Dr Oliver Mathematics
Mathematics: Advanced Higher
2019 Paper
3 hours

The total number of marks available is 100.
You must write down all the stages in your working.

1. (a) Differentiate (2)

$$f(x) = x^6 \cot 5x.$$

Solution

$$u = x^6 \Rightarrow \frac{du}{dx} = 6x^5$$
$$v = \cot 5x \Rightarrow \frac{dv}{dx} = -5\operatorname{cosec}^2 5x$$

$$f(x) = x^6 \cot 5x \Rightarrow f'(x) = x^6 \cdot (-5\operatorname{cosec}^2 5x) + 6x^5 \cdot \cot 5x$$
$$\Rightarrow \underline{\underline{f'(x) = x^5(6\cot 5x - 5\operatorname{cosec}^2 5x)}}.$$

(b) Given (3)

$$y = \frac{2x^3+1}{x^3-4},$$

find $\frac{dy}{dx}$. Simplify your answer.

Solution

$$u = 2x^3 + 1 \Rightarrow \frac{du}{dx} = 6x^2$$
$$v = x^3 - 4 \Rightarrow \frac{dv}{dx} = 3x^2$$

$$y = \frac{2x^3+1}{x^3-4} \Rightarrow \frac{dy}{dx} = \frac{(x^3-4)\cdot(6x^2)-(2x^3+1)\cdot 3x^2}{(x^3-4)^2}$$
$$\Rightarrow \frac{dy}{dx} = \frac{(6x^5-24x^2)-(6x^5+3x^2)}{(x^3-4)^2}$$
$$\Rightarrow \frac{dy}{dx} = \frac{-27x^2}{(x^3-4)^2}.$$

(c) For (3)

$$f(x) = \cos^{-1} 2x,$$

evaluate $f'(\frac{\sqrt{3}}{4})$.

Solution

$$f(x) = \cos^{-1} 2x \Rightarrow f'(x) = \frac{-2}{\sqrt{1-(2x)^2}}$$

and

$$f'\left(\frac{\sqrt{3}}{4}\right) = \frac{-2}{\sqrt{1-(2\cdot\frac{\sqrt{3}}{4})^2}}$$
$$= \frac{-2}{\sqrt{1-(\frac{\sqrt{3}}{2})^2}}$$
$$= \frac{-2}{\sqrt{1-\frac{3}{4}}}$$
$$= \frac{-2}{\sqrt{\frac{1}{4}}}$$
$$= \frac{-2}{\frac{1}{2}}$$
$$= -4.$$

2. Matrix **A** is defined by

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 4 \\ -3 & p & 2 \\ -1 & -2 & 5 \end{pmatrix},$$

where $p \in \mathbb{R}$.

(a) Given that the determinant of $\mathbf{A}$ is 3, find the value of p. (3)

Solution

$$\begin{aligned}\det \mathbf{A} = 3 &\Rightarrow 2(5p+4) - (-15+2) + 4(6+p) = 3 \\ &\Rightarrow 10p + 8 + 13 + 24 + 4p = 3 \\ &\Rightarrow 14p = -42 \\ &\Rightarrow \underline{\underline{p = -3}}.\end{aligned}$$

Matrix $\mathbf{B}$ is defined by

$$\mathbf{B} = \begin{pmatrix} 0 & 1 \\ q & 3 \\ 4 & 0 \end{pmatrix},$$

where $q \in \mathbb{R}$.

(b) Find $\mathbf{AB}$. (2)

Solution

$$\begin{aligned}\mathbf{AB} &= \begin{pmatrix} 2 & 1 & 4 \\ -3 & -3 & 2 \\ -1 & -2 & 5 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ q & 3 \\ 4 & 0 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} q+16 & 5 \\ -3q+8 & -12 \\ -2q+20 & -7 \end{pmatrix}}}.\end{aligned}$$

(c) Explain why $\mathbf{AB}$ does not have an inverse. (1)

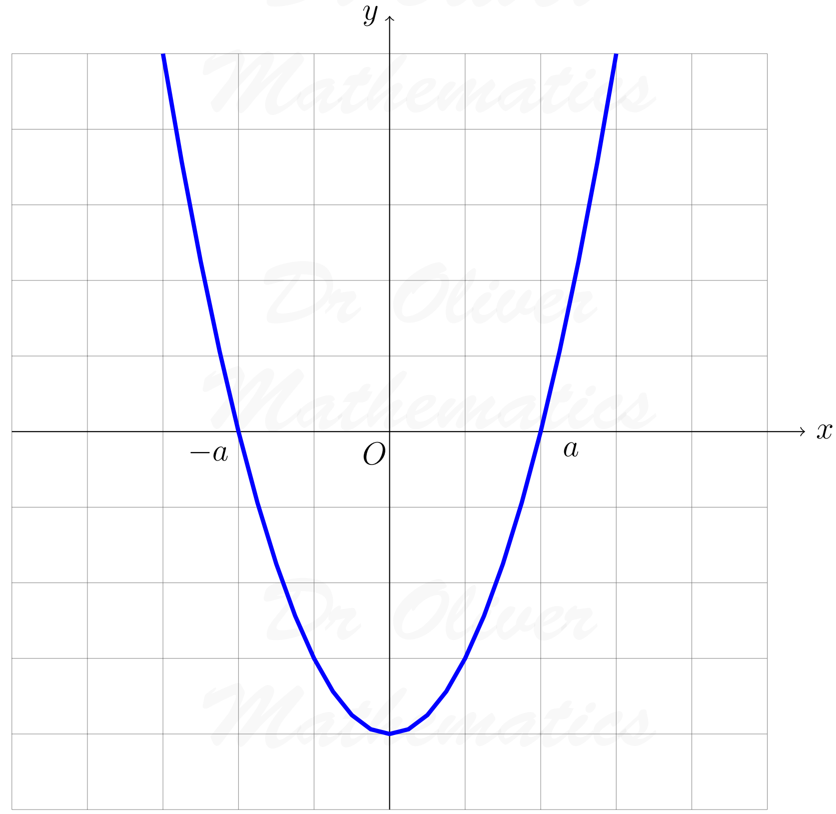
Solution

$\mathbf{AB}$ is a 2×3 matrix and hence is not $n \times n$ matrix; $\mathbf{AB}$ does not have an $\underline{\underline{\text{inverse}}}$.

3. The function diagram $\mathrm{f}(x)$ is defined by

$$\mathrm{f}(x) = x^2 - a^2.$$

The graph of is shown in the diagram.



(a) State whether $\mathrm{f}(x)$ is odd, even or neither. Give a reason for your answer. (1)

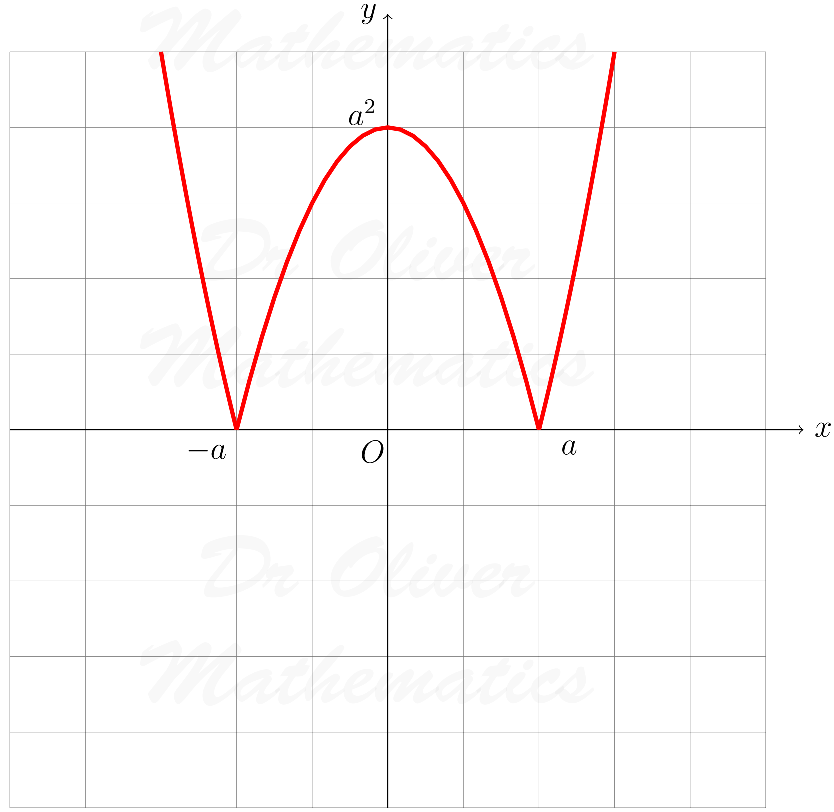
Solution

$$\begin{aligned} \mathrm{f}(-x) &= (-x)^2 - a^2 \\ &= x^2 - a^2 \\ &= \mathrm{f}(x) \end{aligned}$$

and $\mathrm{f}(x)$ is even.

(b) Sketch the graph of $y = |\,\mathrm{f}(x)|$. (1)

Solution



4. (a) Express **(1)**

$$\frac{3x^2 + x - 17}{x^2 - x - 12}$$

in the form

$$p + \frac{qx + r}{x^2 - x - 12}$$

where p, q, and r are integers.

Solution

$$\begin{aligned}\frac{3x^2 + x - 17}{x^2 - x - 12} &= \frac{(3x^2 - 3x - 36) + 4x + 19}{x^2 - x - 12}\\ &= \frac{3(x^2 - x - 12) + 4x + 19}{x^2 - x - 12}\\ &= \underline{\underline{3 + \frac{4x + 19}{x^2 - x - 12}}};\end{aligned}$$

hence, $\underline{\underline{p = 3}}$, $\underline{\underline{q = 4}}$, and $\underline{\underline{r = 19}}$.

(b) Hence express (3)

$$\frac{3x^2 + x - 17}{x^2 - x - 12}$$

with partial fractions.

Solution

$$\left.\begin{array}{ll}\text{add to:} & -1 \\ \text{multiply to:} & -12\end{array}\right\} -4,\ +3$$

$$x^2 - x - 12 = (x - 4)(x + 3).$$

Now,

$$\begin{aligned}\frac{4x + 19}{x^2 - x - 12} &\equiv \frac{A}{(x - 4)} + \frac{B}{(x + 3)} \\ &\equiv \frac{A(x + 3) + B(x - 4)}{(x - 4)(x + 3)}\end{aligned}$$

which means

$$4x + 19 \equiv A(x + 3) + B(x - 4).$$

$\underline{x = 4}$: $35 = 7A \Rightarrow A = 5$.
$\underline{x = -3}$: $7 = -7B \Rightarrow B = -1$.
Hence,

$$\frac{3x^2 + x - 17}{x^2 - x - 12} = \underline{\underline{3 + \frac{5}{(x - 4)} - \frac{1}{(x + 3)}}}.$$

5. For

$$x = \ln(2t + 7) \text{ and } y = t^2,\ t > 0,$$

find

(a) $\frac{\mathrm{d}y}{\mathrm{d}x}$, (2)

Solution

$$x = \ln(2t + 7) \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2}{2t + 7}$$

$$y = t^2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = 2t.$$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\ &= \frac{2t}{\frac{2}{2t+7}} \\ &= \frac{2t(2t+7)}{2} \\ &= \underline{\underline{t(2t+7)}}.\end{aligned}$$

(b) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$. (2)

Solution

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(2t^2+7t\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(2t^2+7t\right) \times \frac{\mathrm{d}t}{\mathrm{d}x} \\ &= (4t+7) \times \frac{2t+7}{2} \\ &= \underline{\underline{\tfrac{1}{2}(2t+7)(4t+7)}}.\end{aligned}$$

6. A spherical balloon of radius r cm, $r > 0$, deflates at a constant rate of 60 $\mathrm{cm}^3\,\mathrm{s}^{-1}$. (3)

Calculate the rate of change of the radius with respect to time when $r = 3$.

Solution

$$V = \tfrac{4}{3}\pi r^3 \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}r} = 4\pi r^2$$

and

$$r = 3 \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}r} = 36\pi.$$

Now,

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t} \Rightarrow -60 = 36\pi \frac{\mathrm{d}r}{\mathrm{d}t}$$
$$\Rightarrow \underline{\underline{\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{3}{5\pi} \text{ cm s}^{-1}}}.$$

7. (a) Find an expression of (1)

$$\sum_{r=1}^{n} (6r + 13)$$

in terms of n.

Solution

$$\begin{aligned}\sum_{r=1}^{n} (6r + 13) &= 6 \sum_{r=1}^{n} r + 13 \sum_{r=1}^{n} 1 \\ &= 6 \cdot \tfrac{1}{2} n(n+1) + 13n \\ &= 3n(n+1) + 13n \\ &= 3n^2 + 3n + 13n \\ &= \underline{\underline{3n^2 + 16n}}.\end{aligned}$$

(b) Hence, or otherwise, find (2)

$$\sum_{r=p+1}^{20} (6r + 13).$$

Solution

$$\begin{aligned}\sum_{r=p+1}^{20} (6r + 13) &= \sum_{r=1}^{20} (6r + 13) - \sum_{r=1}^{p} (6r + 13) \\ &= 1\,520 - (3p^2 + 16p) \\ &= \underline{\underline{1\,520 - 3p^2 - 16p}}.\end{aligned}$$

8. Find the particular solution of the differential equation (5)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 11\frac{\mathrm{d}y}{\mathrm{d}x} + 28y = 0,$$

given that $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 9$, when $x = 0$.

Solution

Complementary function:

$$m^2 + 11m + 28 = 0 \Rightarrow (m+4)(m+7) = 0 \Rightarrow m = -4, -7$$

and hence the complementary function is

$$y = A\mathrm{e}^{-4x} + B\mathrm{e}^{-7x}.$$

Now,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -4A\mathrm{e}^{-4x} - 7B\mathrm{e}^{-7x}$$

and

$$x = 0, y = 0 \Rightarrow 0 = A + B \quad (1)$$
$$x = 0, \frac{\mathrm{d}y}{\mathrm{d}x} = 9 \Rightarrow 9 = -4A - 7B \quad (2)$$

Rearrange (1) and insert it into (2):

$$\begin{aligned} A = -B \Rightarrow 9 &= 4B - 7B \\ \Rightarrow 9 &= -3B \\ \Rightarrow B &= -3 \\ \Rightarrow A &= 3; \end{aligned}$$

hence,

$$\underline{\underline{y = 3\mathrm{e}^{-4x} - 3\mathrm{e}^{-7x}}}.$$

9. (a) Write down and simplify the general term in the binomial expansion of (3)

$$\left(2x^2 - \frac{d}{x^3}\right)^7,$$

where d is a constant.

Solution

$$\begin{aligned}\binom{7}{r}(2x^2)^r\left(-\frac{d}{x^3}\right)^{7-r} &= \binom{7}{r}2^r x^{2r}\left(-dx^{-3}\right)^{7-r}\\ &= \binom{7}{r}2^r x^{2r}(-d)^{7-r}x^{-3(7-r)}\\ &= \binom{7}{r}2^r(-d)^{7-r}x^{2r}x^{3r-21}\\ &= \underline{\underline{\binom{7}{r}2^r(-d)^{7-r}x^{5r-21}}}.\end{aligned}$$

(b) Given that the coefficient of $\frac{1}{x}$ is $-70\,000$, find the value of d. (2)

Solution

$$\begin{aligned}5r-21=-1 &\Rightarrow 5r=20\\ &\Rightarrow r=4\end{aligned}$$

and so the value of d is

$$\begin{aligned}\binom{7}{4}2^4(-d)^3=-70\,000 &\Rightarrow 35\cdot 16(-d^3)=-70\,000\\ &\Rightarrow d^3=125\\ &\Rightarrow \underline{\underline{d=5}}.\end{aligned}$$

10. A curve is defined implicitly by the equation

$$x^2+y^2=xy+12.$$

(a) Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. (3)

Solution

Implicit differentiation:

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}y}{\mathrm{d}x} + y \Rightarrow 2y\frac{\mathrm{d}y}{\mathrm{d}x} - x\frac{\mathrm{d}y}{\mathrm{d}x} = y - 2x$$
$$\Rightarrow (2y - x)\frac{\mathrm{d}y}{\mathrm{d}x} = y - 2x$$
$$\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y - 2x}{2y - x}}}.$$

(b) There are two points where the tangent to the curve has equation $x = k$, $k \in \mathbb{R}$. Find the values of k. (2)

Solution

Well, $x = k$ is a vertical line and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} \text{ is undefined} \Rightarrow 2y - x = 0$$
$$\Rightarrow 2y = x$$

and stick it in to the original equation:

$$(2y)^2 + y^2 = (2y)y + 12 \Rightarrow 5y^2 = 2y^2 + 12$$
$$\Rightarrow 3y^2 = 12$$
$$\Rightarrow y^2 = 4$$
$$\Rightarrow y = \pm 2$$
$$\Rightarrow \underline{\underline{x = \pm 4}}.$$

11. Let n be a positive integer.

(a) Find a counterexample to show that the following statement is false. (1)

$$n^2 + n + 1 \text{ is always a prime number.}$$

Solution

E.g., $n = 4$:

$$4^2 + 4 + 1 = 21$$
$$= 3 \times 7$$

which is composite.

(b) (i) Write down the contrapositive of: (1)

If $n^2 - 2n + 7$ is even, then n is odd.

Solution
If n is even, then $n^2 - 2n + 7$ is odd.

(ii) Use the contrapositive to prove that if $n^2 - 2n + 7$ is even then n is odd. (3)

Solution
Let $n = 2k$ be an even number. Then

$$\begin{aligned} n^2 - 2n + 7 &= (2k)^2 - 2(2k) + 7 \\ &= 4k^2 - 4k + 7 \\ &= 2(2k^2 - 2k + 3) + 1 \\ &= 2 \times \text{some integer} + 1 \end{aligned}$$

and so it is odd. Therefore, if $n^2 - 2n + 7$ is even then n is odd.

12. Express 231_{11} in base 7. (3)

Solution

$$\begin{aligned} 231_{11} &= (2 \times 11^2) + (3 \times 11^1) + (1 \times 11^0) \\ &= 242 + 33 + 1 \\ &= 276_{10}. \end{aligned}$$

Now,

$$\begin{aligned} 276 &= 39 \times 7 + 3 \\ 39 &= 5 \times 7 + 4 \\ 5 &= 0 \times 7 + 5 \end{aligned}$$

and work backwards:

$$231_{11} = 276_{10} = \underline{\underline{543_7}}.$$

13. An electronic device contains a timer circuit that switches off when the voltage, V, reaches a set value. The rate of change of the voltage is given by (5)

$$\frac{\mathrm{d}V}{\mathrm{d}t} = k(12 - V),$$

where k is a constant, t is the time in seconds, and $0 \leqslant V < 12$.

Given that $V = 2$ when $t = 0$, express V in terms of k and t.

Solution

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} = k(12 - V) &\Rightarrow \frac{1}{12 - V}\,\mathrm{d}V = k\,\mathrm{d}t \\ &\Rightarrow \int \frac{1}{12 - V}\,\mathrm{d}V = \int k\,\mathrm{d}t \\ &\Rightarrow \ln(12 - V) = kt + c.\end{aligned}$$

Now,

$$t = 0, V = 2 \Rightarrow \ln 10 = c$$

and

$$\begin{aligned}\ln(12 - V) = kt + \ln 10 &\Rightarrow \ln(12 - V) - \ln 10 = kt \\ &\Rightarrow \ln\left(\frac{12 - V}{10}\right) = kt \\ &\Rightarrow \frac{12 - V}{10} = \mathrm{e}^{kt} \\ &\Rightarrow 12 - V = 10\mathrm{e}^{kt} \\ &\Rightarrow \underline{\underline{V = 12 - 10\mathrm{e}^{kt}}}.\end{aligned}$$

14. Prove by induction that (5)

$$\sum_{r=1}^{n} r!\,r = (n+1)! - 1$$

for all positive integers n.

Solution

$\underline{n = 1}$: LHS $= 1!1 = 1$, RHS $= 2! - 1 = 1$, and so the result is true for $n = 1$.

Suppose that the result is true for $n = k$, i.e.,

$$\sum_{r=1}^{k} r!\, r = (k+1)! - 1.$$

Then

$$\begin{aligned}
\sum_{r=1}^{k+1} r!\, r &= \sum_{r=1}^{k} r!\, r + (k+1)!(k+1) \\
&= (k+1)! - 1 + (k+1)!(k+1) \\
&= (k+1)![1 + (k+1)] - 1 \\
&= (k+1)!(k+2) - 1 \\
&= (k+2)! - 1
\end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the result is true for all positive integers n.

15. The equations of two planes are given below.

$$\begin{aligned}
\pi_1 :&\quad 2x - 3y - z = 9 \\
\pi_2 :&\quad x + y - 3z = 2.
\end{aligned}$$

(a) Verify that the line of intersection, L_1, of these two planes has parametric equations: (2)

$$x = 2\lambda + 3,\ y = \lambda - 1,\ z = \lambda.$$

Solution

$$\begin{aligned}
2x - 3y - z &= 2(2\lambda + 3 - 3(\lambda - 1) - (\lambda) \\
&= 4\lambda + 6 - 3\lambda + 3 - \lambda \\
&= 9
\end{aligned}$$

and

$$\begin{aligned}
x + y - 3z &= (2\lambda + 3) + (\lambda - 1) - 3(\lambda) \\
&= 2;
\end{aligned}$$

hence, the line of intersection, L_1, $\underline{\underline{\text{does}}}$ have the parametric equations.

Let π_3 be the plane with equation

$$-2x + 4y + 3z = 4.$$

(b) Calculate the acute angle between the line L_1 and the plane π_3. (3)

Solution

Let θ be the angle between the line and the plane. The normals are

$$\begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

Now,

$$\sqrt{(-2)^2 + 4^2 + 3^2} = \sqrt{29}$$
$$\sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}.$$

Finally,

$$\begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix} . \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \left| \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix} \right| . \left| \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right| \cos\theta$$
$$\Rightarrow \quad -4 + 4 + 3 = \sqrt{29}\sqrt{6}\cos\theta$$
$$\Rightarrow \quad \cos\theta = \frac{3}{\sqrt{29}\sqrt{6}}$$
$$\Rightarrow \quad \theta = 76.854\,222\,36 \text{ (FCD)}$$

and the angle is

$$90 - 76.854\ldots = \underline{\underline{13.1^\circ \text{ (1 dp)}}}.$$

L_2 is the line perpendicular to π_3 passing through $P(1, 3, -2)$.

(c) Determine whether or not L_1 and L_2 intersect. (4)

Solution

L_2 has parametric equations

$$x = -2\mu + 1,\ y = 4\mu + 3\ z = 3\mu - 2$$

and we pair them up with parametric equations for L_1:

$$2\lambda + 3 = -2\mu + 1 \quad (1)$$
$$\lambda - 1 = 4\mu + 3 \quad (2)$$
$$\lambda = 3\mu - 2 \quad (3).$$

From (3),

$$(3\mu - 2) - 1 = 4\mu + 3 \Rightarrow \mu = -6$$
$$\Rightarrow \lambda = -20.$$

Check in (1):

$$2\lambda + 3 = -37 \text{ and } -2\mu + 1 = 13,$$

so L_1 and L_2 do not intersect.

16. (a) Use integration by parts to find the exact value of (5)

$$\int_0^1 (x^2 - 2x + 1)e^{4x}\, dx.$$

Solution

$$u = x^2 - 2x + 1 \Rightarrow \frac{du}{dx} = 2x - 2$$
$$\frac{dv}{dx} = e^{4x} \Rightarrow v = \tfrac{1}{4}e^{4x}$$

$$\int_0^1 (x^2 - 2x + 1)e^{4x}\, dx = \tfrac{1}{4}(x^2 - 2x + 1)e^{4x} - \tfrac{1}{4}\int_0^1 (2x - 2)e^{4x}\, dx$$

$$u = 2x - 2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{4x} \Rightarrow v = \tfrac{1}{4}\mathrm{e}^{4x}$$

$$\begin{aligned}
&= \tfrac{1}{4}(x^2 - 2x + 1)\mathrm{e}^{4x} - \tfrac{1}{4}\left[\tfrac{1}{4}(2x-2)\mathrm{e}^{4x} - \tfrac{1}{4}\int_0^1 2\mathrm{e}^{4x}\,\mathrm{d}x\right] \\
&= \tfrac{1}{4}(x^2 - 2x + 1)\mathrm{e}^{4x} - \tfrac{1}{16}(2x-2)\mathrm{e}^{4x} + \tfrac{1}{8}\int_0^1 \mathrm{e}^{4x}\,\mathrm{d}x \\
&= \tfrac{1}{16}[(4x^2 - 8x + 4) - (2x - 2)]\mathrm{e}^{4x} + \tfrac{1}{8}\int_0^1 \mathrm{e}^{4x}\,\mathrm{d}x \\
&= \tfrac{1}{16}(4x^2 - 10x + 6)\mathrm{e}^{4x} + \tfrac{1}{32}\mathrm{e}^{4x} + c \\
&= \tfrac{1}{32}[(8x^2 - 20x + 12) + 1]\mathrm{e}^{4x} + c \\
&= \tfrac{1}{32}(8x^2 - 20x + 13)\mathrm{e}^{4x} + c.
\end{aligned}$$

Now,

$$\begin{aligned}
\int_0^1 (x^2 - 2x + 1)\mathrm{e}^{4x}\,\mathrm{d}x &= \left[\tfrac{1}{32}(8x^2 - 20x + 13)\mathrm{e}^{4x}\right]_{x=0}^1 \\
&= \underline{\underline{\tfrac{1}{32}(\mathrm{e}^4 - 13)}}.
\end{aligned}$$

A solid is formed by rotating the curve with equation $y = 4(x-1)\mathrm{e}^{2x}$ between $x = 0$ and $x = 1$ through 2π radians about the x-axis.

(b) Find the exact value of the volume of this solid. (3)

Solution

$$\begin{aligned}
\text{Volume} &= \int_0^1 \pi[4(x-1)\mathrm{e}^{2x}]\,\mathrm{d}x \\
&= 16\pi\int_0^1 (x-1)^2\mathrm{e}^{4x}\,\mathrm{d}x \\
&= 16\pi \cdot \tfrac{1}{32}(\mathrm{e}^4 - 13) \\
&= \underline{\underline{\tfrac{1}{2}\pi(\mathrm{e}^4 - 13)}}.
\end{aligned}$$

17. The first three terms of a sequence are given by

$$5x + 8,\ -2x + 1,\ x - 4.$$

(a) When $x = 11$, show that the first three terms form the start of a geometric sequence, and state the value of the common ratio. (2)

Solution

The terms are $u_1 = 63$, $u_2 = -21$, and $u_3 = 7$. Now,

$$\frac{u_2}{u_1} = \frac{-21}{63} = -\tfrac{1}{3} \text{ and } \frac{u_2}{u_1} = \frac{7}{-21} = -\tfrac{1}{3}.$$

Hence, the three terms do indeed form the start of a geometric sequence and the common ratio is $-\frac{1}{3}$.

(b) Given that the entire sequence is geometric for $x = 11$,

(i) state why the associated series has a sum to infinity, and (1)

Solution

$|-\frac{1}{3}| < 1$.

(ii) calculate this sum to infinity. (2)

Solution

$$\begin{aligned} S_\infty &= \frac{63}{1 - (-\frac{1}{3})} \\ &= \frac{63}{\frac{4}{3}} \\ &= 47\tfrac{1}{4}. \end{aligned}$$

There is a second value for x that also gives a geometric sequence.

(c) For this second sequence

(i) show that (2)

$$x^2 - 8x - 33 = 0,$$

Solution

$$\frac{-2x+1}{5x+8} = \frac{x-4}{-2x+1} \Rightarrow (-2x+1)^2 = (5x+8)(x-4)$$

$\times$	$5x$	$+8$
x	$5x^2$	$+8x$
-4	$-20x$	-32

$\times$	$-2x$	$+1$
$-2x$	$4x^2$	$-2x$
$+1$	$-2x$	$+1$

$$\Rightarrow 4x^2 - 4x + 1 = 5x^2 - 12x - 32$$
$$\Rightarrow \underline{\underline{x^2 - 8x - 33 = 0}},$$

as required.

(ii) find the first three terms, and (2)

Solution

$$\left.\begin{array}{ll}\text{add to:} & -8 \\ \text{multiply to:} & -33\end{array}\right\} -11, +3$$

$$x^2 - 8x - 33 = 0 \Rightarrow (x-11)(x+3) = 0$$
$$\Rightarrow x = 11 \text{ or } x = -3.$$

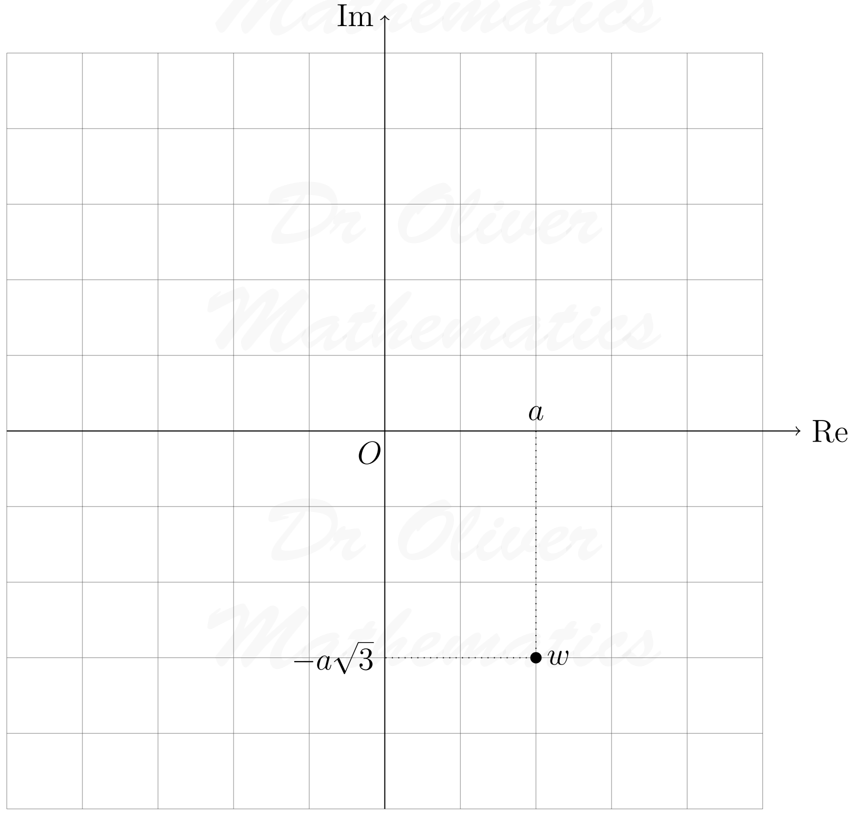
Clearly, $x = -3$ and so the first three terms are $\underline{\underline{-7, 7, -7}}$.

(iii) state the value of S_{2n} and justify your answer. (1)

Solution

$\underline{\underline{S_{2n} = 0}}$ since $2n$ is even and so pairs of terms cancel each other out.

18. The complex number w has been plotted on an Argand diagram, as shown below.



(a) Express w in

(i) Cartesian form, (1)

Solution

$w = a - a\sqrt{3}\mathrm{i}$.

(ii) polar form. (3)

Solution

$$\begin{aligned}\sqrt{a^2 + (-a\sqrt{3})^2} &= \sqrt{a^2 + 3a^2} \\ &= \sqrt{4a^2} \\ &= \sqrt{(2a)^2} \\ &= 2a\end{aligned}$$

and

$$\tan\theta = \frac{a\sqrt{3}}{a} \Rightarrow \tan\theta = \sqrt{3}$$
$$\Rightarrow \theta = \tfrac{1}{3}\pi.$$

Hence,

$$\underline{\underline{w = 2a\left[\cos(-\tfrac{1}{3}\pi) + \mathrm{i}\sin(-\tfrac{1}{3}\pi)\right]}}.$$

The complex number z_1 is a root of $z^3 = w$, where

$$z_1 = k\left(\cos\tfrac{1}{m}\pi + \sin\tfrac{1}{m}\pi\right),$$

for integers k and m.

(b) Given that $a = 4$,

(i) use de Moivre's theorem to obtain the values of k and m, and (4)

Solution

$$\begin{aligned} & z^3 = 8\left[\cos(-\tfrac{1}{3}\pi) + \mathrm{i}\sin(-\tfrac{1}{3}\pi)\right] \\ \Rightarrow\quad & z^3 = 8\left[\cos(-\tfrac{1}{3}\pi + 2n\pi) + \mathrm{i}\sin(-\tfrac{1}{3}\pi + 2n\pi)\right] \\ \Rightarrow\quad & z^3 = 8\left[\cos\frac{(6n-1)\pi}{3} + \mathrm{i}\sin\frac{(6n-1)\pi}{3}\right] \\ \Rightarrow\quad & z = 2\left[\cos\frac{(6n-1)\pi}{9} + \mathrm{i}\sin\frac{(6n-1)\pi}{9}\right]. \end{aligned}$$

Hence, $\underline{\underline{k = 2}}$ and $\underline{\underline{m = -9}}$.

(ii) find the remaining roots. (2)

Solution

$\underline{n = 1}$:

$$\underline{\underline{z_2 = 2\left[\cos\tfrac{5}{9}\pi + \mathrm{i}\sin\tfrac{5}{9}\pi\right]}}.$$

$\underline{n = 2}$:

$$z_3 = 2\left[\cos\tfrac{11}{9}\pi + \mathrm{i}\sin\tfrac{11}{9}\pi\right] = \underline{\underline{2\left[\cos(-\tfrac{7}{9}\pi) + \mathrm{i}\sin(-\tfrac{7}{9}\pi)\right]}}.$$