Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2010 June Paper 1 Variant 2: Calculator
2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

1. Find the coordinates of the points of intersection of the curve (5)

$$y^2 + y = 10x - 8x^2$$

and the straight line

$$y + 4x + 1 = 0.$$

Solution

Well,

$$y + 4x + 1 = 0 \Rightarrow y = -4x - 1$$

and

$\times$	$-4x$	-1
$-4x$	$16x^2$	$+4x$
-1	$+4x$	$+1$

Now,

$$\begin{aligned} y^2 + y = 10x - 8x^2 &\Rightarrow (16x^2 + 8x + 1) + (-4x - 1) = 10x - 8x^2 \\ &\Rightarrow 24x^2 - 6x = 0 \\ &\Rightarrow 6x(4x - 1) = 0 \\ &\Rightarrow x = 0 \text{ or } x = \tfrac{1}{4} \\ &\Rightarrow y = -1 \text{ or } y = -2; \end{aligned}$$

hence, the coordinates of the points are

$$\underline{\underline{(0, -1) \text{ and } (\tfrac{1}{4}, -2)}}.$$

2. The expression (5)

$$6x^3 + ax^2 - (a+1)x + b$$

has a remainder of 15 when divided by $(x+2)$ and a remainder of 24 when divided by $(x+1)$.

Show that $a = 8$ and find the value of b.

Solution

We use synthetic division twice:

$$\begin{array}{r|cccc} -2 & 6 & a & -a-1 & b \\ & \downarrow & -12 & -2a+24 & 6a-46 \\ \hline & 6 & a-12 & -3a+23 & 6a+b-46 \end{array}$$

and

$$\begin{array}{r|cccc} -1 & 6 & a & -a-1 & b \\ & \downarrow & -6 & -a+6 & 2a-5 \\ \hline & 6 & a-6 & -2a+5 & 2a+b-5 \end{array}$$

so

$$\begin{aligned} 6a + b - 46 = 15 &\Rightarrow 6a + b = 61 \quad (1) \\ 2a + b - 5 = 24 &\Rightarrow 2a + b = 29 \quad (2). \end{aligned}$$

Do (1) − (2):

$$4a = 32 \Rightarrow \underline{\underline{a = 8}}$$

and

$$\begin{aligned} 6(8) + b = 61 &\Rightarrow 48 + b = 61 \\ &\Rightarrow \underline{\underline{b = 13}}. \end{aligned}$$

[Check: $2(8) + (13) = 29$ ✓]

3. Given that

$$\overrightarrow{OA} = \begin{pmatrix} -17 \\ 25 \end{pmatrix} \text{ and } \overrightarrow{OB} = \begin{pmatrix} 4 \\ 5 \end{pmatrix},$$

find

(a) the unit vector parallel to $\overrightarrow{AB}$, (3)

Solution

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -\overrightarrow{OA} + \overrightarrow{OB} \\ &= -\begin{pmatrix} -17 \\ 25 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} 21 \\ -20 \end{pmatrix}.\end{aligned}$$

Now,

$$\begin{aligned}AB &= \sqrt{21^2 + (-20)^2} \\ &= \sqrt{441 + 400} \\ &= \sqrt{841} \\ &= 29.\end{aligned}$$

Finally, a unit vector parallel to $\overrightarrow{AB}$ is

$$\underline{\underline{\tfrac{1}{29}\begin{pmatrix} 21 \\ -20 \end{pmatrix}}}.$$

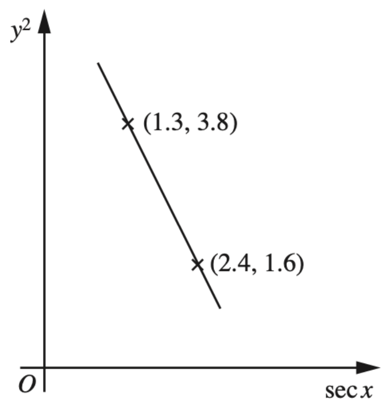
(b) the vector $\overrightarrow{OC}$, such that $\overrightarrow{AC} = 3\overrightarrow{AB}$. (2)

Solution

Now,

$$\begin{aligned}\overrightarrow{OC} &= \overrightarrow{OA} + \overrightarrow{AC} \\ &= \overrightarrow{OA} + 3\overrightarrow{AB} \\ &= \begin{pmatrix} -17 \\ 25 \end{pmatrix} + 3\begin{pmatrix} 21 \\ -20 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 46 \\ -35 \end{pmatrix}}}.\end{aligned}$$

4. Variables x and y are such that, when y^2 is plotted against $\sec x$, a straight line graph passing through the points $(2.4, 1.6)$ and $(1.3, 3.8)$ is obtained.



(a) Express y^2 in terms of $\sec x$. (3)

Solution
Well,

$$m = \frac{3.8 - 1.6}{1.3 - 2.4} = \frac{2.2}{-1.1} = -2$$

and the equation of the line is

$$y^2 - 3.8 = -2(\sec x - 1.3) \Rightarrow y^2 - 3.8 = -2\sec x + 2.6 \Rightarrow \underline{\underline{y^2 = -2\sec x + 6.4}}.$$

(b) Hence find the exact value of $\cos x$ when $y = 2$. (2)

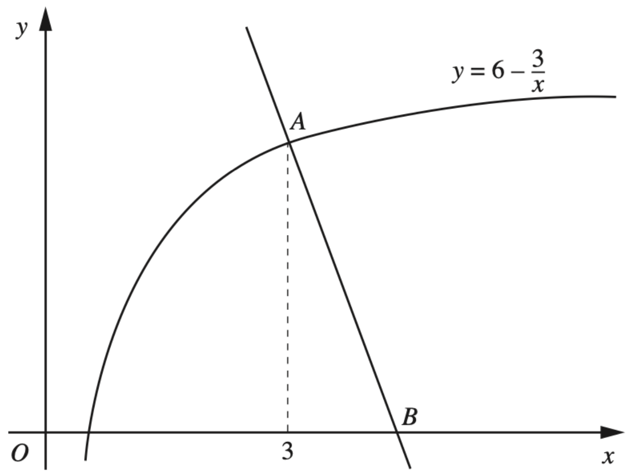
Solution
Now,

$$y = 2 \Rightarrow 2^2 = -2\sec x + 6.4 \Rightarrow 4 = -2\sec x + 6.4 \Rightarrow -2.4 = -2\sec x \Rightarrow \sec x = 1.2 \Rightarrow \underline{\underline{\cos x = \tfrac{5}{6}}}.$$

5. The diagram shows part of the curve (5)

$$y = 6 - \frac{3}{x}$$

which passes through the point A where $x = 3$.



The normal to the curve at the point A meets the x-axis at the point B.

Find the coordinates of the point B.

Solution

Well,

$$x = 3 \Rightarrow y = 5$$

and $A(3, 5)$. Now,

$$\begin{aligned} y = 6 - \frac{3}{x} &\Rightarrow y = 6 - 3x^{-1} \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^{-2}. \end{aligned}$$

Now,

$$\begin{aligned} x = 3 &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{1}{3} \\ &\Rightarrow m_{\text{normal}} = -3. \end{aligned}$$

Next, the equation of the normal is

$$y - 5 = -3(x - 3)$$

and

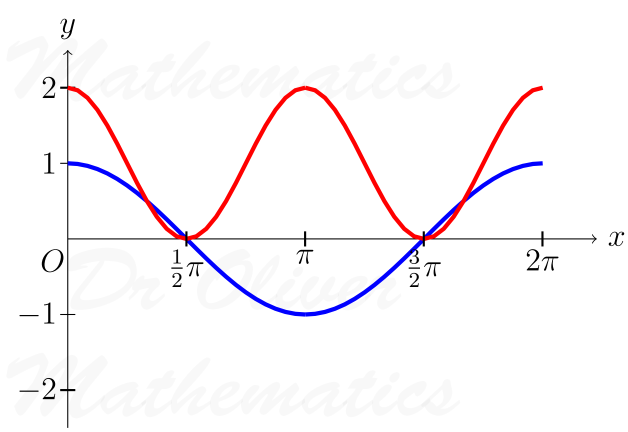
$$\begin{aligned} y = 0 &\Rightarrow -5 = -3x + 9 \\ &\Rightarrow 3x = 14 \\ &\Rightarrow x = 4\tfrac{2}{3}; \end{aligned}$$

hence, $\underline{\underline{B(4\frac{2}{3}, 0)}}$.

6. (a) (i) On the same diagram, sketch the curves (3)

$$y = \cos x \text{ and } y = 1 + \cos 2x \text{ for } 0 \leqslant x \leqslant 2\pi.$$

Solution



(ii) Hence state the **number** of solutions of the equation (1)

$$\cos 2x - \cos x + 1 = 0 \text{ for } 0 \leqslant x \leqslant 2\pi.$$

Solution
Now,

$$\cos x = 1 + \cos 2x \Rightarrow \cos 2x - \cos x + 1 = 0$$

so there are <u>4 solutions</u>.

(b) The function f is given by

$$\mathrm{f}(x) = 5 \sin 3x.$$

Find

(i) the amplitude of f, (1)

Solution
$\underline{\underline{5}}$.

(ii) the period of f. (1)

Solution
$\underline{\underline{\frac{2}{3}\pi}}$.

7. The table shows values of the variables p and v which are related by the equation

$$p = kv^n,$$

where k and n are constants.

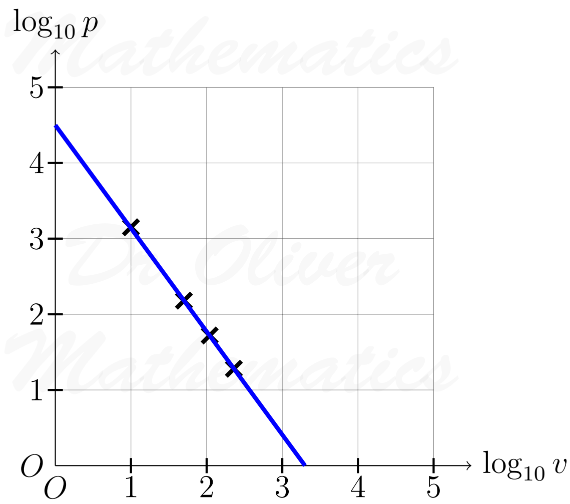
v	10	50	110	230
p	1 412	151	53	19

(a) Using graph paper, plot $\log_{10} p$ against $\log_{10} v$ and draw a straight line graph. (3)

Solution
We will take 2 decimal places:

$\log_{10} v$	1	1.70	2.04	2.36
$\log_{10} p$	3.15	2.18	1.72	1.28

and then we draw the graph:



Use your graph to estimate

(b) the value of n, (2)

Solution

The line goes through $(0, 4.5)$ and $(3.3, 0)$:

$$m = \frac{4.5 - 0}{0 - 3.3} = -\tfrac{15}{11}.$$

Now,

$$\begin{aligned}\log_{10} p - 4.5 = -\tfrac{15}{11}(\log_{10} v - 0) &\Rightarrow \log_{10} p - 4.5 = -\tfrac{15}{11} \log_{10} v \\ &\Rightarrow \log_{10} p + \tfrac{15}{11} \log_{10} v = 4.5 \\ &\Rightarrow \log_{10} p + \log_{10} v^{\frac{15}{11}} = 4.5 \\ &\Rightarrow \log_{10} \left[p v^{\frac{15}{11}} \right] = 4.5 \\ &\Rightarrow p v^{\frac{15}{11}} = 10^{4.5} \\ &\Rightarrow p = 10^{4.5} v^{-\frac{15}{11}};\end{aligned}$$

hence, $n = -\tfrac{15}{11}$.

(c) the value of p when $v = 170$. (2)

Solution

$$\begin{aligned} v = 170 \Rightarrow p &= 10^{4.5} 170^{-\frac{15}{11}} \\ \Rightarrow p &= 28.739\,469\,26 \text{ (FCD)} \\ \Rightarrow \underline{\underline{p = 28.7 \text{ (3 sf)}}}. \end{aligned}$$

8. Given that

$$\mathbf{A} = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} -2 & 0 \\ 1 & 4 \end{pmatrix},$$

find

(a) $3\mathbf{A} - 2\mathbf{B}$, (2)

Solution

$$\begin{aligned} 3\mathbf{A} - 2\mathbf{B} &= 3\begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix} - 2\begin{pmatrix} -2 & 0 \\ 1 & 4 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 16 & 9 \\ 1 & -2 \end{pmatrix}}}. \end{aligned}$$

(b) $\mathbf{A}^{-1}$, (2)

Solution
Well,

$$\det \mathbf{A} = 8 - 3 = 5$$

and so

$$\underline{\underline{\mathbf{A}^{-1} = \tfrac{1}{5}\begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix}}}.$$

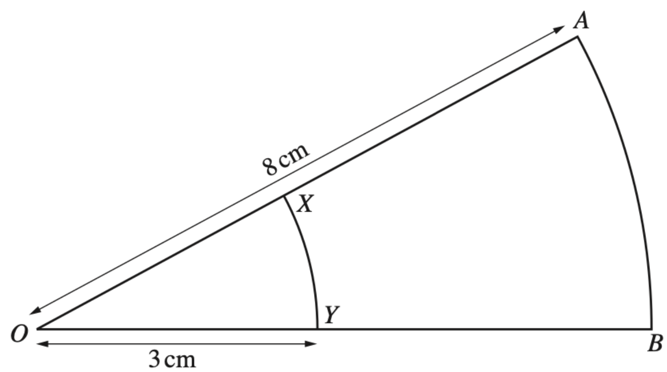
(c) the matrix $\mathbf{X}$ such that (3)

$$\mathbf{X}\mathbf{B}^{-1} = \mathbf{A}.$$

Solution

$$\mathbf{X}\mathbf{B}^{-1} = \mathbf{A} \Rightarrow \mathbf{X} = \mathbf{A}\mathbf{B}$$
$$\Rightarrow \mathbf{X} = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -2 & 0 \\ 1 & 4 \end{pmatrix}$$
$$\Rightarrow \mathbf{X} = \underline{\underline{\begin{pmatrix} -5 & 12 \\ 0 & 8 \end{pmatrix}}}.$$

9. The diagram shows a sector OXY of a circle centre O, radius 3 cm and a sector OAB of a circle centre O, radius 8 cm.



- The point X lies on the line OA and the point Y lies on the line OB.
- The perimeter of the region $XABYX$ is 15.5 cm.

Find

(a) the angle AOB in radians, (3)

Solution
Let $\theta = \angle\, AOB$. Then

$$3\theta + 5 + 8\theta + 5 = 15.5 \Rightarrow 11\theta = 5.5$$
$$\Rightarrow \underline{\underline{\theta = 0.5}}.$$

(b) the ratio of the area of the sector OXY to the area of the region $XABYX$ in the form $p : q$, where p and q are integers. (4)

Solution

Well,

$$\begin{aligned}\text{area of } OXY &= \tfrac{1}{2} \times 3^2 \times 0.5 \\ &= \tfrac{9}{4}\end{aligned}$$

and

$$\begin{aligned}\text{area of } XABYX &= \text{area of } OAB - \text{area of } OXY \\ &= (\tfrac{1}{2} \times 8^2 \times 0.5) - \tfrac{9}{2} \\ &= 16 - \tfrac{9}{4} \\ &= \tfrac{55}{4};\end{aligned}$$

so

$$\tfrac{9}{4} : \tfrac{55}{4} = \underline{\underline{9 : 55}}.$$

10. A music student needs to select 7 pieces of music from 6 classical pieces and 4 modern pieces.

Find the number of different selections that she can make if

(a) there are no restrictions, (1)

Solution

$$\binom{10}{7} = \underline{\underline{120}}.$$

(b) there are to be only 2 modern pieces included, (2)

Solution

$$\binom{6}{5} \times \binom{4}{2} = \underline{\underline{36}}.$$

(c) there are to be more classical pieces than modern pieces. (4)

Solution

$$\begin{aligned}\text{P(more classical pieces)} &= \text{P}(4C, 3M) + \text{P}(5C, 2M) + \text{P}(6C, 1M) \\ &= \left[\binom{6}{4} \times \binom{4}{3}\right] + \left[\binom{6}{5} \times \binom{4}{2}\right] + \left[\binom{6}{6} \times \binom{4}{1}\right] \\ &= 60 + 36 + 4 \\ &= \underline{\underline{100}}.\end{aligned}$$

11. A particle moves in a straight line such that its displacement, x m, from a fixed point O on the line at time t seconds is given by

$$x = 12\ln(2t + 3).$$

Find

(a) the value of t when the displacement of the particle from O is 48 m, (3)

Solution
Well,

$$\begin{aligned}12\ln(2t+3) = 48 &\Rightarrow \ln(2t+3) = 4 \\ &\Rightarrow 2t + 3 = \mathrm{e}^4 \\ &\Rightarrow 2t = \mathrm{e}^4 - 3 \\ &\Rightarrow \underline{\underline{t = \tfrac{1}{2}(\mathrm{e}^4 - 3) \text{ or } 25.8 \text{ s (3 sf)}}}.\end{aligned}$$

(b) the velocity of the particle when $t = 1$, (3)

Solution
Now,

$$x = 12\ln(2t+3) \Rightarrow v = \frac{24}{2t+3}$$

and

$$\begin{aligned}t = 1 &\Rightarrow v = \frac{24}{5} \\ &\Rightarrow \underline{\underline{v = 4.8 \text{ ms}^{-1}}}.\end{aligned}$$

(c) the acceleration of the particle when $t = 1$. (3)

Solution

Well,

$$v = \frac{24}{2t+3} \Rightarrow v = 24(2t+3)^{-1}$$
$$\Rightarrow a = -48(2t+3)^{-2}$$

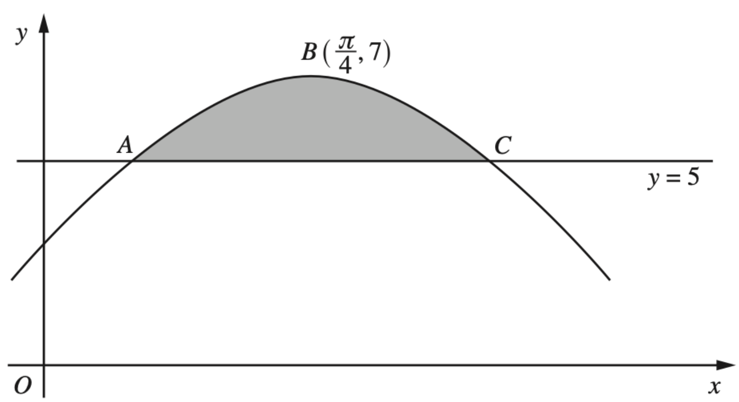
and

$$t = 1 \Rightarrow a = -48(5)^{-2}$$
$$\Rightarrow \underline{\underline{a = -1.92 \text{ ms}^{-2}}}.$$

EITHER

12. The diagram shows part of a curve for which

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 8\cos 2x.$$



- The curve passes through the point $B(\frac{1}{4}\pi, 7)$.
- The line $y = 5$ meets the curve at the points A and C.

(a) Show that the curve has equation (3)

$$y = 3 + 4\sin 2x.$$

Solution

Well,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 8\cos 2x \Rightarrow y = 4\sin 2x + c,$$

for some constant c. Now,

$$\begin{aligned} x = \tfrac{1}{4}\pi, y = 7 &\Rightarrow 7 = 4\sin(\tfrac{1}{2}\pi) + c \\ &\Rightarrow 7 = 4 + c \\ &\Rightarrow c = 3, \end{aligned}$$

so

$$\underline{\underline{y = 3 + 4\sin 2x}}.$$

(b) Find the x-coordinate of the point A and of the point C. (4)

Solution

Well,

$$\begin{aligned} 3 + 4\sin 2x = 5 &\Rightarrow 4\sin 2x = 2 \\ &\Rightarrow \sin 2x = \tfrac{1}{2} \\ &\Rightarrow 2x = \tfrac{1}{6}\pi, \tfrac{5}{6}\pi \\ &\Rightarrow x = \tfrac{1}{12}\pi, \tfrac{5}{12}\pi; \end{aligned}$$

hence, the x-coordinate of the point A is $\underline{\underline{\tfrac{1}{12}\pi}}$ and the x-coordinate of the point C is $\underline{\underline{\tfrac{5}{12}\pi}}$.

(c) Find the area of the shaded region. (5)

Solution

Now,

$$\begin{aligned}
\text{shaded area} &= \int_{\frac{1}{12}\pi}^{\frac{5}{12}\pi} [(3 + 4\sin 2x) - 5]\,\mathrm{d}x \\
&= \int_{\frac{1}{12}\pi}^{\frac{5}{12}\pi} (4\sin 2x - 2)\,\mathrm{d}x \\
&= \left[-2\cos 2x - 2x\right]_{x=\frac{1}{12}\pi}^{\frac{5}{12}\pi} \\
&= \left(\sqrt{3} - \tfrac{5}{6}\pi\right) - \left(-\sqrt{3} - \tfrac{1}{6}\pi\right) \\
&= \underline{\underline{2\sqrt{3} - \tfrac{2}{3}\pi}}.
\end{aligned}$$

OR

13. A curve is such that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6\mathrm{e}^{3x} - 12.$$

The curve passes through the point $(0, 1)$.

(a) Find the equation of the curve. (4)

Solution

Well,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6\mathrm{e}^{3x} - 12 \Rightarrow y = 2\mathrm{e}^{3x} - 12x + c,$$

for some constant c. Now,

$$\begin{aligned}
x = 0, y = 1 &\Rightarrow 1 = 2 - 0 + c \\
&\Rightarrow c = -1,
\end{aligned}$$

so

$$\underline{\underline{y = 2\mathrm{e}^{3x} - 12x - 1}}.$$

(b) Find the coordinates of the stationary point of the curve. (3)

Solution

Now,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 6\mathrm{e}^{3x} - 12 = 0 \\ &\Rightarrow 6\mathrm{e}^{3x} = 12 \\ &\Rightarrow \mathrm{e}^{3x} = 2 \\ &\Rightarrow 3x = \ln 2 \\ &\Rightarrow x = \tfrac{1}{3}\ln 2 \\ &\Rightarrow y = 3 - 4\ln 2;\end{aligned}$$

hence, the coordinates of the stationary point of the curve are

$$\underline{\underline{(\tfrac{1}{3}\ln 2, 3 - 4\ln 2)}}.$$

(c) Determine the nature of the stationary point. (2)

Solution
Next,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 18\mathrm{e}^{3x}$$

and

$$x = \tfrac{1}{3}\ln 2 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 36 > 0;$$

so the stationary point is a minimum.

(d) Find the coordinates of the point where the tangent to the curve at the point $(0, 1)$ meets the x-axis. (3)

Solution
Now,

$$x = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -6$$

and the equation is

$$y - 1 = -6(x - 0).$$

Next,

$$\begin{aligned}y = 0 &\Rightarrow -1 = -6x \\ &\Rightarrow x = \tfrac{1}{6};\end{aligned}$$

hence, the coordinates of the point are

$$\underline{\underline{(\tfrac{1}{6}, 0)}}.$$