Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2010 November Paper 2 Variant 3: Calculator
2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

1. The two variables x and y are such that

$$y = \frac{10}{(x+4)^3}.$$

(a) Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$. (2)

Solution
Well,

$$y = \frac{10}{(x+4)^3} \Rightarrow y = 10(x+4)^{-3}$$
$$\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = -30(x+4)^{-4}}}.$$

(b) Hence find the approximate change in y as x increases from 6 to $6 + p$, where p is small. (2)

Solution
Now,

$$\begin{aligned} \delta y &\approx \frac{\mathrm{d}y}{\mathrm{d}x} \times \delta x \\ &= -30(6+4)^{-4} \times p \\ &= \underline{\underline{-0.003p}}. \end{aligned}$$

2. Find the equation of the curve which passes through the point $(4, 22)$ and for which (4)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x(x-2).$$

Solution

Now,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x(x-2) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 6x$$
$$\Rightarrow y = x^3 - 3x^2 + c,$$

where c is some constant. Next,

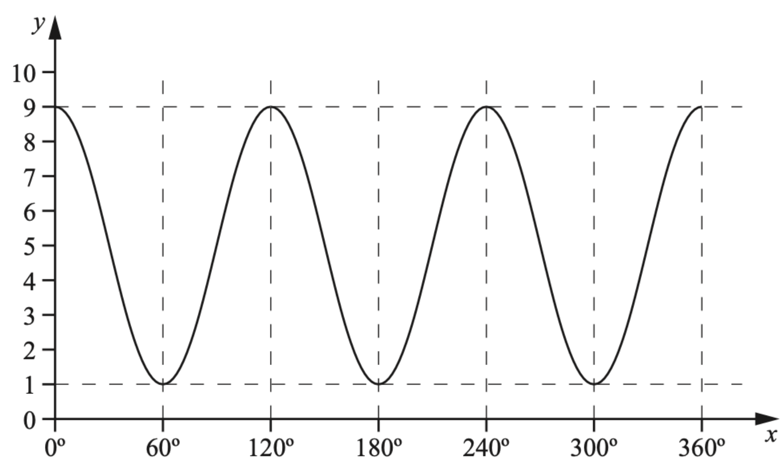
$$x = 4, y = 22 \Rightarrow 22 = 4^3 - 3(4^2) + c$$
$$\Rightarrow 22 = 64 - 48 + c$$
$$\Rightarrow c = 6;$$

hence,

$$\underline{\underline{y = x^3 - 3x^2 + 6}}.$$

3. (a) The diagram shows the curve

$$y = A\cos Bx + C, \text{ for } 0^\circ \leqslant x \leqslant 360^\circ.$$



Find the value of

(b) (i) A, (1)

Solution
$\underline{\underline{A = 4}}$.

(ii) B, (1)

Solution
$\underline{\underline{B = 3}}$.

(iii) C. (1)

Solution
$\underline{\underline{C = 5}}$.

(c) Given that

$$f(x) = 6\sin 2x + 7,$$

state

(i) the period of f, (1)

Solution

$$\frac{360}{2} = \underline{\underline{180^\circ}}.$$

(ii) the amplitude of f. (1)

Solution
$\underline{\underline{6}}$.

4. (a) Find, in ascending powers of x, the first 4 terms of the expansion of (2)

$$(1 + x)^6.$$

Solution
Well,

$$\begin{aligned}(1 + x)^6 &= 1 + \binom{6}{1}x + \binom{6}{2}x^2 + \binom{6}{3}x^3 + \ldots \\ &= \underline{\underline{1 + 6x + 15x^2 + 20x^3 + \ldots}}.\end{aligned}$$

(b) Hence find the coefficient of p^3 in the expansion of (3)

$$(1 + p - p^2)^6.$$

Solution

Now,

$$\begin{aligned}(1+p-p^2)^6 &= [1+(p-p^2)]^6 \\ &= 1 + 6(p-p^2) + 15(p-p^2)^2 + 20(p-p^2)^3 + \ldots.\end{aligned}$$

Next,

$\times$	p	$-p^2$
p	p^2	$-p^3$
$-p^2$	$-p^3$	$+p^4$

and

$\times$	p^2	$-2p^3$	$+p^4$
p	p^3	$\ldots$	$\ldots$
$-p^2$	$\ldots$	$\ldots$	$\ldots$

Hence,

$$\begin{aligned}(1+p-p^2)^6 &= \ldots + 15(\ldots - 2p^3 + \ldots) + 20(p^3 + \ldots) + \ldots \\ &= \ldots - 10p^3 + \ldots;\end{aligned}$$

so, the coefficient of p^3 is $\underline{\underline{-10}}$.

5. (a) Given that (2)

$$\mathbf{A} = \begin{pmatrix} 2 & -4 & 1 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 3 & -1 \\ 0 & 5 \\ -2 & 7 \end{pmatrix},$$

find **AB**.

Solution

Well,

$$\begin{pmatrix} 2 & -4 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 0 & 5 \\ -2 & 7 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 4 & -15 \end{pmatrix}}}.$$

(b) Given that

$$\mathbf{C} = \begin{pmatrix} 3 & 5 \\ -2 & -4 \end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix} 6 & -4 \\ 2 & 8 \end{pmatrix},$$

find

(i) the inverse matrix $\mathbf{C}^{-1}$, (2)

Solution

Now,

$$\det \mathbf{C} = -12 + 10 = -2$$

and

$$\mathbf{C}^{-1} = \underline{\underline{-\tfrac{1}{2} \begin{pmatrix} -4 & -5 \\ 2 & 3 \end{pmatrix}}}.$$

(ii) the matrix $\mathbf{X}$ such that (2)

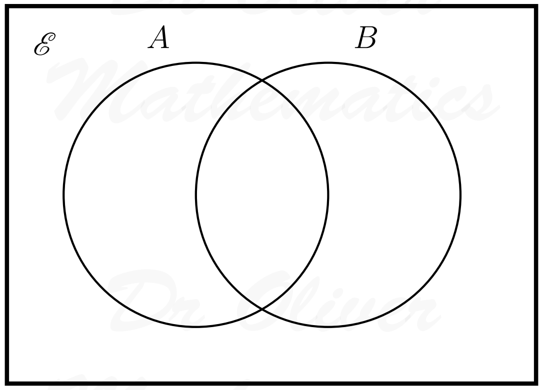
$$\mathbf{CX} = \mathbf{D}.$$

Solution

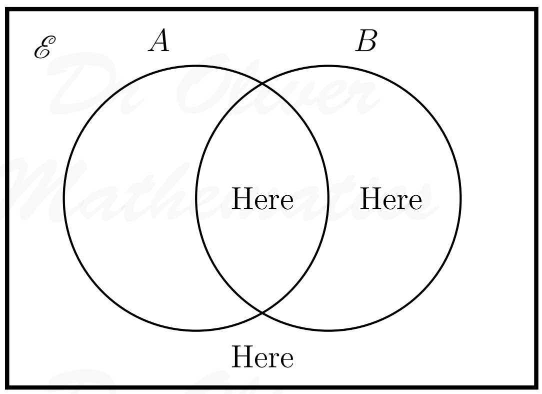
$$\begin{aligned} \mathbf{CX} = \mathbf{D} &\Rightarrow \mathbf{X} = \mathbf{C}^{-1}\mathbf{D} \\ &\Rightarrow \mathbf{X} = -\tfrac{1}{2} \begin{pmatrix} -4 & -5 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 6 & -4 \\ 2 & 8 \end{pmatrix} \\ &\Rightarrow \mathbf{X} = -\tfrac{1}{2} \begin{pmatrix} -34 & -24 \\ 18 & 16 \end{pmatrix} \\ &\Rightarrow \mathbf{X} = \underline{\underline{\begin{pmatrix} 17 & 12 \\ -9 & -8 \end{pmatrix}}}. \end{aligned}$$

6. (a) Copy the diagram above and shade the region which represents the set (1)

$$A' \cup B.$$



Solution

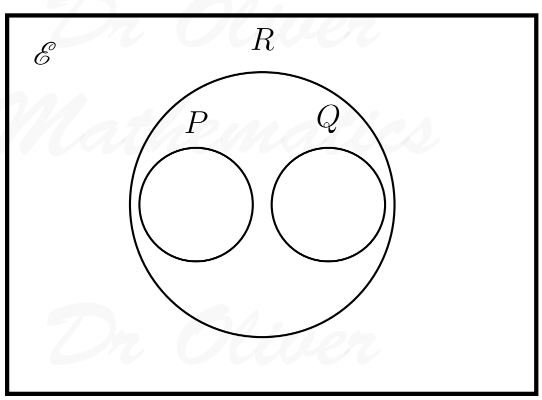


(b) The sets P, Q, and R are such that (2)

$$P \cap Q = \varnothing \text{ and } P \cup Q \subset R.$$

Draw a Venn diagram showing the sets P, Q, and R.

Solution



(c) In a group of 50 students, (3)

- F denotes the set of students who speak French and
- S denotes the set of students who speak Spanish.

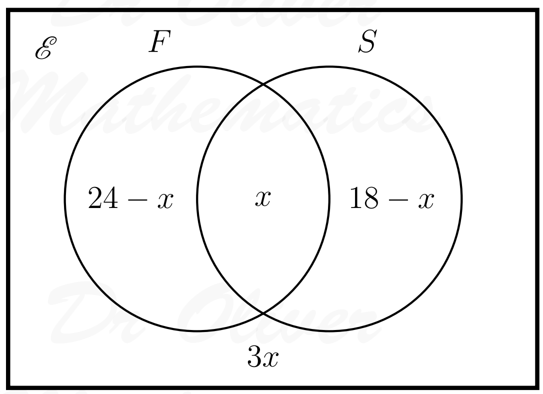
It is given that

- $\mathrm{n}(F) = 24$,
- $\mathrm{n}(S) = 18$,
- $\mathrm{n}(F \cap S) = x$, and
- $\mathrm{n}(F' \cap S') = 3x$.

Write down an equation in x and hence find the number of students in the group who speak neither French nor Spanish.

Solution

Well, this is how the Venn diagram looks:



Now,

$$\begin{aligned}(24 - x) + x(18 - x) + 3x = 50 &\Rightarrow 2x + 42 = 50\\ &\Rightarrow 2x = 8\\ &\Rightarrow 3x = 12;\end{aligned}$$

hence, the number of students in the group who speak neither French nor Spanish is $\underline{\underline{12}}$.

7. The line (7)

$$y = 2x - 6$$

meets the curve

$$4x^2 + 2xy - y^2 = 124$$

at the points A and B.

Find the length of the line AB.

Solution

Well,

$\times$	$2x$	-6
$2x$	$4x^2$	$-12x$
-6	$-12x$	$+36$

and

$$\begin{aligned} 4x^2 + 2xy - y^2 = 124 &\Rightarrow 4x^2 + 2x(2x-6) - (2x-6)^2 = 124 \\ &\Rightarrow 4x^2 + (4x^2x - 12x) - (4x^2 - 24x + 36) = 124 \\ &\Rightarrow 4x^2 + 12x - 160 = 0 \\ &\Rightarrow 4(x^2 + 3x - 40) = 0 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & +3 \\ \text{multiply to:} & -40 \end{array}\right\} + 8, -5$$

$$\begin{aligned} &\Rightarrow 4(x+8)(x-5) = 0 \\ &\Rightarrow x = -8 \text{ or } x = 5 \\ &\Rightarrow y = -22 \text{ or } y = 4; \end{aligned}$$

so, $A(-8, -22)$ and $(B(5, 4)$.

Finally,

$$\begin{aligned} AB &= \sqrt{[5-(-8)]^2 + [4-(-22)]^2} \\ &= \sqrt{13^2 + 26^2} \\ &= \sqrt{169 + 676} \\ &= \sqrt{845} \\ &= \underline{\underline{13\sqrt{5}}}. \end{aligned}$$

8. (a) Show that (1)

$$(5 + 3\sqrt{2})^2 = 43 + 30\sqrt{2}.$$

Solution
Well,

$\times$	5	$+3\sqrt{2}$
5	25	$+15\sqrt{2}$
$+3\sqrt{2}$	$+15\sqrt{2}$	$+18$

and so

$$\begin{aligned}(5 + 3\sqrt{2})^2 &= 25 + 30\sqrt{2} + 18 \\ &= \underline{\underline{43 + 30\sqrt{2}}},\end{aligned}$$

as required.

Hence find, **without using a calculator**, the positive square root of

(b) $86 + 60\sqrt{2}$, giving your answer in the form $a + b\sqrt{2}$, where a and b are integers, (2)

Solution
Now,

$$\begin{aligned}\sqrt{86 + 60\sqrt{2}} &= \sqrt{2(43 + 30\sqrt{2})} \\ &= \sqrt{2} \times \sqrt{43 + 30\sqrt{2}} \\ &= \sqrt{2}\sqrt{(5 + 3\sqrt{2})^2} \\ &= \sqrt{2}(5 + 3\sqrt{2}) \\ &= \underline{\underline{6 + 5\sqrt{2}}};\end{aligned}$$

hence, $\underline{\underline{a = 6}}$ and $\underline{\underline{b = 5}}$.

(c) $43 - 30\sqrt{2}$, giving your answer in the form $c + d\sqrt{2}$, where c and d are integers, (1)

Solution
Well,

$\times$	5	$-3\sqrt{2}$
5	25	$-15\sqrt{2}$
$-3\sqrt{2}$	$-15\sqrt{2}$	$+18$

and so

$$(5-3\sqrt{2})^2 = 43 - 30\sqrt{2}.$$

So,

$$\sqrt{43-30\sqrt{2}} = \underline{\underline{5-3\sqrt{2}}};$$

hence, $\underline{\underline{c=6}}$ and $\underline{\underline{d=-5}}$.

(d) $\dfrac{1}{43+30\sqrt{2}}$, giving your answer in the form (3)

$$\frac{f+g\sqrt{2}}{h},$$

where f, g, and h are integers.

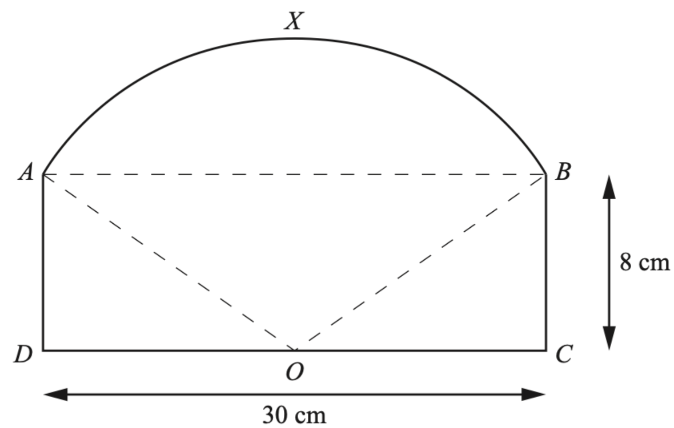
Solution

$$\begin{aligned}\frac{1}{\sqrt{43+30\sqrt{2}}} &= \frac{1}{5+3\sqrt{2}} \\ &= \frac{1}{5+3\sqrt{2}} \times \frac{5-3\sqrt{2}}{5-3\sqrt{2}}\end{aligned}$$

$\times$	5	$+3\sqrt{2}$
5	25	$+15\sqrt{2}$
$-3\sqrt{2}$	$-15\sqrt{2}$	-18

$$\begin{aligned} &= \frac{5-3\sqrt{2}}{25-18} \\ &= \frac{5-3\sqrt{2}}{7} \\ &= \tfrac{5}{7} - \tfrac{3}{7}\sqrt{2}\end{aligned}$$

9. The diagram shows a rectangle $ABCD$ and an arc AXB of a circle with centre at O, the mid-point of DC.



The lengths of DC and BC are 30 cm and 8 cm respectively.

Find

(a) the length of OA, (2)

Solution

Pythagoras' theorem:

$$\begin{aligned} OA &= \sqrt{OD^2 + AD^2} \\ &= \sqrt{15^2 + 8^2} \\ &= \sqrt{225 + 64} \\ &= \sqrt{289} \\ &= \underline{\underline{17\text{ cm}}}. \end{aligned}$$

(b) the angle AOB, in radians, (2)

Solution

Well,

$$\tan = \frac{\text{opp}}{\text{adj}} \Rightarrow \tan AOD = \tfrac{8}{15}$$

and

$$\begin{aligned}\text{angle } AOB &= \pi - 2\tan^{-1}\left(\tfrac{8}{15}\right)\\ &= 2.161\,678\,001 \text{ (FCD)}\\ &= \underline{\underline{2.16 \text{ radians (3 sf)}}}.\end{aligned}$$

(c) the perimeter of figure $ADOCBXA$, (2)

Solution

Well,

$$\begin{aligned}\text{perimeter} &= DA + \text{arc } AXB + BC + CD\\ &= 8 + (17 \times 2.161\ldots) + 8 + 30\\ &= 82.748\,526\,02 \text{ (FCD)}\\ &= \underline{\underline{82.7 \text{ cm (3 sf)}}}.\end{aligned}$$

(d) the area of figure $ADOCBXA$. (2)

Solution

Now,

$$\begin{aligned}\text{area} &= (2 \times \text{area of } OAD) + \text{area of } OAXBO\\ &= (2 \times \tfrac{1}{2} \times 8 \times 15) + (\tfrac{1}{2} \times 17^2 \times 2.161\ldots)\\ &= 432.362\,471\,2 \text{ (FCD)}\\ &= \underline{\underline{432 \text{ cm}^2 \text{ (3 sf)}}}.\end{aligned}$$

10. The equation of a curve is (9)

$$y = x^2 \mathrm{e}^x.$$

- The tangent to the curve at the point $P(1, \mathrm{e})$ meets the y-axis at the point A.
- The normal to the curve at P meets the x-axis at the point B.

Find the area of the triangle OAB, where O is the origin.

Solution

Product rule:

$$u = x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$$
$$v = \mathrm{e}^x \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x$$

so

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= (x^2)(\mathrm{e}^x) + (2x)(\mathrm{e}^x)\\ &= x\mathrm{e}^x(x+2).\end{aligned}$$

Now,

$$\begin{aligned}x = 1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= 3\mathrm{e}\\ \Rightarrow m_{\text{normal}} &= -\frac{1}{3\mathrm{e}}.\end{aligned}$$

<u>Tangent</u> :

$$y - \mathrm{e} = 3\mathrm{e}(x-1)$$

and so

$$\begin{aligned}x = 0 \Rightarrow y - \mathrm{e} &= 3\mathrm{e}(0-1)\\ \Rightarrow y &= -2\mathrm{e};\end{aligned}$$

hence, $A(0, -2\mathrm{e})$.

<u>Normal</u> :

$$y - \mathrm{e} = -\frac{1}{3\mathrm{e}}(x-1)$$

and so

$$\begin{aligned}y = 0 \Rightarrow -\mathrm{e} &= -\frac{1}{3\mathrm{e}}(x-1)\\ \Rightarrow 3\mathrm{e}^2 &= x - 1\\ \Rightarrow x &= 3\mathrm{e}^2 + 1;\end{aligned}$$

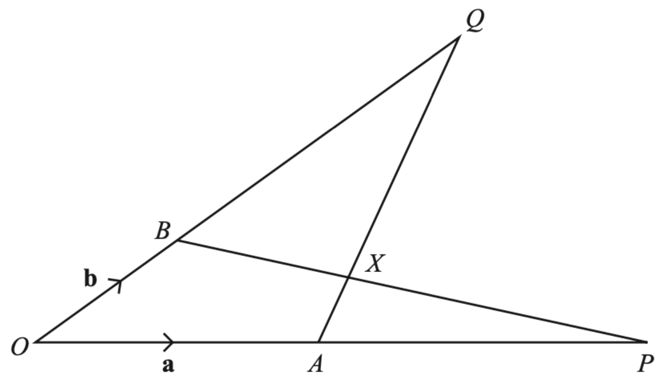
hence, $B(3\mathrm{e}^2 + 1, 0)$.

Finally,

$$\begin{aligned}\text{area of the triangle} &= \tfrac{1}{2} \times OA \times OB\\ &= \tfrac{1}{2} \times 2\mathrm{e} \times (3\mathrm{e}^2 + 1)\\ &= \underline{\underline{\mathrm{e}(3\mathrm{e}^2 + 1)}}.\end{aligned}$$

11. In the diagram,

- $\overrightarrow{OA} = \mathbf{a}$,
- $\overrightarrow{OB} = \mathbf{b}$,
- $\overrightarrow{OP} = 2\mathbf{a}$, and
- $\overrightarrow{OQ} = 3\mathbf{b}$.



(a) Given that

$$\overrightarrow{AX} = \mu\overrightarrow{AQ},$$

express $\overrightarrow{OX}$ in terms of μ, $\mathbf{a}$, and $\mathbf{b}$. (3)

Solution

$$\begin{aligned}\overrightarrow{OX} &= \overrightarrow{OA} + \overrightarrow{AX} \\ &= \overrightarrow{OA} + \mu(\overrightarrow{AO} + \overrightarrow{OQ}) \\ &= \overrightarrow{OA} + \mu(-\overrightarrow{OA} + \overrightarrow{OQ}) \\ &= \underline{\underline{\mathbf{a} + \mu(-\mathbf{a} + 3\mathbf{b})}}. \quad (1)\end{aligned}$$

(b) Given that

$$\overrightarrow{BX} = \lambda\overrightarrow{BP},$$

express $\overrightarrow{OX}$ in terms of λ, $\mathbf{a}$, and $\mathbf{b}$. (3)

Solution

$$\begin{aligned}\overrightarrow{OX} &= \overrightarrow{OB} + \overrightarrow{BX} \\ &= \overrightarrow{OB} + \lambda(\overrightarrow{BO} + \overrightarrow{OP}) \\ &= \overrightarrow{OB} + \lambda(-\overrightarrow{OB} + \overrightarrow{OP}) \\ &= \underline{\underline{\mathbf{b} + \lambda(2\mathbf{a} - \mathbf{b})}}. \quad (2)\end{aligned}$$

(c) Hence find the value of μ and of λ. (3)

Solution

From (1),

$$\overrightarrow{OX} = (1-\mu)\mathbf{a} + 3\mu\mathbf{b}$$

and, from (2),

$$\overrightarrow{OX} = 2\lambda\mathbf{a} + (1-\lambda)\mathbf{b}.$$

So,

$$\begin{aligned}1 - \mu &= 2\lambda \quad (3) \\ 3\mu &= 1 - \lambda. \quad (4)\end{aligned}$$

Do $2 \times (4)$:

$$6\mu = 2 - 2\lambda \quad (5)$$

and add $(3) + (5)$:

$$\begin{aligned}1 + 5\mu = 2 \Rightarrow 5\mu &= 1 \\ \Rightarrow \underline{\underline{\mu = \tfrac{1}{5}}} \\ \Rightarrow \underline{\underline{\lambda = \tfrac{2}{5}}}.\end{aligned}$$

EITHER

12. The table shows values of the variables v and p which are related by the equation

$$p = \frac{a}{v^2} + \frac{b}{v},$$

where a and b are constants.

v	2	4	6	8
p	6.22	2.84	1.83	1.35

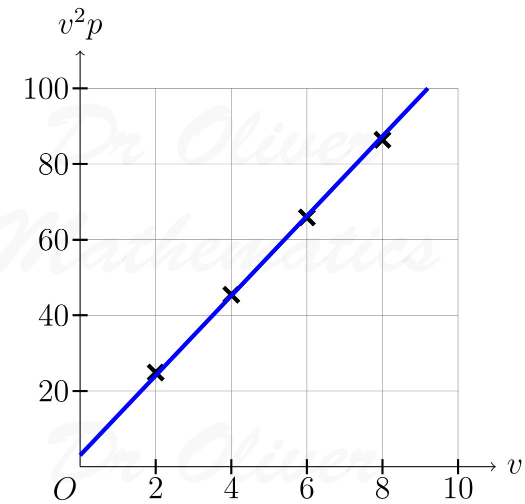
(a) Using graph paper, plot v^2p on the y-axis against v on the x-axis and draw a straight line graph. (2)

Solution

Well,

v	2	4	6	8
v^2p	24.88	45.44	65.88	86.4

and so we plot the graph:



(b) Use your graph to estimate the value of a and of b. (4)

Solution

Now, the straight line goes $(0, 3)$ and $(9.2, 100)$ and

$$m = \frac{100 - 3}{9.2 - 0} = 10.543\ldots;$$

we will say $m = 10.5$. Next, the equation of the line is

$$v^2p - 3 = 10.5(v - 0) \Rightarrow v^2p - 3 = 10.5v \Rightarrow v^2p = 10.5v + 3 \Rightarrow p = \frac{3}{v^2} + \frac{10.5}{v};$$

hence, $\underline{\underline{a = 3}}$ and $\underline{\underline{b = 10.5}}$.

In another method of finding a and b from a straight line graph, $\frac{1}{v}$ is plotted along the x-axis.

In this case, and without drawing a second graph,

(c) state the variable that should be plotted on the y-axis, (2)

Solution
Well,

$$p = \frac{3}{v^2} + \frac{10.5}{v} \Rightarrow pv = \frac{3}{v} + 10.5,$$

so the variable that should be plotted on the y-axis is $\underline{\underline{pv}}$.

(d) explain how the values of a and of b could be obtained. (2)

Solution
The gradient is $\underline{\underline{a}}$ and the y-intercept is $\underline{\underline{b}}$.

OR

13. The table shows experimental values of two variables r and t.

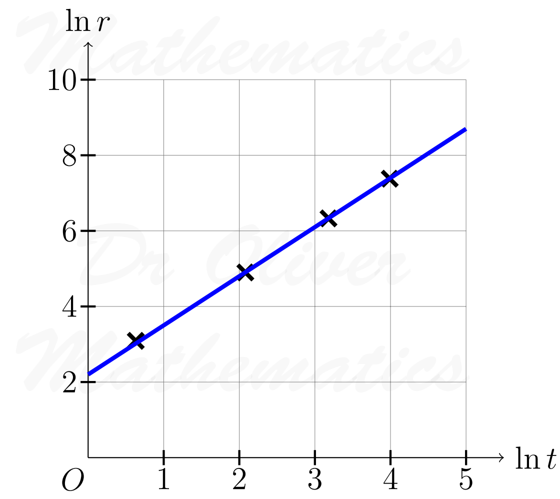
t	2	8	24	54
r	22	134	560	1 608

(a) Using the y-axis for $\ln r$ and the x-axis for $\ln t$, plot $\ln r$ against $\ln t$ to obtain a straight line graph. (2)

Solution
We will use 2 decimal places:

$\ln t$	0.63	2.08	3.18	3.99
$\ln r$	3.09	4.90	6.33	7.38

and so we plot the graph:



(b) Find the gradient and the intercept on the y-axis of this graph and express r in terms of t. (6)

Solution

Now, the straight line goes $(0, 2.2)$ and $(5, 8.7)$ and

$$\begin{aligned} m &= \frac{8.7 - 2.2}{5 - 0} \\ &= 1.3 \end{aligned}$$

Next, the equation of the line is

$$\begin{aligned} \ln r - 8.7 = 1.3(\ln t - 5) &\Rightarrow \ln r - 8.7 = 1.3 \ln t - 6.5 \\ &\Rightarrow \ln r - = 1.3 \ln t + 2.2 \\ &\Rightarrow \ln r - \ln t^{1.3} + 2.2 \\ &\Rightarrow \ln r - \ln t^{1.3} = 2.2 \\ &\Rightarrow \ln\left(\frac{r}{t^{1.3}}\right) = 2.2 \\ &\Rightarrow \frac{r}{t^{1.3}} = \mathrm{e}^{2.2} \\ &\Rightarrow \underline{\underline{r = \mathrm{e}^{2.2} t^{1.3}}}. \end{aligned}$$

Another method of finding the relationship between r and t from a straight line graph is to plot $\log_{10} r$ on the y-axis and $\log_{10} t$ on the x-axis.

(c) Without drawing this second graph, find the value of the gradient and of the intercept on the y-axis for this graph. (2)

Solution

The gradient is $\underline{\underline{1.3}}$ (does not make a difference which log we take) but the y-intercept is

$$\underline{\underline{\log_{10}(e^{2.2}) = 0.955 \text{ (3 sf)}}}.$$