf{r} &= 8\mathbf{i} + 12\mathbf{j} + 14\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} - \mathbf{k}) \\ &= (8 + \lambda)\mathbf{i} + (12 + \lambda)\mathbf{j} + (14 - \lambda)\mathbf{k}, \end{aligned}$$

and

$$\begin{pmatrix} 8+\lambda \\ 12+\lambda \\ 14-\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0 \Rightarrow (8+\lambda) + (12+\lambda) - (14-\lambda) = 0$$
$$\Rightarrow 3\lambda + 6 = 0$$
$$\Rightarrow 3\lambda = -6$$
$$\Rightarrow \lambda = -2.$$

Finally, the coordinates of P are $\underline{\underline{(6, 10, 16)}}$.

(c) Hence find the distance OP, giving your answer as a simplified surd. (2)

Solution

Now,

$$\begin{aligned} OP &= \sqrt{6^2 + 10^2 + 16^2} \\ &= \sqrt{36 + 100 + 256} \\ &= \sqrt{392} \\ &= \underline{\underline{14\sqrt{2}}}. \end{aligned}$$

3. The point A, with coordinates $(0, a, b)$ lies on the line l_1, which has equation

$$\mathbf{r} = 6\mathbf{i} + 19\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}).$$

(a) Find the values of a and b. (3)

Solution

For the **i**th component of A,

$$6 + \lambda = 0 \Rightarrow \lambda = -6.$$

Now,

$$\begin{aligned} \mathbf{r} &= 6\mathbf{i} + 19\mathbf{j} - \mathbf{k} - 6(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) \\ &= -\mathbf{j} + 11\mathbf{k}; \end{aligned}$$

so, $\underline{\underline{a = -5}}$ and $\underline{\underline{b = 11}}$.

The point P lies on l_1 and is such that OP is perpendicular to l_1, where O is the origin.

(b) Find the position vector of point P. (6)

Solution

Well,

$$\begin{aligned}\mathbf{r} &= 6\mathbf{i} + 19\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) \\ &= (6+\lambda)\mathbf{i} + (19+4\lambda)\mathbf{j} + (-1-2\lambda)\mathbf{k},\end{aligned}$$

and

$$\begin{aligned}\begin{pmatrix} 6+\lambda \\ 19+4\lambda \\ -1-2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = 0 &\Rightarrow (6+\lambda) + 4(19+4\lambda) - 2(-1-2\lambda) = 0 \\ &\Rightarrow 21\lambda = -84 \\ &\Rightarrow \lambda = -4.\end{aligned}$$

Finally, the coordinates of P are

$$\underline{\underline{\begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}}}.$$

Given that B has coordinates $(5, 15, 1)$,

(c) show that the points A, P, and B are collinear and find the ratio $AP : PB$. (4)

Solution

Well,

$$\begin{aligned}\overrightarrow{AP} &= \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} - \begin{pmatrix} 0 \\ -5 \\ 11 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 8 \\ -4 \end{pmatrix} \\ &= 2\begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}\end{aligned}$$

and

$$\overrightarrow{PB} = \begin{pmatrix} 5 \\ 15 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 3 \\ 12 \\ -6 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}.$$

Hence, the points A, P, and B are collinear and

$$AP : PB = \underline{\underline{2 : 3}}.$$

4. The point A has position vector

$$\mathbf{a} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

and the point B has position vector

$$\mathbf{b} = \mathbf{i} + \mathbf{j} - 4\mathbf{k},$$

relative to an origin O.

(a) Find the position vector of the point C, with position vector $\mathbf{c}$, given by (1)

$$\mathbf{c} = \mathbf{a} + \mathbf{b}.$$

Solution

Well,

$$\begin{aligned}\mathbf{c} &= \mathbf{a} + \mathbf{b} \\ &= (2\mathbf{i} + 2\mathbf{j} + \mathbf{k}) + (\mathbf{i} + \mathbf{j} - 4\mathbf{k}) \\ &= \underline{\underline{3\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}}}.\end{aligned}$$

(b) Show that $OACB$ is a rectangle, and find its exact area. (6)

Solution

Now,

$$\begin{aligned}\overrightarrow{OA}\cdot\overrightarrow{OB} &= (2\mathbf{i}+2\mathbf{j}+\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}-4\mathbf{k})\\ &= 2+2-4\\ &= 0;\end{aligned}$$

so, OA is perpendicular to OB.
Hence, $OACB$ is a rectangle.

Next,

$$\begin{aligned}OA &= \sqrt{2^2+2^2+1^2}\\ &= \sqrt{9}\\ &= 3\end{aligned}$$

and

$$\begin{aligned}OB &= \sqrt{1^2+1^2+(-4)^2}\\ &= \sqrt{18}\\ &= 3\sqrt{2}.\end{aligned}$$

Finally,

$$\begin{aligned}\text{area of } OACB &= OA\times OB\\ &= 3\times 3\sqrt{2}\\ &= 9\sqrt{2}.\end{aligned}$$

The diagonals of the rectangle, AB and OC, meet at the point D.

(c) Write down the position vector of the point D. (1)

Solution

Well,

$$\begin{aligned}\overrightarrow{OD} &= \tfrac{1}{2}\overrightarrow{OC}\\ &= \tfrac{1}{2}(3\mathbf{i}+3\mathbf{j}-3\mathbf{k})\\ &= (\tfrac{3}{2}\mathbf{i}+\tfrac{3}{2}\mathbf{j}-\tfrac{3}{2}\mathbf{k}).\end{aligned}$$

(d) Find the size of the angle ADC. (6)

Solution
Well,

$$\begin{aligned}\overrightarrow{DA} &= (2\mathbf{i} + 2\mathbf{j} + \mathbf{k}) - (\tfrac{3}{2}\mathbf{i} + \tfrac{3}{2}\mathbf{j} - \tfrac{3}{2}\mathbf{k}) \\ &= \tfrac{1}{2}\mathbf{i} + \tfrac{1}{2}\mathbf{j} + \tfrac{5}{2}\mathbf{k}\end{aligned}$$

and

$$\begin{aligned}\overrightarrow{DC} &= (3\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}) - (\tfrac{3}{2}\mathbf{i} + \tfrac{3}{2}\mathbf{j} - \tfrac{3}{2}\mathbf{k}) \\ &= \tfrac{3}{2}\mathbf{i} + \tfrac{3}{2}\mathbf{j} - \tfrac{3}{2}\mathbf{k}.\end{aligned}$$

Now,

$$\begin{aligned}DA &= \sqrt{(\tfrac{1}{2})^2 + (\tfrac{1}{2})^2 + (\tfrac{5}{2})^2} \\ &= \tfrac{3}{2}\sqrt{3} \\ &= DC\end{aligned}$$

and

$$\begin{aligned}\overrightarrow{DA} \cdot \overrightarrow{DC} &= \tfrac{3}{4} + \tfrac{3}{4} - \tfrac{15}{4} \\ &= -\tfrac{9}{4}.\end{aligned}$$

Finally,

$$\begin{aligned}& \overrightarrow{DA} \cdot \overrightarrow{DC} = DA \times DC \times \cos ADC \\ \Rightarrow\quad & -\tfrac{9}{4} = (\tfrac{3}{2}\sqrt{3}) \times (\tfrac{3}{2}\sqrt{3}) \times \cos ADC \\ \Rightarrow\quad & \cos ADC = -\tfrac{1}{3} \\ \Rightarrow\quad & \angle ADC = 109.471\,220\,6 \text{ (FCD)} \\ \Rightarrow\quad & \underline{\underline{\angle ADC = 109^\circ \text{ (3 sf)}}}.\end{aligned}$$

5. The line l_1 has equation

$$\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

and the line l_2 has equation

$$\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$$

(a) Show that l_1 and l_2 do not meet. (4)

Solution

Well,

$$\begin{aligned} 1 + \lambda &= 1 + 2\mu \quad (1) \\ \lambda &= 3 + \mu \quad (2) \\ -1 &= 6 - \mu \quad (3). \end{aligned}$$

From (3),

$$1 = 6 - \mu \Rightarrow \mu = 7$$

and, from (2),

$$\lambda = 3 + 7 = 10.$$

What about (1)? Now,

$$1 + 10 = 11 \text{ but } 1 + 2 \times 7 = 15;$$

hence, l_1 and l_2 do not meet.

The point A is on l_1 where $\lambda = 1$ and the point B is on l_2 where $\mu = 2$.

(b) Find the cosine of the acute angle between AB and l_1. (6)

Solution

Well, $A(2, 1, -1)$ and $B(5, 5, 4)$. Now,

$$\begin{aligned} \overrightarrow{AB} &= \begin{pmatrix} 5 \\ 5 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}. \end{aligned}$$

Next, the l_1 has direction

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Then,

$$\overrightarrow{AB} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 3 + 4 + 0$$
$$= 7,$$

$$|\overrightarrow{AB}| = \sqrt{3^2 + 4^2 + 5^2}$$
$$= \sqrt{50}$$
$$= 5\sqrt{2},$$

and

$$|\mathbf{d_1}| = \sqrt{1^2 + 1^2 + 0^2}$$
$$= \sqrt{2}.$$

Finally,

$$\overrightarrow{AB} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = |\overrightarrow{AB}| \times |\mathbf{d_1}| \times \cos\theta$$
$$\Rightarrow \quad 7 = 5\sqrt{2} \times \sqrt{2} \times \cos\theta$$
$$\Rightarrow \quad 7 = 10\cos\theta$$
$$\Rightarrow \quad \underline{\underline{\cos\theta = \tfrac{7}{10}}}.$$

6. The points A and B have position vectors

$$2\mathbf{i} + 6\mathbf{j} - \mathbf{k} \text{ and } 3\mathbf{i} + 4\mathbf{j} + \mathbf{k},$$

respectively.

The line l_1 passes through the points A and B.

(a) Find the vector $\overrightarrow{AB}$. (2)

Solution

Well,

$$\begin{aligned}\overrightarrow{AB} &= (3\mathbf{i} + 4\mathbf{j} + \mathbf{k}) - (2\mathbf{i} + 6\mathbf{j} - \mathbf{k}) \\ &= \underline{\underline{\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}}}.\end{aligned}$$

(b) Find a vector equation for the line l_1. (2)

Solution

Now, a vector equation for the line l_1 is, e.g.,

$$\underline{\underline{\mathbf{r} = (2\mathbf{i} + 6\mathbf{j} - \mathbf{k}) + \lambda(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})}},$$

for some scalar λ.

A second line l_2 passes through the origin and is parallel to the vector $\mathbf{i} + \mathbf{k}$.

The line l_1 meets the line l_2 at the point C.

(c) Find the acute angle between and l_1 and l_2. (3)

Solution

Well, a vector equation for the line l_2 is, e.g.,

$$\mathbf{r} = \mu(\mathbf{i} + \mathbf{k}),$$

for some scalar μ. Now,

$$\begin{aligned}\overrightarrow{AB} \cdot |\mathbf{d_2}| &= (\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) \cdot (\mathbf{i} + \mathbf{k}) \\ &= 1 + 0 + 2 \\ &= 3,\end{aligned}$$

$$\begin{aligned}\overrightarrow{AB} &= \sqrt{1^2 + (-2)^2 + 2^2} \\ &= 3,\end{aligned}$$

and

$$\begin{aligned}|\mathbf{d_2}| &= \sqrt{1^2 + 0 + 1^2} \\ &= \sqrt{2}.\end{aligned}$$

Finally,

$$\begin{aligned} & \overrightarrow{AB} \cdot |\mathbf{d_2}| = |\overrightarrow{AB}||\mathbf{d_2}| \cos\theta \\ \Rightarrow\quad & 3 = 3 \times \sqrt{2} \times \cos\theta \\ \Rightarrow\quad & \cos\theta = \tfrac{1}{\sqrt{2}} \\ \Rightarrow\quad & \underline{\underline{\theta = 45^\circ}}. \end{aligned}$$

(d) Find the position vector of the point C. (4)

Solution

Well,

$$(2\mathbf{i} + 6\mathbf{j} - \mathbf{k}) + \lambda(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) = \mu(\mathbf{i} + \mathbf{k})$$

and

$$\begin{aligned} 2 + \lambda &= \mu \quad (1) \\ 6 - 2\lambda &= 0 \quad (2) \\ -1 + 2\lambda &= \mu \quad (3). \end{aligned}$$

From (2),

$$6 - 2\lambda = 0 \Rightarrow \lambda = 3$$

and, from (1),

$$\mu = 5.$$

What about (3)?

$$1 + 2 \times 3 = 5 \checkmark.$$

Finally, the position vector of the point C is

$$(2\mathbf{i} + 6\mathbf{j} - \mathbf{k}) + 3(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) = \underline{\underline{5\mathbf{i} + 5\mathbf{k}}}.$$

7. With respect to a fixed origin O, the lines l_1 and l_2 are given by the equations:

$$\begin{aligned} l_1: &\quad \mathbf{r} = (-9\mathbf{i} + 10\mathbf{k}) + \lambda(2\mathbf{i} + \mathbf{j} - \mathbf{k}), \\ l_2: &\quad \mathbf{r} = (3\mathbf{i} + \mathbf{j} + 17\mathbf{k}) + \mu(3\mathbf{i} - \mathbf{j} + 5\mathbf{k}), \end{aligned}$$

where λ and μ are scalar parameters.

(a) Show that l_1 and l_2 meet and find the position vector of their point of intersection. (6)

Solution

Well, if l_1 and l_2 do intersect, then

$$\begin{aligned} -9 + 2\lambda &= 3 + 3\mu \quad (1) \\ \lambda &= 1 - \mu \quad (2) \\ 10 - \lambda &= 17 + 5\mu \quad (3) \end{aligned}$$

Do (2) + (3):

$$\begin{aligned} 10 = 18 + 4\mu \Rightarrow 4\mu &= -8 \\ \Rightarrow \mu &= -2 \\ \Rightarrow \lambda &= 3. \end{aligned}$$

What about (1)?

$$-9 + 2(3) = -3 \text{ and } 3 + 3(-2) = -3 \checkmark.$$

So, l_1 and l_2 meet and the position vector of their point of intersection is

$$(-9\mathbf{i} + 10\mathbf{k}) + 3(2\mathbf{i} + \mathbf{j} - \mathbf{k}) = \underline{\underline{-3\mathbf{i} + 3\mathbf{j} + 7\mathbf{k}}}.$$

(b) Show that l_1 and l_2 are perpendicular to each other. (2)

Solution

Now,

$$\begin{aligned} \mathbf{d_1} \cdot \mathbf{d_2} &= (2\mathbf{i} + \mathbf{j} - \mathbf{k}) \cdot 3\mathbf{i} - \mathbf{j} + 5\mathbf{k} \\ &= 6 - 1 - 5 \\ &= 0; \end{aligned}$$

hence, l_1 and l_2 are perpendicular to each other.

The point A has position vector

$$5\mathbf{i} + 7\mathbf{j} + 3\mathbf{k}.$$

(c) Show that A lies on l_1. (1)

Solution

Well,

$$\begin{aligned} -9 + 2\lambda = 5 &\Rightarrow 2\lambda = 14 \\ &\Rightarrow \lambda = -2; \end{aligned}$$

so,

$$\underline{\underline{5\mathbf{i} + 7\mathbf{j} + 3\mathbf{k}}} = (-9\mathbf{i} + 10\mathbf{k}) + 7(2\mathbf{i} + \mathbf{j} - \mathbf{k}).$$

The point B is the image of A after reflection in the line l_2.

(d) Find the position vector of B. (3)

Solution

Let $X(-3, 3, 7)$. Then,

$$\begin{aligned} \overrightarrow{AX} &= \overrightarrow{OX} - \overrightarrow{OA} \\ &= (-3\mathbf{i} + 3\mathbf{j} + 7\mathbf{k}) - (5\mathbf{i} + 7\mathbf{j} + 3\mathbf{k}) \\ &= -8\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}. \end{aligned}$$

Now,

$$\begin{aligned} \overrightarrow{OB} &= \overrightarrow{OA} + \overrightarrow{AB} \\ &= \overrightarrow{OA} + 2\overrightarrow{AX} \\ &= (5\mathbf{i} + 7\mathbf{j} + 3\mathbf{k}) + 2(-8\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}) \\ &= \underline{\underline{-11\mathbf{i} - \mathbf{j} + 11\mathbf{k}}}. \end{aligned}$$

8. With respect to a fixed origin O, the lines l_1 and l_2 are given by the equations:

$$\begin{aligned} l_1 : &\quad \begin{pmatrix} 11 \\ 2 \\ 17 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix}, \\ l_2 : &\quad \begin{pmatrix} -5 \\ 11 \\ p \end{pmatrix} + \mu \begin{pmatrix} q \\ 2 \\ 2 \end{pmatrix}, \end{aligned}$$

where λ and μ are parameters and p and q are constants.

Given that l_1 and l_2 are perpendicular,

(a) show that $q = -3$. (2)

Solution
Well,

$$\begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} q \\ 2 \\ 2 \end{pmatrix} = 0 \Rightarrow -2q + 2 - 8 = 0$$
$$\Rightarrow -2q = 6$$
$$\Rightarrow \underline{\underline{q = -3}},$$

as required.

Given further that l_1 and l_2 intersect, find

(b) the value of p, (6)

Solution
l_1 and l_2 intersect so

$$11 - 2\lambda = -5 - 3\mu \quad (1)$$
$$2 + \lambda = 11 + 2\mu \quad (2)$$
$$17 - 4\lambda = p + 2\mu \quad (3).$$

Do $2 \times (2)$:

$$4 + 2\lambda = 22 + 4\mu \quad (4)$$

and do $(1) + (4)$:

$$15 = 17 + \mu \Rightarrow \mu = -2$$
$$\Rightarrow \lambda = 5.$$

Finally,

$$17 - 4(5) = p + 2(-2) \Rightarrow -3 = p - 4$$
$$\Rightarrow \underline{\underline{p = 1}}.$$

(c) the coordinates of the point of intersection. (2)

Solution

Well,

$$\begin{pmatrix} 11 \\ 2 \\ 17 \end{pmatrix} + 5 \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix};$$

hence, the coordinates of the point of intersection are $\underline{\underline{(1, 5, -3)}}$.

The point A lies on l_1 and has position vector

$$\begin{pmatrix} 9 \\ 3 \\ 13 \end{pmatrix}.$$

The point C lies on l_2.

Given that a circle, with centre C, cuts the line l_1 at the points A and B,

(d) find the position vector of B. (3)

Solution

Let

$$\overrightarrow{OX} = \begin{pmatrix} 1 \\ 7 \\ -3 \end{pmatrix}$$

be the point of intersection. Then

$$\begin{aligned} \overrightarrow{AX} &= \overrightarrow{OX} - \overrightarrow{OA} \\ &= \begin{pmatrix} 1 \\ 7 \\ -3 \end{pmatrix} - \begin{pmatrix} 9 \\ 3 \\ 13 \end{pmatrix} \\ &= \begin{pmatrix} -8 \\ 4 \\ -16 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned}\overrightarrow{OB} &= \overrightarrow{OA} + \overrightarrow{AB} \\ &= \overrightarrow{OA} + 2\overrightarrow{AX} \\ &= \begin{pmatrix} 9 \\ 3 \\ 13 \end{pmatrix} + 2\begin{pmatrix} -8 \\ 4 \\ -16 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} -7 \\ 11 \\ -19 \end{pmatrix}}}.\end{aligned}$$

9. Relative to a fixed origin O, the point A has position vector

$$8\mathbf{i} + 13\mathbf{j} - 2\mathbf{k},$$

the point B has position vector

$$10\mathbf{i} + 14\mathbf{j} - 4\mathbf{k},$$

and the point C has position vector

$$9\mathbf{i} + 9\mathbf{j} + 6\mathbf{k}.$$

The line l passes through the points A and B.

(a) Find a vector equation for the line l. (3)

Solution

Well,

$$\begin{aligned}\overrightarrow{AB} &= (10\mathbf{i} + 14\mathbf{j} - 4\mathbf{k}) - (8\mathbf{i} + 13\mathbf{j} - 2\mathbf{k}) \\ &= 2\mathbf{i} + \mathbf{j} - 2\mathbf{k},\end{aligned}$$

and a vector equation for the line l is, e.g.,

$$\underline{\underline{\mathbf{r} = (8\mathbf{i} + 13\mathbf{j} - 2\mathbf{k}) + \lambda(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})}},$$

for some λ.

(b) Find $|\overrightarrow{CB}|$. (2)

Solution

Now,

$$\begin{aligned}|\overrightarrow{CB}| &= |\overrightarrow{OB} - \overrightarrow{OC}| \\ &= |(10\mathbf{i} + 14\mathbf{j} - 4\mathbf{k}) - (9\mathbf{i} + 9\mathbf{j} + 6\mathbf{k})| \\ &= |\mathbf{i} + 5\mathbf{j} - 10\mathbf{k}| \\ &= \sqrt{1^2 + 5^2 + (-10)^2} \\ &= \underline{\underline{3\sqrt{14}}}.\end{aligned}$$

(c) Find the size of the acute angle between the line segment CB and the line l, giving your answer in degrees to 1 decimal place. (3)

Solution

Well,

$$\begin{aligned}\overrightarrow{CB} \cdot \overrightarrow{AB} &= (\mathbf{i} + 5\mathbf{j} - 10\mathbf{k}) \cdot (2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \\ &= 2 + 5 + 20 \\ &= 27\end{aligned}$$

and

$$\begin{aligned}|\overrightarrow{AB}| &= \sqrt{2^2 + 1^2 + (-2)^2} \\ &= 3.\end{aligned}$$

So,

$$\begin{aligned}& \overrightarrow{CB} \cdot \overrightarrow{AB} = |\overrightarrow{CB}||\overrightarrow{AB}| \cos\theta \\ \Rightarrow\quad & 27 = 3\sqrt{14} \times 3 \times \cos\theta \\ \Rightarrow\quad & \cos\theta = \tfrac{3}{\sqrt{14}} \\ \Rightarrow\quad & \theta = 36.699\,225\,2 \text{ (FCD)} \\ \Rightarrow\quad & \underline{\underline{\theta = 36.7^\circ \text{ (3 sf)}}}.\end{aligned}$$

(d) Find the shortest distance from the point C to the line l. (3)

Solution

Let d be the shortest distance. Then

$$\begin{aligned} \sin = \frac{\text{opp}}{\text{hyp}} &\Rightarrow \sin 36.699\ldots^\circ = \frac{d}{3\sqrt{14}} \\ &\Rightarrow d = 3\sqrt{14}\sin 36.699\ldots^\circ \\ &\Rightarrow \underline{\underline{d = 3\sqrt{5}\text{ cm}}}. \end{aligned}$$

The point X lies on l.

Given that the vector $\overrightarrow{CX}$ is perpendicular to l,

(e) find the area of the triangle CXB, giving your answer to 3 significant figures. (3)

Solution

Pythagoras' theorem:

$$\begin{aligned} CX^2 + BX^2 = BC^2 &\Rightarrow (6.708\ldots)^2 + BX^2 = (3\sqrt{14})^2 \\ &\Rightarrow BX^2 = 81 \\ &\Rightarrow BX = 9. \end{aligned}$$

Finally,

$$\begin{aligned} \text{area of } BCX &= \tfrac{1}{2} \times BX \times CX \\ &= \tfrac{1}{2} \times 9 \times 3\sqrt{5} \\ &= \underline{\underline{\tfrac{27}{2}\sqrt{5}\text{ cm}^2}}. \end{aligned}$$

10. The line l_1 has vector equation

$$\mathbf{r} = \begin{pmatrix} -6 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$$

and the line l_2 has vector equation

$$\mathbf{r} = \begin{pmatrix} -6 \\ 4 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix},$$

where λ and μ are parameters.

The lines l_1 and l_2 intersect at the point A and the acute angle between l_1 and l_2 is θ.

(a) Write down the coordinates of A. (1)

Solution
$\underline{\underline{A(-6, 4, -1)}}$.

(b) Find the value of $\cos\theta$. (3)

Solution
Well,

$$\begin{aligned}\begin{pmatrix}4\\-1\\3\end{pmatrix}\cdot\begin{pmatrix}3\\-4\\1\end{pmatrix} &= 12+4+3\\ &= 19,\end{aligned}$$

$$\begin{aligned}\left|\begin{pmatrix}4\\-1\\3\end{pmatrix}\right| &= \sqrt{4^2+(-1)^2+3^2}\\ &= \sqrt{26},\end{aligned}$$

and

$$\begin{aligned}\left|\begin{pmatrix}3\\-4\\1\end{pmatrix}\right| &= \sqrt{3^2+(-4)^2+1^2}\\ &= \sqrt{26}.\end{aligned}$$

Hence,

$$\begin{pmatrix}4\\-1\\3\end{pmatrix}\cdot\begin{pmatrix}3\\-4\\1\end{pmatrix} = \left|\begin{pmatrix}4\\-1\\3\end{pmatrix}\right|\left|\begin{pmatrix}3\\-4\\1\end{pmatrix}\right|\cos\theta$$

$$\Rightarrow \quad 19 = \sqrt{26}\times\sqrt{26}\times\cos\theta$$

$$\Rightarrow \quad \underline{\underline{\cos\theta = \tfrac{19}{26}}}.$$

The point X lies on l_1 where $\lambda = 4$.

(c) Find the coordinates of X. (1)

Solution
$\underline{\underline{X(10, 0, 11)}}$.

(d) Find the vector $\overrightarrow{AX}$. (2)

Solution
Well,

$$\begin{aligned}\overrightarrow{AX} &= \overrightarrow{AO} + \overrightarrow{OX} \\ &= -\overrightarrow{OA} + \overrightarrow{OX} \\ &= -\begin{pmatrix} -6 \\ 4 \\ -1 \end{pmatrix} + \begin{pmatrix} 10 \\ 0 \\ 11 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 16 \\ -4 \\ 12 \end{pmatrix}}}.\end{aligned}$$

(e) Hence, or otherwise, show that (2)

$$|\overrightarrow{AX}| = 4\sqrt{26}.$$

Solution
Now,

$$\begin{aligned}|\overrightarrow{AX}| &= \sqrt{16^2 + (-4)^2 + 12^2} \\ &= \sqrt{416} \\ &= \sqrt{16 \times 26} \\ &= \sqrt{16} \times \sqrt{26} \\ &= \underline{\underline{4\sqrt{26}}},\end{aligned}$$

as required.

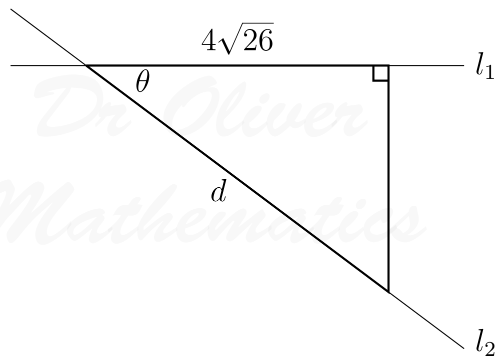
The point Y lies on l_2.

Given that the vector YX is perpendicular to l_1,

(f) find the length of AY, giving your answer to 3 significant figures. (3)

Solution

We will draw a picture:



and

$$\begin{aligned}\cos = \frac{\text{adj}}{\text{hyp}} &\Rightarrow \tfrac{19}{26} = \frac{4\sqrt{26}}{d} \\ &\Rightarrow d = \frac{4\sqrt{26}}{\tfrac{19}{26}} \\ &\Rightarrow \underline{\underline{d = \tfrac{104}{19}\sqrt{26}}}.\end{aligned}$$

11. The line l_1 has equation

$$\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix},$$

where λ is a scalar parameter.

The line l_2 has equation

$$\mathbf{r} = \begin{pmatrix} 0 \\ 9 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix},$$

where mu is a scalar parameter.

Given that l_1 and l_2 meet at the point C, find

(a) the coordinates of C. (3)

Solution

Well,

$$\begin{aligned} 2+\lambda &= 5\mu \quad (1) \\ 3+2\lambda &= 9 \quad (2) \\ -4+\lambda &= -3+2\mu \quad (3) \end{aligned}$$

From (2),

$$\begin{aligned} 3+2\lambda = 9 &\Rightarrow 2\lambda = 6 \\ &\Rightarrow \lambda = 3 \end{aligned}$$

and, from (1),

$$\begin{aligned} 2+\lambda = 5\mu &\Rightarrow 5 = 5\mu \\ &\Rightarrow \mu = 1. \end{aligned}$$

What about (3)?

$$-4+3 = -1 \text{ and } -3+2(1) = -1 \checkmark.$$

So,

$$\begin{aligned} \mathbf{r} &= \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ 9 \\ -1 \end{pmatrix}; \end{aligned}$$

hence, the coordinates of $\underline{\underline{C(5, 9, -1)}}$.

The point A is the point on l_1 where $\lambda = 0$ and the point B is the point on l_2 where $\mu = -1$.

(b) Find the size of the angle ACB. (4)
Give your answer in degrees to 2 decimal places.

Solution

Well, $A(2,3,-4)$ and $B(-5,9,-5)$. Now,

$$\begin{aligned}\overrightarrow{CA} &= \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} - \begin{pmatrix} 5 \\ 9 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ -6 \\ -3 \end{pmatrix},\end{aligned}$$

$$\begin{aligned}|\overrightarrow{CA}| &= \sqrt{(-3)^2 + (-6)^2 + (-3)^2} \\ &= 3\sqrt{6},\end{aligned}$$

$$\begin{aligned}\overrightarrow{CB} &= \begin{pmatrix} -5 \\ 9 \\ -5 \end{pmatrix} - \begin{pmatrix} 5 \\ 9 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} -10 \\ 0 \\ -4 \end{pmatrix},\end{aligned}$$

$$\begin{aligned}|\overrightarrow{CB}| &= \sqrt{(-10)^2 + 0^2 + (-4)^2} \\ &= 2\sqrt{29},\end{aligned}$$

and

$$\begin{aligned}\overrightarrow{CA} \cdot \overrightarrow{CB} &= 30 + 0 + 12 \\ &= 42.\end{aligned}$$

Hence,

$$\begin{aligned} & \overrightarrow{CA} \cdot \overrightarrow{CB} = |\overrightarrow{CA}||\overrightarrow{CB}| \cos\theta \\ \Rightarrow \quad & 42 = 3\sqrt{6} \times 2\sqrt{29} \times \cos\theta \\ \Rightarrow \quad & \cos\theta = \tfrac{7}{\sqrt{174}} \\ \Rightarrow \quad & \theta = 57.949\,357\,36 \text{ (FCD)} \\ \Rightarrow \quad & \underline{\underline{\theta = 57.9^\circ \text{ (3 sf)}}}.\end{aligned}$$

(c) Hence, or otherwise, find the area of the triangle ABC. (5)

Solution
Well,

$$\begin{aligned}\text{opp}^2 + \text{adj}^2 = \text{hyp}^2 &\Rightarrow \text{opp}^2 + (\tfrac{7}{\sqrt{174}})^2 = 1 \\ &\Rightarrow \text{opp}^2 + \tfrac{49}{174} = 1 \\ &\Rightarrow \text{opp}^2 = \tfrac{125}{174} \\ &\Rightarrow \text{opp} = \sqrt{\tfrac{125}{174}}\end{aligned}$$

and

$$\begin{aligned}\text{area of } ABC &= \tfrac{1}{2} \times 3\sqrt{6} \times 2\sqrt{29} \times \sqrt{\tfrac{125}{174}} \\ &= \underline{\underline{15\sqrt{5}}}.\end{aligned}$$

12. Relative to a fixed origin O, the point A has position vector

$$\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$$

and the point B has position vector

$$-2\mathbf{i} + 2\mathbf{j} - \mathbf{k}.$$

The points A and B lie on a straight line l.

(a) Find $\overrightarrow{AB}$. (2)

Solution
Well,

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= (-2\mathbf{i} + 2\mathbf{j} - \mathbf{k}) - (\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) \\ &= \underline{\underline{-3\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}}}.\end{aligned}$$

(b) Find a vector equation of l. (2)

Solution
Now, a vector equation of l is

$$\underline{\underline{\mathbf{r} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k} + \lambda(-3\mathbf{i} + 5\mathbf{j} - 3\mathbf{k})}},$$

for some scalar λ.

The point C has position vector

$$2\mathbf{i} + p\mathbf{j} - 4\mathbf{k}$$

with respect to O, where p is a constant.

Given that AC is perpendicular to l, find

(c) the value of p, (4)

Solution

Well,

$$\begin{aligned}\overrightarrow{AC} &= \overrightarrow{OC} - \overrightarrow{OA} \\ &= (2\mathbf{i} + p\mathbf{j} - 4\mathbf{k}) - (\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) \\ &= \mathbf{i} + (p+3)\mathbf{j} - 6\mathbf{k}.\end{aligned}$$

Now,

$$\begin{aligned}& \overrightarrow{AB} \cdot \overrightarrow{AC} = 0 \\ \Rightarrow\ & [-3\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}] \cdot [\mathbf{i} + (p+3)\mathbf{j} - 6\mathbf{k}] = 0 \\ \Rightarrow\ & -3 + 5(p+3) + 18 = 0 \\ \Rightarrow\ & 5(p+3) = -15 \\ \Rightarrow\ & p + 3 = -3 \\ \Rightarrow\ & \underline{\underline{p = -6}}.\end{aligned}$$

(d) the distance AC. (2)

Solution

Well,

$$\overrightarrow{AC} = \mathbf{i} - 3\mathbf{j} - 6\mathbf{k}$$

and

$$\begin{aligned}AC &= \sqrt{1^2 + (-3)^2 + (-6)^2} \\ &= \underline{\underline{\sqrt{46}}}.\end{aligned}$$

13. With respect to a fixed origin O, the lines l_1 and l_2 are given by the equations:

$$l_1: \quad \mathbf{r} = \begin{pmatrix} 6 \\ -3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix},$$

$$l_2: \quad \mathbf{r} = \begin{pmatrix} -5 \\ 15 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix},$$

where λ and μ are scalar parameters.

(a) Show that l_1 and l_2 meet and find the position vector of their point of intersection A. (6)

Solution

Well,

$$\begin{aligned} 6 - \lambda &= -5 + 2\mu \quad (1) \\ -3 + 2\lambda &= 15 - 3\mu \quad (2) \\ -2 + 3\lambda &= 3 + \mu \quad (3). \end{aligned}$$

Do $3 \times (1)$ and $2 \times (3)$:

$$\begin{aligned} 18 - 3\lambda &= -15 + 6\mu \quad (4) \\ -6 + 4\lambda &= 30 - 6\mu \quad (5) \end{aligned}$$

Do $(4) + (5)$:

$$\begin{aligned} \lambda + 12 = 15 \Rightarrow \lambda = 3 \\ &\Rightarrow 6 - 3 = -5 + 2\mu \\ &\Rightarrow 2\mu = 8 \\ &\Rightarrow \mu = 4. \end{aligned}$$

What about (3)?

$$-2 + 3(3) = 7 \text{ and } 3 + 4 = 7 \checkmark.$$

Hence, l_1 and l_2 meet and the position vector is

$$\begin{pmatrix} 6 \\ -3 \\ -2 \end{pmatrix} + 3 \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 7 \end{pmatrix}.$$

(b) Find, to the nearest 0.1°, the acute angle between l_1 and l_2. (3)

Solution
Because the directions are ± 1, ± 2, and ± 2,

$$\begin{aligned}\left|\begin{pmatrix}-1\\2\\3\end{pmatrix}\right| = \left|\begin{pmatrix}2\\-3\\1\end{pmatrix}\right| &= \sqrt{2^2+1^2+3^2}\\ &= \sqrt{14}\end{aligned}$$

and

$$\begin{aligned}\begin{pmatrix}-1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}2\\-3\\1\end{pmatrix} &= -2-6+3\\ &= -5.\end{aligned}$$

Now,

$$\begin{aligned}&\begin{pmatrix}-1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}2\\-3\\1\end{pmatrix} = \left|\begin{pmatrix}-1\\2\\3\end{pmatrix}\right|\left|\begin{pmatrix}2\\-3\\1\end{pmatrix}\right|\cos\theta\\ \Rightarrow\quad & -5 = \sqrt{14}\times\sqrt{14}\times\cos\theta\\ \Rightarrow\quad & \cos\theta = -\tfrac{5}{14}\\ \Rightarrow\quad & \theta = 110.924\,832\,4 \text{ (FCD)},\end{aligned}$$

so the angle is

$$\begin{aligned}180 - 110.924\ldots &= 69.075\,167\,57 \text{ (FCD)}\\ &= \underline{\underline{69.1^\circ \text{ (nearest 0.1)}}}.\end{aligned}$$

The point B has position vector

$$\begin{pmatrix}5\\-1\\1\end{pmatrix}.$$

(c) Show that B lies on l_1. (1)

Solution
Let $\lambda = 1$:

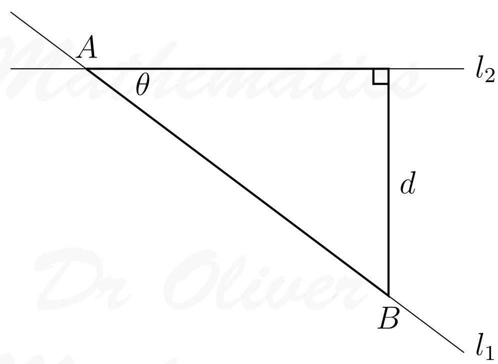
$$\begin{pmatrix}6\\-3\\-2\end{pmatrix}+\begin{pmatrix}-1\\2\\3\end{pmatrix}=\begin{pmatrix}5\\-1\\1\end{pmatrix};$$

hence, B lies on l_1.

(d) Find the shortest distance from B to the line l_2, giving your answer to 3 significant figures. (4)

Solution

We will draw a picture:



Let d be shortest distance from B to l_2. Then

$$\overrightarrow{AB} = \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ -6 \end{pmatrix},$$

and

$$AB = \sqrt{2^2 + (-4)^2 + (-6)^2} = \sqrt{56} = 2\sqrt{14}.$$

Next,

$$\text{opp}^2 + \text{adj}^2 = \text{hyp}^2 \Rightarrow \text{opp}^2 + (-\tfrac{5}{14})^2 = 1$$
$$\Rightarrow \text{opp}^2 + \tfrac{25}{196} = 1$$
$$\Rightarrow \text{opp}^2 = \tfrac{171}{196}$$
$$\Rightarrow \text{opp} = \tfrac{3}{14}\sqrt{19}$$

and, finally,

$$\sin = \frac{\text{opp}}{\text{hyp}} \Rightarrow \tfrac{3}{14}\sqrt{19} = \tfrac{d}{2\sqrt{14}}$$
$$\Rightarrow \underline{\underline{d = \tfrac{3}{7}\sqrt{266}}}.$$

14. Relative to a fixed origin O, the point A has position vector

$$(2\mathbf{i} - \mathbf{j} + 5\mathbf{k}),$$

the point B has position vector

$$(5\mathbf{i} + 2\mathbf{j} + 10\mathbf{k}),$$

and the point D has position vector

$$(-\mathbf{i} + \mathbf{j} + 4\mathbf{k}).$$

The line l passes through the points A and B.

(a) Find the vector $\overrightarrow{AB}$. (2)

Solution
Well,

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= (5\mathbf{i} + 2\mathbf{j} + 10\mathbf{k}) - (2\mathbf{i} - \mathbf{j} + 5\mathbf{k}) \\ &= \underline{\underline{3\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}}}. \end{aligned}$$

(b) Find a vector equation for the line l. (2)

Solution
E.g.,

$$\underline{\underline{\mathbf{r} = (2\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}) + \lambda(3\mathbf{i} + 3\mathbf{j} + 5\mathbf{k})}},$$

for some scalar λ.

(c) Show that the size of the angle BAD is 109°, to the nearest degree. (4)

Solution

Well,

$$\begin{aligned}\overrightarrow{AD} &= \overrightarrow{OD} - \overrightarrow{OA} \\ &= (-\mathbf{i} + \mathbf{j} + 4\mathbf{k}) - (2\mathbf{i} - \mathbf{j} + 5\mathbf{k}) \\ &= -3\mathbf{i} + 2\mathbf{j} - \mathbf{k}.\end{aligned}$$

Now,

$$\begin{aligned}\overrightarrow{AB} \cdot \overrightarrow{AD} &= (3\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}) \cdot (-3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \\ &= -9 + 6 - 5 \\ &= -8\end{aligned}$$

$$\begin{aligned}|\overrightarrow{AB}| &= \sqrt{3^2 + 3^2 + 5^2} \\ &= \sqrt{43},\end{aligned}$$

and

$$\begin{aligned}|\overrightarrow{AD}| &= \sqrt{(-3)^2 + 2^2 + (-1)^2} \\ &= \sqrt{14}.\end{aligned}$$

Next,

$$\begin{aligned} & \overrightarrow{AB} \cdot \overrightarrow{AD} = |\overrightarrow{AB}||\overrightarrow{AD}| \cos\theta \\ \Rightarrow \quad & -8 = \sqrt{43} \times \sqrt{14} \times \cos\theta \\ \Rightarrow \quad & \cos\theta = -\tfrac{8}{\sqrt{602}} \\ \Rightarrow \quad & \theta = 109.029\,544 \text{ (FCD)} \\ \Rightarrow \quad & \underline{\underline{\theta = 109^\circ \text{ (nearest degree)}}}.\end{aligned}$$

The points A, B, and D, together with a point C, are the vertices of the parallelogram $ABCD$, where

$$\overrightarrow{AB} = \overrightarrow{DC}.$$

(d) Find the position vector of C. (2)

Solution

Well,

$$\begin{aligned}\overrightarrow{OC} &= \overrightarrow{OD} + \overrightarrow{DC} \\ &= \overrightarrow{OD} + \overrightarrow{AB} \\ &= (-\mathbf{i} + \mathbf{j} + 4\mathbf{k}) + (3\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}) \\ &= \underline{\underline{2\mathbf{i} + 4\mathbf{j} + 9\mathbf{k}}}.\end{aligned}$$

(e) Find the area of the parallelogram $ABCD$, giving your answer to 3 significant figures. (3)

Solution

Now,

$$\begin{aligned}\text{area of the parallelogram} &= 2 \times \text{area of } ABD \\ &= 2 \times \tfrac{1}{2} \times \sqrt{43} \times \sqrt{14} \times \sin 109.029\ldots^\circ \\ &= 23.194\,827\,01 \text{ (FCD)} \\ &= \underline{\underline{23.2 \ \ (3 \text{ sf})}}.\end{aligned}$$

(f) Find the shortest distance from the point D to the line l, giving your answer to 3 significant figures. (2)

Solution

Well, we need

$$180 - 109.029\ldots = 70.970\,456 \text{ (FCD)}$$

and

$$\begin{aligned}\sin = \frac{\text{opp}}{\text{hyp}} &\Rightarrow \sin 70.970\ldots^\circ = \tfrac{d}{\sqrt{14}} \\ &\Rightarrow d = \sqrt{14} \sin 70.970\ldots^\circ \\ &\Rightarrow d = 3.537\,177\,958 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{d = 3.54 \text{ cm (3 sf)}}}.\end{aligned}$$

15. Relative to a fixed origin O, the point A has position vector

$$(10\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$$

and the point B has position vector

$$(8\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}).$$

The line l passes through the points A and B.

(a) Find the vector $\overrightarrow{AB}$. (2)

Solution

Well,

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= (8\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) - (10\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \\ &= \underline{\underline{-2\mathbf{i} + \mathbf{j} + \mathbf{k}}}.\end{aligned}$$

(b) Find a vector equation for the line l. (2)

Solution

E.g.,

$$\underline{\underline{\mathbf{r} = (10\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \lambda(-2\mathbf{i} + \mathbf{j} + \mathbf{k})}},$$

for some scalar λ.

The point C has position vector

$$(3\mathbf{i} + 12\mathbf{j} + 3\mathbf{k}).$$

The point P lies on l.

Given that the vector $\overrightarrow{CP}$ is perpendicular to l,

(c) find the position vector of the point P. (6)

Solution

Let

$$\overrightarrow{OP} = (10\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \lambda(-2\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

Then,

$$\begin{aligned}\overrightarrow{CP} &= \overrightarrow{OP} - \overrightarrow{OC} \\ &= [(10 - 2\lambda)\mathbf{i} + (2 + \lambda)\mathbf{j} + (3 + \lambda)\mathbf{k}] - (3\mathbf{i} + 12\mathbf{j} + 3\mathbf{k}) \\ &= (7 - 2\lambda)\mathbf{i} + (-10 + \lambda)\mathbf{j} + \lambda\mathbf{k}.\end{aligned}$$

Now,

$$\begin{aligned}
&\overrightarrow{CP} \cdot \mathbf{d_1} = 0 \\
\Rightarrow\quad & [(7-2\lambda)\mathbf{i} + (-10+\lambda)\mathbf{j} + \lambda\mathbf{k}] \cdot [-2\mathbf{i} + \mathbf{j} + \mathbf{k}] = 0 \\
\Rightarrow\quad & -2(7-2\lambda) + (-10+\lambda) + \lambda = 0 \\
\Rightarrow\quad & -14 + 4\lambda - 10 + \lambda + \lambda = 0 \\
\Rightarrow\quad & 6\lambda = 24 \\
\Rightarrow\quad & \lambda = 4.
\end{aligned}$$

Finally,

$$\begin{aligned}
\overrightarrow{OP} &= (10\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + 4(-2\mathbf{i} + \mathbf{j} + \mathbf{k}) \\
&= \underline{\underline{2\mathbf{i} + 6\mathbf{j} + 7\mathbf{k}}}.
\end{aligned}$$

16. With respect to a fixed origin O, the lines l_1 and l_2 are given by the equations

$$\begin{aligned}
l_1:\quad & \mathbf{r} = (9\mathbf{i} + 13\mathbf{j} - 3\mathbf{k}) + \lambda(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) \\
l_2:\quad & \mathbf{r} = (2\mathbf{i} - \mathbf{j} + \mathbf{k}) + \mu(2\mathbf{i} + \mathbf{j} + \mathbf{k}),
\end{aligned}$$

where λ and μ are scalar parameters.

(a) Given that l_1 and l_2 meet, find the position vector of their point of intersection. (5)

Solution
Well,

$$\begin{aligned}
9 + \lambda &= 2 + 2\mu \quad (1) \\
13 + 4\lambda &= -1 + \mu \quad (2) \\
-3 - 2\lambda &= 1 + \mu \quad (3)
\end{aligned}$$

Do (2) − (3):

$$\begin{aligned}
16 + 6\lambda = -2 \Rightarrow 6\lambda &= -18 \\
\Rightarrow \lambda &= -3 \\
\Rightarrow \mu &= 2.
\end{aligned}$$

What about (1)?

$$9 + (-3) = 6 \text{ and } 2 + 2(2) = 6\,\checkmark.$$

So, l_1 and l_2 meet and position vector of their point of intersection is

$$\begin{aligned}\mathbf{r} &= (9\mathbf{i} + 13\mathbf{j} - 3\mathbf{k}) - 3(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) \\ &= \underline{\underline{6\mathbf{i} + \mathbf{j} + 3\mathbf{k}}}.\end{aligned}$$

(b) Find the acute angle between l_1 and l_2, giving your answer in degrees to 1 decimal place. (3)

Solution

Now,

$$\begin{aligned}|\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}| &= \sqrt{1^2 + 4^2 + (-2)^2} \\ &= \sqrt{21},\end{aligned}$$

$$\begin{aligned}|2\mathbf{i} + \mathbf{j} + \mathbf{k}| &= \sqrt{2^2 + 1^2 + 1^2} \\ &= \sqrt{6},\end{aligned}$$

and

$$\begin{aligned}(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) \cdot (2\mathbf{i} + \mathbf{j} + \mathbf{k}) &= 2 + 4 - 2 \\ &= 4.\end{aligned}$$

Finally,

$$\begin{aligned}& (\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) \cdot (2\mathbf{i} + \mathbf{j} + \mathbf{k}) = |\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}||2\mathbf{i} + \mathbf{j} + \mathbf{k}| \cos\theta \\ \Rightarrow\ & 4 = \sqrt{21} \times \sqrt{6} \times \cos\theta \\ \Rightarrow\ & \cos\theta = \tfrac{2}{21}\sqrt{14} \\ \Rightarrow\ & \theta = 69.123\,897\,4 \text{ (FCD)} \\ \Rightarrow\ & \underline{\underline{\theta = 69.1^\circ \text{ (3 sf)}}}.\end{aligned}$$

Given that the point A has position vector

$$(4\mathbf{i} + 16\mathbf{j} - 3\mathbf{k})$$

and that the point P lies on l_1 such that AP is perpendicular to l_1,

(c) find the exact coordinates of P. (6)

Solution

Well,

$$\begin{aligned}\overrightarrow{AP} &= \overrightarrow{OP} - \overrightarrow{OA} \\ &= (9\mathbf{i} + 13\mathbf{j} - 3\mathbf{k}) + \lambda(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})] - (4\mathbf{i} + 16\mathbf{j} - 3\mathbf{k}) \\ &= (5 + \lambda)\mathbf{i} + (-3 + 4\lambda)\mathbf{j} - 2\lambda\mathbf{k}.\end{aligned}$$

Now,

$$\begin{aligned} & \overrightarrow{AP} \cdot \mathbf{d_1} = 0 \\ \Rightarrow \quad & [(5 + \lambda)\mathbf{i} + (-3 + 4\lambda)\mathbf{j} - 2\lambda\mathbf{k}] \cdot (\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) \\ \Rightarrow \quad & 5 + \lambda + 4(-3 + 4\lambda) + 4\lambda = 0 \\ \Rightarrow \quad & 5 + \lambda - 12 + 16\lambda + 4\lambda = 0 \\ \Rightarrow \quad & 21\lambda = 7 \\ \Rightarrow \quad & \lambda = \tfrac{1}{3}.\end{aligned}$$

Finally,

$$\begin{aligned}\overrightarrow{OP} &= (9\mathbf{i} + 13\mathbf{j} - 3\mathbf{k}) + \tfrac{1}{3}(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) \\ &= \tfrac{28}{3}\mathbf{i} + \tfrac{43}{3}\mathbf{j} - \tfrac{11}{3}\mathbf{k};\end{aligned}$$

hence, the exact coordinates of $\underline{\underline{P(\frac{28}{3}, \frac{43}{3}, -\frac{11}{3})}}$.

17. With respect to a fixed origin O, the line l has equation

$$\mathbf{r} = \begin{pmatrix} 13 \\ 8 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix},$$

where λ is a scalar parameter.

The point A lies on l and has coordinates $(3, -2, 6)$.

The point P has position vector

$$(-p\mathbf{i} + 2p\mathbf{k}),$$

relative to O, where p is a constant.

Given that vector $\overrightarrow{PA}$ is perpendicular to l,

(a) find the value of p. (4)

Solution

Well,

$$\begin{aligned}\overrightarrow{PA} &= \overrightarrow{OA} - \overrightarrow{OP} \\ &= (3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}) - (-p\mathbf{i} + 2p\mathbf{k}) \\ &= (3 + p)\mathbf{i} - 2\mathbf{j} + (6 - 2p)\mathbf{k}.\end{aligned}$$

Now,

$$\begin{aligned} & \overrightarrow{PA} \cdot \mathbf{d_1} = 0 \\ \Rightarrow\quad & 2(3 + p) - 4 - (6 - 2p) = 0 \\ \Rightarrow\quad & 6 + 2p - 4 - 6 + 2p = 0 \\ \Rightarrow\quad & 4p = 4 \\ \Rightarrow\quad & \underline{\underline{p = 1}}.\end{aligned}$$

Given also that B is a point on l such that $\angle BPA = 45°$,

(b) find the coordinates of the two possible positions of B. (5)

Solution

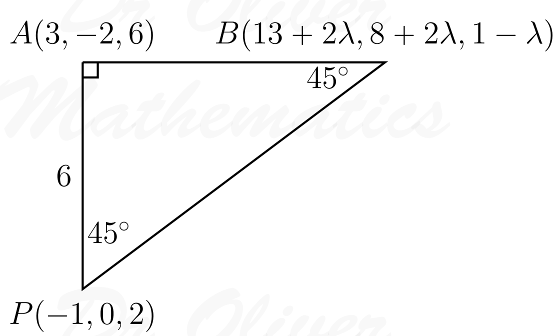
So,

$$\overrightarrow{PA} = 4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$$

and

$$\begin{aligned}PA &= \sqrt{4^2 + (-2)^2 + 4^2} \\ &= 6.\end{aligned}$$

We will draw a picture:



which means

$$AB = 6.$$

$$\begin{aligned}\overrightarrow{AB} &= [(13+2\lambda)\mathbf{i} + (8+2\lambda)\mathbf{j} + (1-\lambda)\mathbf{k}] - (\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}) \\ &= (10+2\lambda)\mathbf{i} + (10+2\lambda)\mathbf{j} + (-5-\lambda)\mathbf{k},\end{aligned}$$

and

$$\begin{aligned}AB = 6 \Rightarrow AB^2 &= 36 \\ &\Rightarrow (10+2\lambda)^2 + (10+2\lambda)^2 + (-5-\lambda)^2 = 36 \\ &\Rightarrow [2(5+\lambda)]^2 + [2(5+\lambda)]^2 + (5+\lambda)^2 = 36 \\ &\Rightarrow 9(5+\lambda)^2 = 36 \\ &\Rightarrow (5+\lambda)^2 = 4 \\ &\Rightarrow 5+\lambda = -2 \text{ or } 5+\lambda = 2 \\ &\Rightarrow \lambda = -7 \text{ or } \lambda = -3.\end{aligned}$$

Hence, the coordinates of the two possible positions are

$$\underline{\underline{B_1(7,2,4) \text{ and } B_2(-1,-6,8)}}.$$

18. Relative to a fixed origin O, the point A has position vector

$$21\mathbf{i} - 17\mathbf{j} + 6\mathbf{k}$$

and the point B has position vector

$$25\mathbf{i} - 14\mathbf{j} + 18\mathbf{k}.$$

The line l has equation

$$\mathbf{r} = \begin{pmatrix} a \\ b \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ c \\ -1 \end{pmatrix},$$

where a, b, and c are constants and λ is a scalar parameter.

Given that the point A lies on the line l,

(a) find the value of a. (3)

Solution

Take the $\mathbf{k}$th component:

$$10 - \lambda = 6 \Rightarrow \lambda = 4,$$

and take the $\mathbf{i}$th component

$$\begin{aligned} a + 6(4) = 21 &\Rightarrow a + 24 = 21 \\ &\Rightarrow \underline{\underline{a = -3}}. \end{aligned}$$

Given also that the vector $\overrightarrow{AB}$ is perpendicular to l,

(b) find the value of b and c, (5)

Solution

Well,

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= (25\mathbf{i} - 14\mathbf{j} + 18\mathbf{k}) - (21\mathbf{i} - 17\mathbf{j} + 6\mathbf{k}) \\ &= 4\mathbf{i} + 3\mathbf{j} + 12\mathbf{k}. \end{aligned}$$

Now,

$$\begin{aligned} & \overrightarrow{AB} \cdot \mathbf{d} = 0 \\ \Rightarrow\ & (4\mathbf{i} + 3\mathbf{j} + 12\mathbf{k}) \cdot (6\mathbf{i} + c\mathbf{j} - \mathbf{k}) = 0 \\ \Rightarrow\ & 24 + 3c - 12 = 0 \\ \Rightarrow\ & 3c = -12 \\ \Rightarrow\ & \underline{\underline{c = -4}}. \end{aligned}$$

Take the $\mathbf{j}$th component of A:

$$\begin{aligned} b + (4)(-4) = -17 &\Rightarrow b - 16 = -17 \\ &\Rightarrow \underline{\underline{b = -1}}. \end{aligned}$$

(c) find the distance AB. (2)

Solution

Well,

$$\begin{aligned} AB &= \sqrt{4^2 + 3^2 + 12^2} \\ &= \underline{\underline{13}}. \end{aligned}$$

The image of the point B after reflection in the line l is the point B'.

(d) Find the position vector of the point B'. (2)

Solution

Now,

$$\begin{aligned} \overrightarrow{OB'} &= \overrightarrow{OB} + \overrightarrow{BB'} \\ &= \overrightarrow{OB} + 2\overrightarrow{BA} \\ &= \overrightarrow{OB} - 2\overrightarrow{AB} \\ &= (25\mathbf{i} - 14\mathbf{j} + 18\mathbf{k}) - 2(4\mathbf{i} + 3\mathbf{j} + 12\mathbf{k}) \\ &= \underline{\underline{17\mathbf{i} - 20\mathbf{j} - 6\mathbf{k}}}. \end{aligned}$$

19. Relative to a fixed origin O, the point A has position vector

$$\begin{pmatrix} -2 \\ 4 \\ 7 \end{pmatrix}$$

and the point B has position vector

$$\begin{pmatrix} -1 \\ 3 \\ 8 \end{pmatrix}.$$

The line l_1 passes through the points A and B.

(a) Find the vector $\overrightarrow{AB}$. (2)

Solution

Well,

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= \begin{pmatrix} -1 \\ 3 \\ 8 \end{pmatrix} - \begin{pmatrix} -2 \\ 4 \\ 7 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}}.\end{aligned}$$

(b) Hence find a vector equation for the line l_1. (1)

Solution

E.g.,

$$\underline{\underline{\mathbf{r} = \begin{pmatrix} -2 \\ 4 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}},$$

for some scalar λ.

The point P has position vector

$$\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}.$$

Given that angle PBA is θ,

(c) show that $\cos\theta = \frac{1}{3}$. (3)

Solution

Well,

$$\begin{aligned}\overrightarrow{BP} &= \overrightarrow{OP} - \overrightarrow{OB} \\ &= \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \\ 8 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -1 \\ -5 \end{pmatrix},\end{aligned}$$

$$\begin{aligned}\overrightarrow{BA} &= -\overrightarrow{AB} \\ &= -\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix},\end{aligned}$$

$$\begin{aligned}BP &= \sqrt{1^2 + (-1)^2 + 5^2} \\ &= 3\sqrt{3},\end{aligned}$$

and

$$\begin{aligned}BA &= \sqrt{(-1)^2 + 1^2 + (-1)^2} \\ &= \sqrt{3}.\end{aligned}$$

Now,

$$\begin{aligned}\overrightarrow{BP} \cdot \overrightarrow{BP} &= \begin{pmatrix} 1 \\ -1 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \\ &= -1 - 1 + 5 \\ &= 3.\end{aligned}$$

Finally,

$$\begin{aligned} & \overrightarrow{BP} \cdot \overrightarrow{BA} = |\overrightarrow{BP}||\overrightarrow{BA}| \cos\theta \\ \Rightarrow \quad & 3 = 3\sqrt{3} \times \sqrt{3} \times \cos\theta \\ \Rightarrow \quad & \underline{\underline{\cos\theta = \tfrac{1}{3}}},\end{aligned}$$

as required.

The line l_2 passes through the point P and is parallel to the line l_1.

(d) Find a vector equation for the line l_2. (2)

Solution

E.g.,

$$\underline{\underline{\mathbf{r} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}},$$

for some scalar μ.

The points C and D both lie on the line l_2.

Given that

$$AB = PC = DP$$

and the x-coordinate of C is positive,

(e) find the coordinates of C and the coordinates of D. (3)

Solution

Well,

$$\begin{aligned} \overrightarrow{OC} &= \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \end{aligned}$$

and

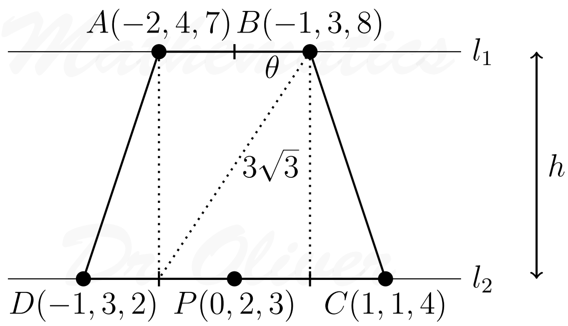
$$\begin{aligned} \overrightarrow{OD} &= \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}; \end{aligned}$$

hence, $\underline{\underline{C(1, 1, 2)}}$ and $\underline{\underline{D(-1, 3, 4)}}$.

(f) find the exact area of the trapezium $ABCD$, giving your answer as a simplified surd. (4)

Solution

We will draw a picture:



Well,

$$\begin{aligned}\sin^2 + \cos^2 = 1 &\Rightarrow \sin^2\theta + (\tfrac{1}{3})^2 = 1\\ &\Rightarrow \sin^2\theta + \tfrac{1}{9} = 1\\ &\Rightarrow \sin^2\theta = \tfrac{8}{9}\\ &\Rightarrow \sin\theta = \tfrac{2}{3}\sqrt{2}.\end{aligned}$$

and

$$\begin{aligned}\sin = \frac{\text{opp}}{\text{hyp}} &\Rightarrow \tfrac{2}{3}\sqrt{2} = \frac{h}{3\sqrt{3}}\\ &\Rightarrow h = 2\sqrt{6}.\end{aligned}$$

Finally,

$$\begin{aligned}\text{area of the trapezium} &= \tfrac{1}{2}(AB + CD)h\\ &= \tfrac{1}{2}(2\sqrt{3} + \sqrt{3})(2\sqrt{6})\\ &= (3\sqrt{3})(\sqrt{6})\\ &= \underline{\underline{9\sqrt{2}}}.\end{aligned}$$

20. With respect to a fixed origin, the point A with position vector

$$\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$$

lies on the line l_1 with equation

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix},$$

for some scalar λ, and the point B with position vector

$$4\mathbf{i} + p\mathbf{j} + 3\mathbf{k},$$

where p is a constant, lies on the line l_2 with equation

$$\mathbf{r} = \begin{pmatrix} 7 \\ 0 \\ 7 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix},$$

for some scalar μ.

(a) Find the value of the constant p. (1)

Solution

Well,

$$\begin{aligned} 7 + 3\mu = 4 &\Rightarrow 3\mu = -3 \\ &\Rightarrow \mu = -1, \end{aligned}$$

and so, from the **j**th component,

$$\mu = -1 \Rightarrow \underline{\underline{p = 5}}.$$

(b) Show that l_1 and l_2 intersect and find the position vector of their point of intersection, C. (4)

Solution

Well,

$$\begin{aligned} 1 &= 7 + 3\mu \quad (1) \\ 2 + 2\lambda &= -5\mu \quad (2) \\ 3 - \lambda &= 7 + 4\mu \quad (3) \end{aligned}$$

From (1),

$$\begin{aligned} 7 + 3\mu = 1 &\Rightarrow 3\mu = -6 \\ &\Rightarrow \mu = -2 \\ &\Rightarrow 2 + 2\lambda = -5(-2) \\ &\Rightarrow 2 + 2\lambda = 10 \\ &\Rightarrow 2\lambda = 8 \\ &\Rightarrow \lambda = 4. \end{aligned}$$

What about (3)?

$$3 - 4 = -1 \text{ and } 7 + 4(-2) = -1 \checkmark.$$

So, l_1 and l_2 intersect and the position vector of their point of intersection is

$$\begin{aligned} \overrightarrow{OC} &= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 1 \\ 10 \\ -1 \end{pmatrix}}}. \end{aligned}$$

(c) Find the size of the angle ACB, giving your answer in degrees to 3 significant figures. (3)

Solution

Well,

$$\begin{aligned} \overrightarrow{CA} &= \overrightarrow{OA} - \overrightarrow{OC} \\ &= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 10 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -8 \\ 4 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \overrightarrow{CB} &= \overrightarrow{OB} - \overrightarrow{OC} \\ &= \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 10 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} CA &= \sqrt{0^2 + (-8)^2 + 4^2} \\ &= 4\sqrt{5}, \end{aligned}$$

$$CB = \sqrt{3^2 + (-5)^2 + 4^2}$$
$$= 5\sqrt{2},$$

and

$$\overrightarrow{CA} \cdot \overrightarrow{CB} = \begin{pmatrix} 0 \\ -8 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}$$
$$= 0 + 40 + 16$$
$$= 56.$$

Finally,

$$\overrightarrow{CA} \cdot \overrightarrow{CB} = |\overrightarrow{CA}||\overrightarrow{CB}| \cos\theta$$
$$\Rightarrow \quad 56 = 4\sqrt{5} \times 5\sqrt{2} \times \cos\theta$$
$$\Rightarrow \quad \cos\theta = \tfrac{7}{25}\sqrt{10}$$
$$\Rightarrow \quad \theta = 27.694\,561\,45 \text{ (FCD)}$$
$$\Rightarrow \quad \underline{\underline{\theta = 27.7^\circ \text{ (3 sf)}}}.$$

(d) Find the area of the triangle ABC, giving your answer to 3 significant figures. (2)

Solution

Well,

$$\sin^2 + \cos^2 = 1 \Rightarrow \sin^2\theta + (\tfrac{7}{25}\sqrt{10})^2 = 1$$
$$\Rightarrow \sin^2\theta + \tfrac{98}{125} = 1$$
$$\Rightarrow \sin^2\theta = \tfrac{27}{125}$$
$$\Rightarrow \sin\theta = \tfrac{3}{25}\sqrt{15},$$

and, finally,

$$\text{area of } ACB = \tfrac{1}{2} \times CA \times CB \times \sin ACB$$
$$= \tfrac{1}{2} \times 4\sqrt{5} \times 5\sqrt{2} \times \tfrac{3}{25}\sqrt{15}$$
$$= \underline{\underline{6\sqrt{6}}}.$$

21. With respect to a fixed origin O, the lines l_1 and l_2 are given by the equations:

$$l_1: \quad \mathbf{r} = \begin{pmatrix} 5 \\ -3 \\ p \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ -13 \end{pmatrix},$$
$$l_2: \quad \mathbf{r} = \begin{pmatrix} 8 \\ 5 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix},$$

where λ and μ are scalar parameters and p is a constant.

The lines l_1 and l_2 intersect at the point A.

(a) Find the coordinates of A. (2)

Solution

Well,

$$\begin{aligned} 5 &= 8 + 3\mu \quad (1) \\ -3 + \lambda &= 5 + 4\mu \quad (2) \\ p - 3\lambda &= -2 - 5\mu \quad (3) \end{aligned}$$

so, from (1),

$$\begin{aligned} 5 = 8 + 3\mu &\Rightarrow 3\mu = -3 \\ &\Rightarrow \mu = -1 \\ &\Rightarrow \lambda = 4; \end{aligned}$$

so

$$\begin{aligned} \mathbf{r} &= \begin{pmatrix} 8 \\ 5 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix}; \end{aligned}$$

hence, the coordinates are $\underline{\underline{A(5, 1, 3)}}$.

(b) Find the value of the constant p. (3)

Solution

From (3),

$$p - 3(4) = -2 - 5(-1) \Rightarrow p - 12 = 3$$
$$\Rightarrow \underline{\underline{p = 15}}.$$

(c) Find the acute angle between l_1 and l_2, giving your answer in degrees to 2 decimal places. (3)

Solution

Now,

$$|\mathbf{d_1}| = \sqrt{0^2 + 1^2 + (-3)^2} = \sqrt{10},$$

$$|\mathbf{d_2}| = \sqrt{3^2 + 4^2 + (-5)^2} = 5\sqrt{2},$$

and

$$\mathbf{d_1} \cdot \mathbf{d_2} = 0 + 4 + 15 = 19.$$

Finally,

$$\mathbf{d_1} \cdot \mathbf{d_2} = |\mathbf{d_1}||\mathbf{d_2}| \cos\theta$$
$$\Rightarrow \quad 19 = \sqrt{10} \times 5\sqrt{2} \times \cos\theta$$
$$\Rightarrow \quad \cos\theta = \tfrac{19}{50}\sqrt{5}$$
$$\Rightarrow \quad \theta = 31.820\,311\,6 \text{ (FCD)}$$
$$\Rightarrow \quad \underline{\underline{\theta = 31.82^\circ \text{ (2 dp)}}}.$$

The point B lies on l_2 where $\mu = 1$.

(d) Find the shortest distance from the point B to the line l_1, giving your answer to 3 significant figures. (3)

Solution

Well, $B(11, 9, -7)$ and

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= \begin{pmatrix} 11 \\ 9 \\ -7 \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 6 \\ 8 \\ -10 \end{pmatrix}.\end{aligned}$$

Now,

$$\begin{aligned}|\overrightarrow{AB}| &= \sqrt{6^2 + 8^2 + (-10)^2} \\ &= 10\sqrt{2}.\end{aligned}$$

Next,

$$\begin{aligned}\sin^2 + \cos^2 = 1 &\Rightarrow \sin^2\theta + (\tfrac{19}{50}\sqrt{5})^2 = 1 \\ &\Rightarrow \sin^2\theta + \tfrac{361}{500} = 1 \\ &\Rightarrow \sin^2\theta = \tfrac{139}{125} \\ &\Rightarrow \sin\theta = \tfrac{\sqrt{695}}{50}.\end{aligned}$$

Let d be the shortest distance. Then,

$$\begin{aligned}\sin = \frac{\text{opp}}{\text{hyp}} &\Rightarrow \tfrac{\sqrt{695}}{50} = \frac{d}{10\sqrt{2}} \\ &\Rightarrow d = 7.456\,540\,753 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{d = 7.46 \quad (3 \text{ sf})}}.\end{aligned}$$

22. With respect to a fixed origin O, the line l_1 is given by the equation

$$\mathbf{r} = \begin{pmatrix} 8 \\ 1 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix},$$

where μ is scalar parameter.

The point A lies on l_1 where $\mu = 1$.

(a) Find the coordinates of A. (1)

Solution
Well,

$$\begin{aligned}\overrightarrow{OA} &= \begin{pmatrix} 8 \\ 1 \\ -3 \end{pmatrix} + \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix};\end{aligned}$$

hence, the coordinates are $\underline{\underline{A(3, 5, 0)}}$.

The point P has position vector

$$\begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}.$$

The line l_2 passes through the point P and is parallel to the line l_1.

(b) Write down a vector equation for the line l_2. (2)

Solution
E.g.,

$$\underline{\underline{\mathbf{r} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix}}},$$

where λ is scalar parameter.

(c) Find the exact value of the distance AP. (2)
Give your answer in the form $k\sqrt{2}$, where k is a constant to be determined.

Solution
Now,

$$\begin{aligned}\overrightarrow{AP} &= \overrightarrow{OP} - \overrightarrow{OA} \\ &= \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix},\end{aligned}$$

and

$$AP = \sqrt{(-2)^2 + 0^2 + 2^2} = \underline{\underline{2\sqrt{2}}};$$

hence, $\underline{\underline{k = 2}}$.

The acute angle between AP and l_2 is θ.

(d) Find the value of $\cos\theta$. (3)

Solution
Well,

$$\overrightarrow{AP} \cdot \mathbf{d} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix} = 10 + 0 + 6 = 16$$

and

$$|\mathbf{d}| = \sqrt{(-5)^2 + 4^2 + 3^2} = 5\sqrt{2}.$$

Now,

$$\overrightarrow{AP} \cdot \mathbf{d} = |\overrightarrow{AP}||\mathbf{d}|\cos\theta$$
$$\Rightarrow \quad 16 = 2\sqrt{2} \times 5\sqrt{2} \times \cos\theta$$
$$\Rightarrow \quad \underline{\underline{\cos\theta = \tfrac{4}{5}}}.$$

A point E lies on the line l_2.

Given that $AP = PE$,

(e) find the area of triangle APE, (3)

Solution

Well,

$$\begin{aligned}\sin^2 + \cos^2 = 1 &\Rightarrow \sin^2\theta + (\tfrac{4}{5})^2 = 1\\ &\Rightarrow \sin^2\theta + \tfrac{16}{25} = 1\\ &\Rightarrow \sin^2\theta = \tfrac{9}{25}\\ &\Rightarrow \sin\theta = \tfrac{3}{5},\end{aligned}$$

and

$$\begin{aligned}\text{area of } APE &= \tfrac{1}{2} \times 2\sqrt{2} \times 2\sqrt{2} \times \tfrac{3}{5}\\ &= \underline{\underline{2.4}}.\end{aligned}$$

(f) find the coordinates of the two possible positions of E. (5)

Solution

Well, $PE = 2\sqrt{2}$ and

$$\overrightarrow{PE} = \lambda \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix}.$$

Now,

$$\begin{aligned}\lambda^2[(-5)^2 + 4^2 + 3^2] = (2\sqrt{2})^2 &\Rightarrow 50\lambda^2 = 8\\ &\Rightarrow \lambda^2 = \tfrac{4}{25}\\ &\Rightarrow \lambda = \pm\tfrac{2}{5}.\end{aligned}$$

So, the vector positions are

$$\begin{aligned}\overrightarrow{OE} &= \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} + \tfrac{2}{5}\begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix} \text{ or } \overrightarrow{OE} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} - \tfrac{2}{5}\begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix}\\ \Rightarrow \quad \overrightarrow{OE} &= \begin{pmatrix} -1 \\ 6.6 \\ 3.2 \end{pmatrix} \text{ or } \overrightarrow{OE} = \begin{pmatrix} 3 \\ 3.4 \\ 0.8 \end{pmatrix};\end{aligned}$$

hence, the the coordinates of the two possible positions are $\underline{\underline{E_1(-1, 6.6, 3.2)}}$ and $\underline{\underline{E_2(3, 3.4, 0.8)}}$.

23. With respect to a fixed origin O, the lines l_1 and l_2 are given by the equations

$$l_1: \quad \mathbf{r} = \begin{pmatrix} 4 \\ 28 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -5 \\ 1 \end{pmatrix},$$
$$l_2: \quad \mathbf{r} = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix},$$

where λ and μ are scalar parameters.

The lines l_1 and l_2 intersect at the point X.

(a) Find the coordinates of the point X. (3)

Solution

Well,

$$\begin{aligned} 4 - \lambda &= 5 + 3\mu \quad (1) \\ 28 - 5\lambda &= 3 \quad (2) \\ 4 + \lambda &= 1 - 4\mu \quad (3). \end{aligned}$$

Now, from (2),

$$\begin{aligned} 28 - 5\lambda = 3 &\Rightarrow 5\lambda = 25 \\ &\Rightarrow \lambda = 5, \end{aligned}$$

and, from (1),

$$\begin{aligned} 4 - 5 = 5 + 3\mu &\Rightarrow 3\mu = -6 \\ &\Rightarrow \mu = -2. \end{aligned}$$

The vector equation is

$$\begin{aligned} \overrightarrow{OX} &= \begin{pmatrix} 4 \\ 28 \\ 4 \end{pmatrix} + 5 \begin{pmatrix} -1 \\ -5 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 3 \\ 9 \end{pmatrix}; \end{aligned}$$

hence, $\underline{\underline{X(-1, 3, 9)}}$.

(b) Find the size of the acute angle between l_1 and l_2, giving your answer in degrees to 2 decimal places. (3)

Solution

Well,

$$\begin{aligned}|\mathbf{d_1}| &= \sqrt{(-1)^2 + (-5)^2 + 1^2} \\ &= 3\sqrt{3},\end{aligned}$$

$$\begin{aligned}|\mathbf{d_1}| &= \sqrt{3^2 + 0^2 + (-4)^2} \\ &= 5,\end{aligned}$$

and

$$\begin{aligned}\mathbf{d_1} \cdot \mathbf{d_2} &= \begin{pmatrix} -1 \\ -5 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix} \\ &= -3 + 0 - 4 \\ &= -7.\end{aligned}$$

Now,

$$\begin{aligned} & \mathbf{d_1} \cdot \mathbf{d_2} = |\mathbf{d_1}||\mathbf{d_2}| \cos\theta \\ \Rightarrow \quad & -7 = 3\sqrt{3} \times 5 \times \cos\theta \\ \Rightarrow \quad & \cos\theta = -\tfrac{7}{45}\sqrt{3} \\ \Rightarrow \quad & \theta = 105.630\,358\,8 \text{ (FCD)} \\ \Rightarrow \quad & 180 - \theta = 74.369\,641\,17 \text{ (FCD)};\end{aligned}$$

so, the angle is 74.37° (2 dp).

The point A lies on l_1 and has position vector

$$\begin{pmatrix} 2 \\ 18 \\ 6 \end{pmatrix}.$$

(c) Find the distance AX, giving your answer as a surd in its simplest form. (2)

Solution

Well,

$$\begin{aligned}\overrightarrow{AX} &= \overrightarrow{OX} - \overrightarrow{OA} \\ &= \begin{pmatrix} -1 \\ 3 \\ 9 \end{pmatrix} - \begin{pmatrix} 2 \\ 18 \\ 6 \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ -15 \\ 3 \end{pmatrix},\end{aligned}$$

and

$$\begin{aligned}AX &= \sqrt{(-3)^2 + (-15)^2 + 3^2} \\ &= \underline{\underline{9\sqrt{3}}}.\end{aligned}$$

The point Y lies on l_2.

Given that the vector $\overrightarrow{YA}$ is perpendicular to the line l_1,

(d) find the distance YA, giving your answer to one decimal place. (2)

Solution

Well,

$$\begin{aligned}\tan = \frac{\text{opp}}{\text{adj}} &\Rightarrow \tan 74.369\ldots^\circ = \frac{YA}{9\sqrt{3}} \\ &\Rightarrow YA = 9\sqrt{3}\tan 74.369\ldots^\circ \\ &\Rightarrow YA = 55.717\,582\,32 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{YA = 55.7 \text{ (1 dp)}}}.\end{aligned}$$

The point B lies on l_1 where

$$|\overrightarrow{AX}| = 2|\overrightarrow{AB}|.$$

(e) Find the two possible position vectors of B. (3)

Solution

Well, the vector equation of the line l_1 passing through A is

$$\begin{aligned}\overrightarrow{AB} &= \begin{pmatrix} 4 \\ 28 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -5 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 18 \\ 6 \end{pmatrix} \\ &= \begin{pmatrix} 2-\lambda \\ 10-5\lambda \\ -2+\lambda \end{pmatrix}\end{aligned}$$

and

$$\overrightarrow{AX} = \begin{pmatrix} -3 \\ -15 \\ 3 \end{pmatrix}.$$

Now,

$$\begin{aligned}|\overrightarrow{AX}| = 2|\overrightarrow{AB}| &\Rightarrow |\overrightarrow{AX}|^2 = 4|\overrightarrow{AB}|^2 \\ &\Rightarrow 4[(2-\lambda)^2 + (10-5\lambda)^2 + (-2+\lambda)^2] = 243 \\ &\Rightarrow 4[(2-\lambda)^2 + [5(2-\lambda)]^2 + (2-\lambda)^2] = 243 \\ &\Rightarrow 4[(2-\lambda)^2 + 25(2-\lambda)^2 + (2-\lambda)^2] = 243 \\ &\Rightarrow 108(2-\lambda)^2 = 243 \\ &\Rightarrow (2-\lambda)^2 = \tfrac{9}{4} \\ &\Rightarrow 2-\lambda = -\tfrac{3}{2} \text{ or } 2-\lambda = \tfrac{3}{2} \\ &\Rightarrow \lambda = \tfrac{7}{2} \text{ or } \lambda = \tfrac{1}{2}.\end{aligned}$$

Finally,

$$\begin{aligned}\overrightarrow{OB} &= \begin{pmatrix} 4 \\ 28 \\ 4 \end{pmatrix} + \tfrac{7}{2} \begin{pmatrix} -1 \\ -5 \\ 1 \end{pmatrix} \text{ or } \overrightarrow{OB} = \begin{pmatrix} 4 \\ 28 \\ 4 \end{pmatrix} + \tfrac{1}{2} \begin{pmatrix} -1 \\ -5 \\ 1 \end{pmatrix} \\ \Rightarrow \overrightarrow{OB} &= \begin{pmatrix} 0.5 \\ 10.5 \\ 7.5 \end{pmatrix} \text{ or } \overrightarrow{OB} = \begin{pmatrix} 3.5 \\ 25.5 \\ 4.5 \end{pmatrix}.\end{aligned}$$

24. The point A with coordinates $(-3, 7, 2)$ lies on a line l_1.

    The point B also lies on the line l_1.

Given that

$$\overrightarrow{AB} = \begin{pmatrix} 4 \\ -6 \\ 2 \end{pmatrix},$$

(a) find the coordinates of point B. (2)

Solution

Well,

$$\begin{aligned} \overrightarrow{OB} &= \overrightarrow{OA} + \overrightarrow{AB} \\ &= \begin{pmatrix} -3 \\ 7 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ -6 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}; \end{aligned}$$

hence, $\underline{\underline{B(1, 1, 4)}}$.

The point P has coordinates $(9, 1, 8)$.

(b) Find the cosine of the angle PAB, giving your answer as a simplified surd. (3)

Solution

Now,

$$\begin{aligned} \overrightarrow{AP} &= \overrightarrow{OP} + \overrightarrow{OA} \\ &= \begin{pmatrix} 9 \\ 1 \\ 8 \end{pmatrix} - \begin{pmatrix} -3 \\ 7 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 12 \\ -6 \\ 6 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} AP &= \sqrt{12^2 + (-6)^2 + 6^2} \\ &= 6\sqrt{6}, \end{aligned}$$

$$\begin{aligned} AB &= \sqrt{4^2 + (-6)^2 + 2^2} \\ &= 2\sqrt{14}, \end{aligned}$$

and

$$\overrightarrow{AP} \cdot \overrightarrow{AB} = \begin{pmatrix} 12 \\ -6 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -6 \\ 2 \end{pmatrix}$$
$$= 48 + 36 + 12$$
$$= 96.$$

Finally,

$$\begin{aligned} & \overrightarrow{AP} \cdot \overrightarrow{AB} = |\overrightarrow{AP}||\overrightarrow{AB}| \cos\theta \\ \Rightarrow \quad & 96 = 6\sqrt{6} \times 2\sqrt{14} \times \cos\theta \\ \Rightarrow \quad & \underline{\underline{\cos\theta = \tfrac{4}{21}\sqrt{21}}}. \end{aligned}$$

(c) Find the exact area of triangle PAB, giving your answer in its simplest form. (3)

Solution

Well,

$$\begin{aligned} \sin^2 + \cos^2 = 1 &\Rightarrow \sin^2\theta + (\tfrac{4}{21}\sqrt{21})^2 = 1 \\ &\Rightarrow \sin^2\theta + \tfrac{16}{21} = 1 \\ &\Rightarrow \sin^2\theta = \tfrac{5}{21} \\ &\Rightarrow \sin\theta = \tfrac{\sqrt{105}}{21}, \end{aligned}$$

and

$$\begin{aligned} \text{area of triangle } PAB &= \tfrac{1}{2} \times 6\sqrt{6} \times 2\sqrt{14} \times \tfrac{\sqrt{105}}{21} \\ &= \tfrac{12}{42} \times \sqrt{6 \times 14 \times 105} \\ &= \tfrac{2}{7} \times \sqrt{8\,820} \\ &= \tfrac{2}{7} \times \sqrt{1\,764 \times 5} \\ &= \tfrac{2}{7} \times 42 \times \sqrt{5} \\ &= \underline{\underline{12\sqrt{5}}}. \end{aligned}$$

The line l_2 passes through the point P and is parallel to the line l_1.

(d) Find a vector equation for the line l_2. (2)

Solution

E.g.,

$$\underline{\underline{\mathbf{r} = \begin{pmatrix} 9 \\ 1 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -6 \\ 2 \end{pmatrix}}},$$

for scalar μ.

The point Q lies on the line l_2.

Given that the line segment AP is perpendicular to the line segment BQ,

(e) find the coordinates of the point Q. (5)

Solution

Well,

$$\begin{aligned} \overrightarrow{BQ} &= \overrightarrow{OQ} - \overrightarrow{OB} \\ &= \begin{pmatrix} 9 \\ 1 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -6 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 8 + 4\mu \\ -6\mu \\ 4 + 2\mu \end{pmatrix}. \end{aligned}$$

Now,

$$\begin{aligned} \overrightarrow{BQ} \cdot \overrightarrow{AP} = 0 &\Rightarrow \begin{pmatrix} 8 + 4\mu \\ -6\mu \\ 4 + 2\mu \end{pmatrix} \cdot \begin{pmatrix} 12 \\ -6 \\ 6 \end{pmatrix} = 0 \\ &\Rightarrow 12(8 + 4\mu) + 36\mu + 6(4 + 2\mu) = 0 \\ &\Rightarrow 96 + 48\mu + 36\mu + 24 + 12\mu = 0 \\ &\Rightarrow 96\mu = -120 \\ &\Rightarrow \mu = -\tfrac{5}{4}. \end{aligned}$$

Next,

$$\begin{aligned} \mu = -\tfrac{5}{4} \Rightarrow \overrightarrow{OQ} &= \begin{pmatrix} 9 \\ 1 \\ 8 \end{pmatrix} - \tfrac{5}{4} \begin{pmatrix} 4 \\ -6 \\ 2 \end{pmatrix} \\ \Rightarrow \overrightarrow{OQ} &= \begin{pmatrix} 4 \\ 8.5 \\ 5.5 \end{pmatrix}; \end{aligned}$$

hence, the coordinates are $\underline{\underline{Q(4, 8.5, 5.5)}}$.

25. Figure 1 shows a sketch of triangle OAB.

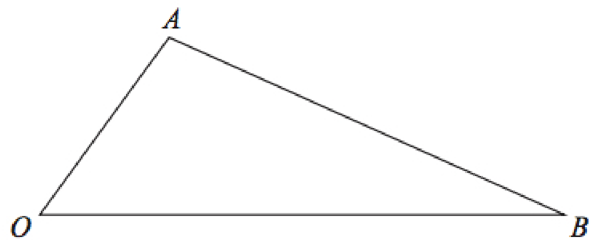


Figure 1: triangle OAB

The point C is such that $\overrightarrow{OC} = 2\overrightarrow{OA}$.
The point M is the midpoint of AB.
The straight line through C and M cuts OB at the point N.
Given $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(a) Find $\overrightarrow{CM}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. (2)

Solution

$$\begin{aligned}\overrightarrow{CM} &= \overrightarrow{CO} + \overrightarrow{OA} + \overrightarrow{AM} \\ &= -2\mathbf{a} + \mathbf{a} + \tfrac{1}{2}\overrightarrow{AB} \\ &= -\mathbf{a} + \tfrac{1}{2}(-\mathbf{a} + \mathbf{b}) \\ &= \underline{\underline{-\tfrac{3}{2}\mathbf{a} + \tfrac{1}{2}\mathbf{b}}}.\end{aligned}$$

(b) Show that (2)

$$\overrightarrow{ON} = (2 - \tfrac{3}{2}\lambda)\mathbf{a} + \tfrac{1}{2}\lambda\mathbf{b},$$

where λ is a scalar constant.

Solution

$$\begin{aligned}\overrightarrow{ON} &= \overrightarrow{OC} + \overrightarrow{CN} \\ &= \overrightarrow{OC} + \lambda\overrightarrow{CM} \\ &= 2\mathbf{a} + \lambda(-\tfrac{3}{2}\mathbf{a} + \tfrac{1}{2}\mathbf{b}) \\ &= \underline{\underline{(2 - \tfrac{3}{2}\lambda)\mathbf{a} + \tfrac{1}{2}\lambda\mathbf{b}}},\end{aligned}$$

as required.

(c) Hence prove that $ON : NB = 2 : 1$. (2)

Solution

Well,

$$\begin{aligned}2 - \tfrac{3}{2}\lambda = 0 &\Rightarrow \tfrac{3}{2}\lambda = 2 \\ &\Rightarrow \lambda = \tfrac{4}{3}\end{aligned}$$

and so

$$\overrightarrow{ON} = \tfrac{2}{3}\mathbf{b}$$

and

$$\overrightarrow{NB} = \tfrac{1}{3}\mathbf{b}.$$

Finally,

$$ON : NB = \tfrac{2}{3}\mathbf{b} : \tfrac{1}{3}\mathbf{b} = \underline{\underline{2 : 1}},$$

as required.

26. Figure 2 shows a sketch of triangle ABC.

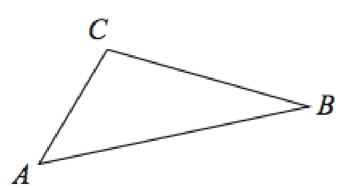


Figure 2: a sketch of triangle ABC

Given that

- $\overrightarrow{AB} = -3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k}$ and
- $\overrightarrow{BC} = \mathbf{i} + \mathbf{j} + 4\mathbf{k}$,

(a) find $\overrightarrow{AC}$, (2)

Solution

$$\begin{aligned}\overrightarrow{AC} &= \overrightarrow{AB} + \overrightarrow{BC} \\ &= (-3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k}) + (\mathbf{i} + \mathbf{j} + 4\mathbf{k}) \\ &= \underline{\underline{-2\mathbf{i} - 3\mathbf{j} - \mathbf{k}}}.\end{aligned}$$

(b) show that (3)

$$\cos ABC = \tfrac{9}{10}.$$

Solution

Well,

$$\begin{aligned}a &= BC \\ &= \sqrt{1^2 + 1^2 + 4^2} \\ &= 3\sqrt{2}, \\ b &= AC \\ &= \sqrt{(-2)^2 + (-3)^2 + (-1)^2} \\ &= \sqrt{14}, \text{ and} \\ c &= AB \\ &= \sqrt{(-3)^2 + (-4)^2 + (-5)^2} \\ &= 5\sqrt{2}.\end{aligned}$$

Now,

$$\begin{aligned}b^2 = a^2 + c^2 - 2bc\cos ABC &\Rightarrow 14 = 18 + 50 - 2 \times 3\sqrt{2} \times 5\sqrt{2} \times \cos ABC \\ &\Rightarrow -54 = -60\cos ABC \\ &\Rightarrow \underline{\underline{\cos ABC = \tfrac{9}{10}}},\end{aligned}$$

as required.

27. Figure 3 shows a sketch of a parallelogram $PQRS$.

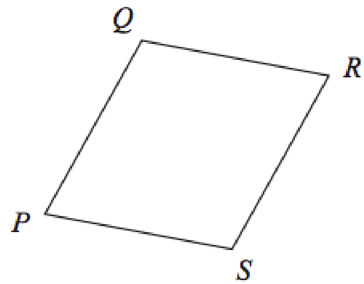


Figure 3: a parallelogram $PQRS$

Given that

- $\overrightarrow{PQ} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ and
- $\overrightarrow{QR} = 5\mathbf{i} - 2\mathbf{k}$,

(a) show that parallelogram $PQRS$ is a rhombus. (2)

Solution
Well,

$$\begin{aligned} |\overrightarrow{PQ}| &= \sqrt{2^2 + 3^2 + (-4)^2} \\ &= \sqrt{4 + 9 + 16} \\ &= \sqrt{29} \end{aligned}$$

and

$$\begin{aligned} |\overrightarrow{QR}| &= \sqrt{5^2 + (-2)^2} \\ &= \sqrt{25 + 4} \\ &= \sqrt{29}; \end{aligned}$$

so, the parallelogram $PQRS$ is a rhombus as all four sides are the same length.

(b) Find the exact area of the rhombus $PQRS$. (4)

Solution
Now,

$$\begin{aligned} \overrightarrow{PR} &= \overrightarrow{PQ} + \overrightarrow{QR} \\ &= (2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) + (5\mathbf{i} - 2\mathbf{k}) \\ &= 7\mathbf{i} + 3\mathbf{j} - 6\mathbf{k} \end{aligned}$$

and

$$
\begin{aligned}
|\overrightarrow{PR}| &= \sqrt{7^2 + 3^2 + (-6)^2} \\
&= \sqrt{49 + 9 + 36} \\
&= \sqrt{94}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
\overrightarrow{QS} &= \overrightarrow{QP} + \overrightarrow{PS} \\
&= -\overrightarrow{PQ} + \overrightarrow{QR} \\
&= -(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) + (5\mathbf{i} - 2\mathbf{k}) \\
&= -2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) + 5\mathbf{i} - 2\mathbf{k} \\
&= 3\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}
\end{aligned}
$$

and

$$
\begin{aligned}
|\overrightarrow{QS}| &= \sqrt{3^2 + (-3)^2 + 2^2} \\
&= \sqrt{9 + 9 + 4} \\
&= \sqrt{22}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\text{area of } PQRS &= \tfrac{1}{2}|\overrightarrow{PR}||\overrightarrow{QS}| \\
&= \tfrac{1}{2} \times \sqrt{94} \times \sqrt{22} \\
&= \underline{\underline{\sqrt{517}}}.
\end{aligned}
$$