Dr Oliver Mathematics
AQA Further Maths Level 2
June 2019 Paper 2
2 hours

The total number of marks available is 105.
You must write down all the stages in your working.
You are permitted to use a scientific or graphical calculator in this paper.

1. (a) (3)

$$a \begin{pmatrix} 3 \\ 5 \end{pmatrix} = 4 \begin{pmatrix} 2a + 3 \\ b \end{pmatrix}.$$

Work out the values of a and b.

Solution

Well,

$$3a = 4(2b + 3) \Rightarrow 3a = 8b + 12 \quad (1)$$

and

$$5a = 4b \quad (2).$$

Now, do $2 \times (2)$:

$$10a = 8b \quad (3)$$

and do $(3) - (1)$:

$$\begin{aligned} 5a = -12 &\Rightarrow \underline{\underline{a = -\tfrac{12}{5}}} \\ &\Rightarrow 5(-\tfrac{12}{5}) = 4b \\ &\Rightarrow -12 = 4b \\ &\Rightarrow \underline{\underline{b = -3}}. \end{aligned}$$

(b) (2)

$$\begin{pmatrix} m & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ -2 & -1 \end{pmatrix} = \mathbf{I},$$

where $\mathbf{I}$ is the identity matrix.

Work out the value of m.

Solution

Now,

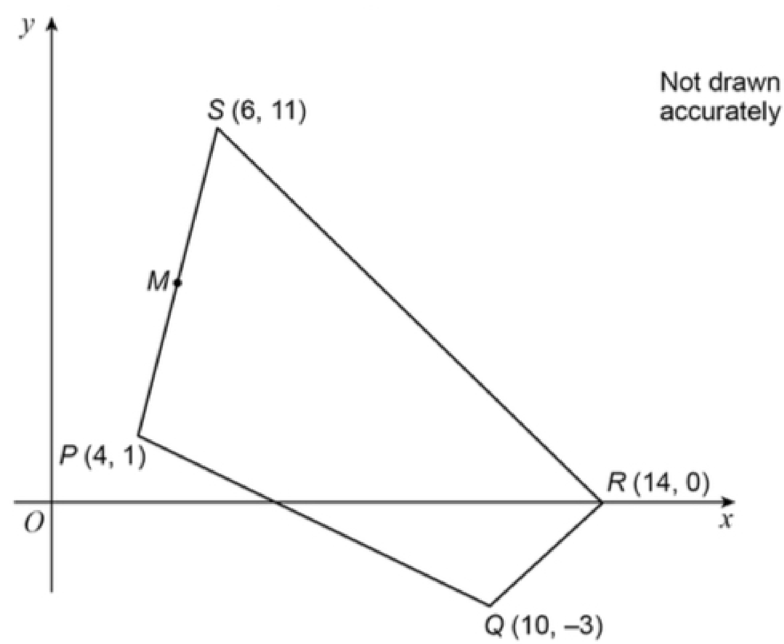
$$\mathbf{I} = \begin{pmatrix} m & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} 2m+2 & 2m+1 \\ 0 & 1 \end{pmatrix}.$$

Choose the $(1, 1)$ element:

$$2m + 2 = 1 \Rightarrow 2m = -1 \Rightarrow \underline{\underline{m = -\tfrac{1}{2}}}.$$

2. Here is a sketch of quadrilateral $PQRS$. (3)

M is the midpoint of PS.



Use gradients to show that MR is parallel to PQ.

Solution

Now,

$$\begin{aligned} m_{PQ} &= \frac{-3-1}{10-4} \\ &= \frac{-4}{6} \\ &= -\tfrac{2}{3}. \end{aligned}$$

Next, M is the point

$$\left(\frac{4+6}{2}, \frac{1+11}{2}\right) = (5, 6)$$

and

$$\begin{aligned} m_{MR} &= \frac{0-6}{14-5} \\ &= \frac{-6}{9} \\ &= -\tfrac{2}{3}; \end{aligned}$$

hence, MR is parallel to PQ as the gradients are the same.

3. (4)

$$-2 < a < 0 \text{ and } -1 < b < 1.$$

Tick the correct box for each statement.

	Always true	**Sometimes true**	**Never true**
$a^2 < 0$			
$-1 < b^3 < 1$			
$\frac{b}{a} < 0$			
$a - b > 0$			

Solution

	Always true	**Sometimes true**	**Never true**
$a^2 < 0$			✓
$-1 < b^3 < 1$	✓		
$\dfrac{b}{a} < 0$		✓	
$a - b > 0$		✓	

4. P is a point on a curve. (4)

The curve has gradient function
$$\frac{x^5 - 17}{10}.$$
The tangent to the curve at P is parallel to the line
$$3x - 2y = 9.$$
Work out the x-coordinate of P.

Solution

Tangent:
$$\begin{aligned} 3x - 2y = 9 &\Rightarrow 2y = 3x - 9 \\ &\Rightarrow y = \tfrac{3}{2}x - \tfrac{9}{2} \end{aligned}$$
and so the gradient is $\frac{3}{2}$.

Now,
$$\begin{aligned} \frac{x^5 - 17}{10} = \tfrac{3}{2} &\Rightarrow x^5 - 17 = 15 \\ &\Rightarrow x^5 = 32 \\ &\Rightarrow x = \sqrt[5]{32} \\ &\Rightarrow \underline{\underline{x = 2}}. \end{aligned}$$

5. (a) Write (2)
$$\sqrt[4]{a \times a^{-9}}$$
as an integer power of a.

Solution

$$\sqrt[4]{a \times a^{-9}} = (a^{-8})^{\frac{1}{4}}$$
$$= \underline{\underline{a^{-2}}}.$$

(b) Simplify fully (3)

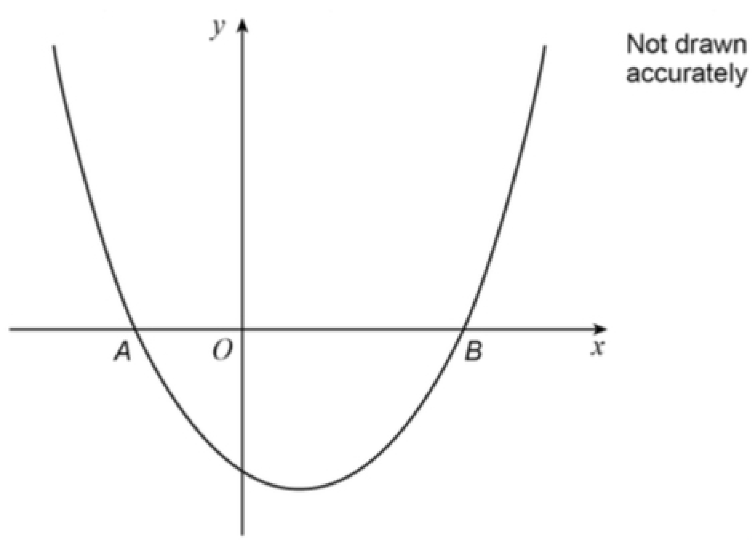
$$\frac{(4cd^2)^3}{2cd^4}.$$

Solution

$$\frac{(4cd^2)^3}{2cd^4} = \frac{64c^3d^6}{2cd^4}$$
$$= \underline{\underline{32c^2d^2}}.$$

6. Here is a sketch of the curve

$$y = (2x + 3)(x - 2).$$



The curve intersects the x-axis at A and B.

(a) Complete the coordinates of A and B. (2)

$$A(\quad ,0) \qquad B(\quad ,0)$$

Solution

$$(2x+3)(x-2) = 0 \Rightarrow 2x+3 = 0 \text{ or } x-2 = 0$$
$$\Rightarrow x = -\tfrac{3}{2} \text{ or } x = 2.$$

Hence,

$$\underline{\underline{A(-\tfrac{3}{2}, 0)}} \qquad \underline{\underline{B(2, 0)}}$$

(b) Write down the range of values for x for which (1)

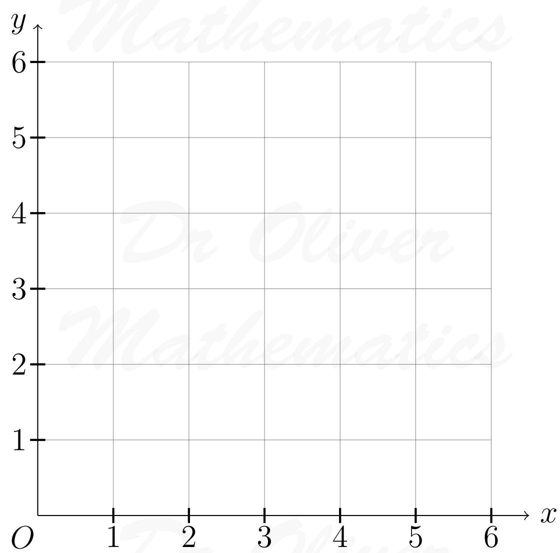
$$(2x+3)(x-2) < 0.$$

Solution

$$\underline{\underline{-\tfrac{3}{2} < x < 2.}}$$

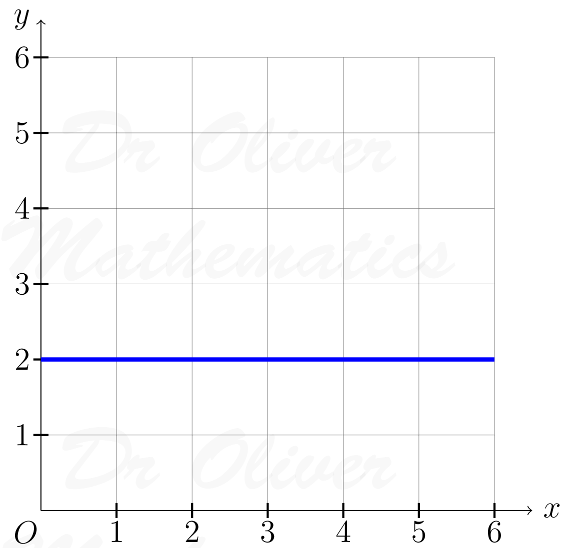
7. (a) On the grid, sketch a graph for which (1)

the rate of change of y with respect to x is always zero.



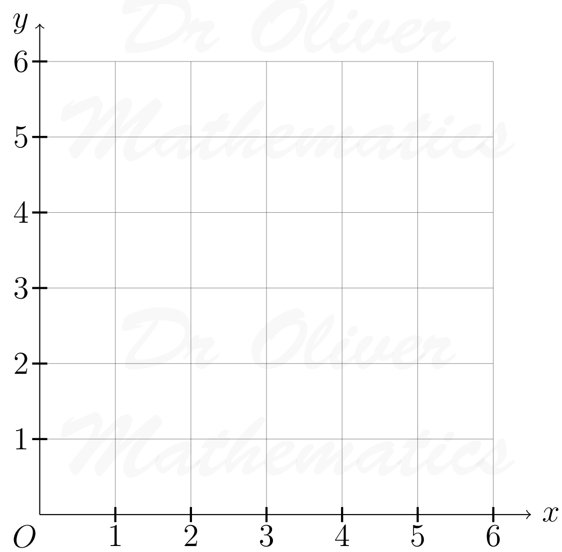
Solution

E.g.,



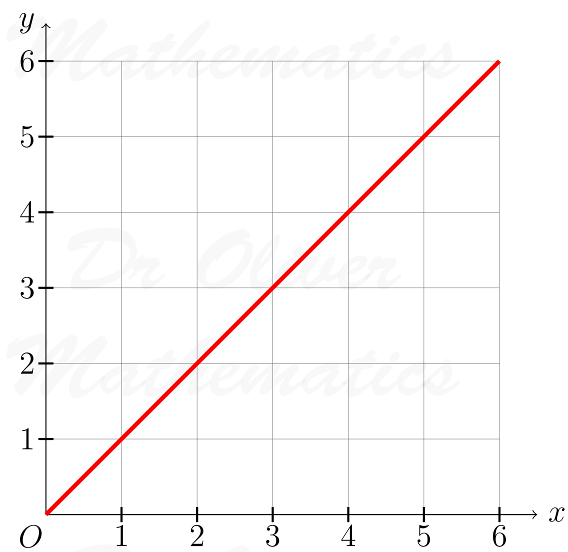
(b) On the grid, sketch a graph for which (1)

the rate of change of y with respect to x is always a positive constant.



Solution

E.g.,



8. (a) A linear sequence has first term (2)

$$7 + 12\sqrt{5}.$$

The term-to-term rule is

$$\text{add } 9 - 2\sqrt{5}.$$

One term of the sequence is an integer.

Work out the value of this integer.

Solution

Well,

$$x = \frac{12}{2} = 6$$

and is the *seventh* term (1+6=7):

$$\begin{aligned}(7 + 12\sqrt{5}) + 6(9 - 2\sqrt{5}) &= (7 + 12\sqrt{5}) + (54 - 12\sqrt{5}) \\ &= \underline{\underline{61}}.\end{aligned}$$

(b) The nth term of a different sequence is (2)

$$\frac{3n^2 - 1}{n^2 + 1}.$$

Work out the sum of the first three terms.

Solution

$$n = 1 \Rightarrow \frac{3-1}{1+1} = \frac{2}{2} = 1,$$
$$n = 2 \Rightarrow \frac{12-1}{4+1} = \frac{11}{5},$$
$$n = 3 \Rightarrow \frac{27-1}{9+1} = \frac{26}{10} = \frac{13}{5};$$

the sum of the first three terms is

$$1 + \frac{11}{5} + \frac{13}{5} = \underline{\underline{5\tfrac{4}{5}}}.$$

(c) The first four terms of a quadratic sequence are (3)

$$-3 \qquad 3 \qquad 13 \qquad 27.$$

Work out an expression for the nth term.

Solution

Let the

$$n\text{th term} = an^2 + bn + c.$$

Then

-3		3		13		27
	6		10		14	
		4		4		

and

$a+b+c$		$4a+2b+c$		$9a+3b+c$		$16a+4b+c$
	$3a+b$		$5a+b$		$7a+b$	
		$2a$		$2a$		

We compare terms:

$$2a = 4 \Rightarrow a = 2,$$

$$3a + b = 6 \Rightarrow 3 \times 2 + b = 6$$
$$\Rightarrow b = 0,$$

and

$$\begin{aligned} a+b+c=0 &\Rightarrow 2+0+c=-3 \\ &\Rightarrow c=-5; \end{aligned}$$

hence,

$$n\text{th term} = \underline{\underline{2n^2-5}}.$$

9. Factorise fully (2)

$$(p+6)^{11}-(p+6)^{10}.$$

Solution

$$\begin{aligned} (p+6)^{11}-(p+6)^{10} &= (p+6)^{10}[(p+6)-1] \\ &= \underline{\underline{(p+5)(p+6)^{10}}}. \end{aligned}$$

10. (a) (2)

$$\mathrm{f}(x)=x^3-2.$$

The domain of $\mathrm{f}(x)$ is $x \leqslant 3$.

Work out the range of $\mathrm{f}(x)$.

Solution
Well,

$$\mathrm{f}(3)=3^3-2=25$$

and the range of $\mathrm{f}(x)$ is

$$\underline{\underline{\mathrm{f}(x) \leqslant 25}}.$$

(b) (2)

$$\mathrm{g}(x)=5-x^2.$$

The domain of $\mathrm{g}(x)$ is $-2 \leqslant x \leqslant 1$.

Work out the range of $\mathrm{g}(x)$.

Solution

So,

$$g(-2) = 5 - (-2)^2 = 1$$

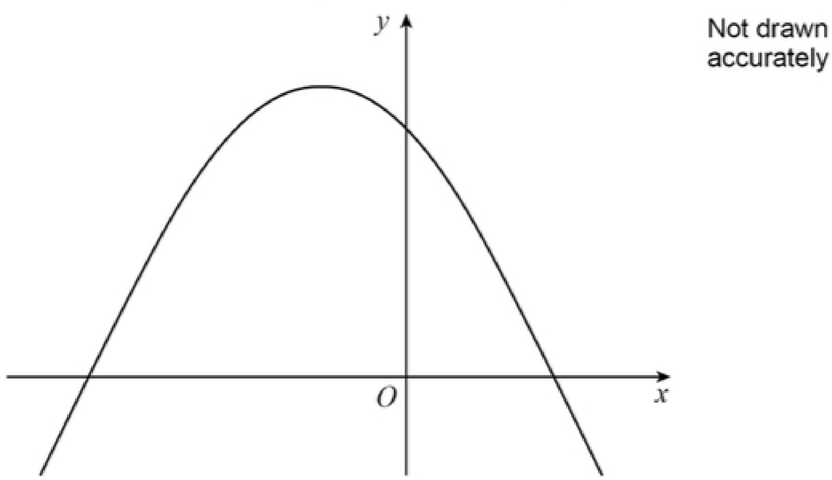
and

$$g(0) = 5 - 0^2 = 5.$$

Hence, the range of $g(x)$ is

$$\underline{\underline{1 \leqslant g(x) \leqslant 5}}.$$

11. Here is a sketch of a quadratic curve which has a maximum point at $(-2, 5)$. (1)



What is the equation of the normal to the curve at the maximum point?
Circle your answer.

$$x = -2 \qquad y = 5 \qquad x = 5 \qquad y = -2$$

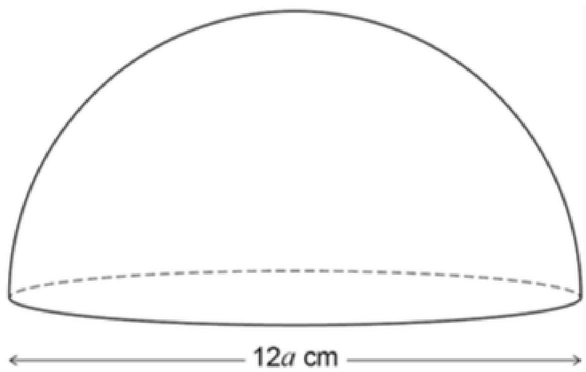
Solution

It is either $x = 5$ or $x = -2$ (why?). Hence, the equation of the normal to the curve at the maximum point is

$$\underline{\underline{x = -2}} \qquad y = 5 \qquad x = 5 \qquad y = -2$$

12. The diagram shows a solid hemisphere. (3)

- The diameter is $12a$ cm.
- The volume is 486π cm^3.



Work out the value of a.

Solution

Well, the volume of the sphere would have be

$$2 \times 486\pi = 972\pi$$

and

$$\begin{aligned} \tfrac{4}{3} \times \pi \times (6a)^3 = 972\pi &\Rightarrow 288a^3 = 972 \\ &\Rightarrow a^3 = \tfrac{27}{8} \\ &\Rightarrow \underline{\underline{a = 1\tfrac{1}{2}}}. \end{aligned}$$

13. Simplify fully (4)

$$\frac{x - x^3}{2x + 2x^2}.$$

You **must** show your working.

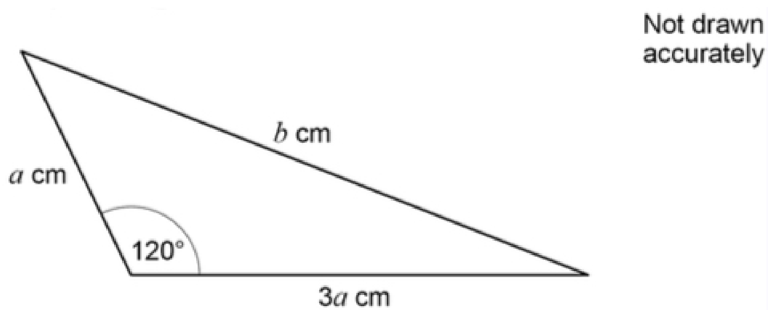
Solution

$$\frac{x - x^3}{2x + 2x^2} = \frac{x(1 - x^2)}{2x(1 + x)}$$

difference of two squares:

$$= \frac{x(1 + x)(1 - x)}{2x(1 + x)}$$
$$= \underline{\underline{\frac{x(1 - x)}{2x}}}.$$

14. Here is a triangle. (3)



Use the cosine rule to work out the ratio

$$b^2 : a^2.$$

Solution

Cosine rule:

$$\begin{aligned} & b^2 = a^2 + (3a)^2 - 2(a)(3a)(\cos 120^\circ) \\ \Rightarrow\ & b^2 = a^2 + 9a^2 - 6a^2(-\tfrac{1}{2}) \\ \Rightarrow\ & b^2 = 10a^2 + 3a^2 \\ \Rightarrow\ & b^2 = 13a^2; \end{aligned}$$

hence, the ratio is

$$\underline{\underline{13 : 1}}.$$

15. Rearrange (4)

$$m = \frac{2p+1}{p} + \frac{p+5}{3p}$$

to make p the subject.

Solution

$$\begin{aligned} m = \frac{2p+1}{p} + \frac{p+5}{3p} &\Rightarrow m = \frac{3(2p+1)}{3p} + \frac{p+5}{3p} \\ &\Rightarrow m = \frac{(6p+3)+(p+5)}{3p} \\ &\Rightarrow m = \frac{7p+8}{3p} \\ &\Rightarrow 3mp = 7p+8 \\ &\Rightarrow 3mp - 7p = 8 \\ &\Rightarrow p(3m-7) = 8 \\ &\Rightarrow \underline{\underline{p = \frac{8}{3m-7}}}. \end{aligned}$$

16. The curve (3)

$$y = 2\sqrt{x-a} + 5$$

passes through the point $(1, 8)$

Work out the value of a.

Solution

$$\begin{aligned} x = 1, y = 8 &\Rightarrow 8 = 2\sqrt{1-a} + 5 \\ &\Rightarrow 3 = 2\sqrt{1-a} \\ &\Rightarrow \tfrac{3}{2} = \sqrt{1-a} \\ &\Rightarrow (\tfrac{3}{2})^2 = 1-a \\ &\Rightarrow \tfrac{9}{4} = 1-a \\ &\Rightarrow \underline{\underline{a = -1\tfrac{1}{4}}}. \end{aligned}$$

17. Show that (5)

$$(x+1)(x+3)(x+4) - x(x^2+7x+11)$$

can be written in the form

$$(x+a)(x+b),$$

where a and b are positive integers.

Solution

$\times$	x	$+1$
x	x^2	$+x$
$+3$	$+3x$	$+3$

so

$$(x+1)(x+3) = x^2+4x+3$$

and

$\times$	x^2	$+4x$	$+3$
x	x^3	$+4x^2$	$+3x$
$+4$	$+4x^2$	$+16x$	$+12$

so

$$(x^2+4x+3)(x+4) = x^3+8x^2+19x+12.$$

Finally,

$$\begin{aligned} & (x+1)(x+3)(x+4) - x(x^2+7x+11) \\ = \; & (x^3+8x^2+19x+12) - (x^3+7x^2+11x) \\ = \; & x^2+8x+12 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & +8 \\ \text{multiply to:} & +12 \end{array}\right\} +6,\ +2$$

$$= \underline{\underline{(x+6)(x+2)}};$$

hence,

$$\underline{\underline{a=6}} \text{ and } \underline{\underline{b=2}}$$

or vice versa.

18. Solve (3)

$$4(x-5)^2 = k^2,$$

where k is a constant.

Give your answers in their simplest form in terms of k.

Solution

$$\begin{aligned} 4(x-5)^2 = k^2 &\Rightarrow (x-5)^2 = \tfrac{1}{4}k^2 \\ &\Rightarrow (x-5)^2 = (\tfrac{1}{2}k)^2 \\ &\Rightarrow x-5 = \pm\tfrac{1}{2}k \\ &\Rightarrow \underline{\underline{x = 5 \pm \tfrac{1}{2}k}}. \end{aligned}$$

19.
- ABC is a right-angled triangle.
- ACD is an isosceles triangle.
- All dimensions are in centimetres.

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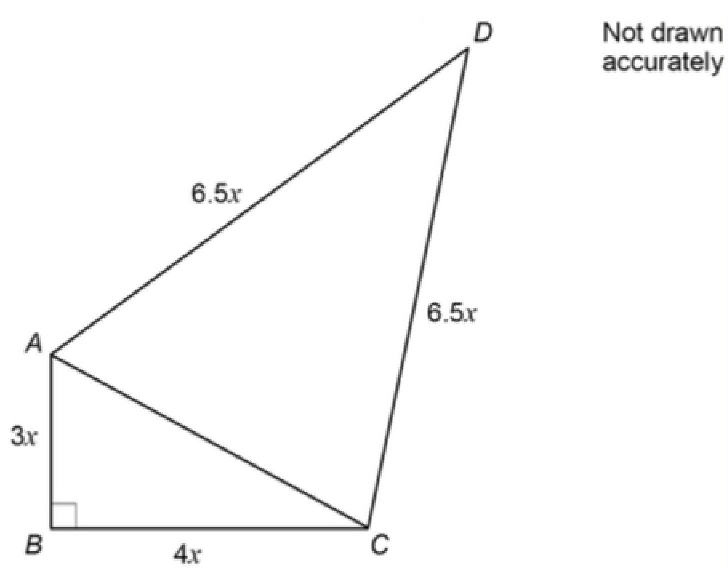


(a) Show that $AC = 5x$. (1)

Solution

$$\begin{aligned} AC^2 = AB^2 + BC^2 &\Rightarrow AC^2 = (3x)^2 + (4x)^2 \\ &\Rightarrow AC^2 = 9x^2 + 16x^2 \\ &\Rightarrow AC^2 = 25x^2 \\ &\Rightarrow \underline{\underline{AC = 5x}}, \end{aligned}$$

as required.

(b) Work out an expression, in cm^2, for the area of quadrilateral $ABCD$. (5)

Give your answer in the form px^2, where p is an integer.

Solution

Let E be the midpoint of AC. Then $AE = 2.5x$ and

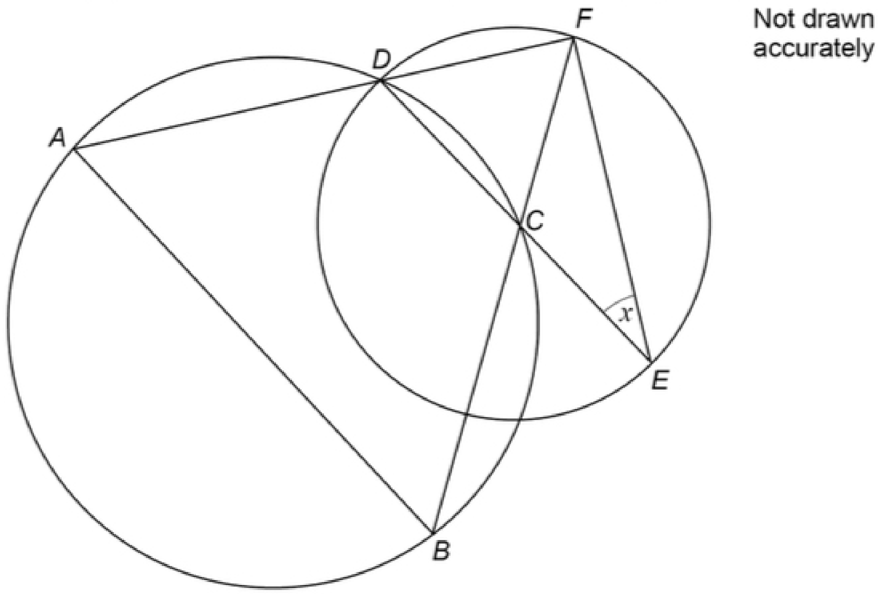
$$\begin{aligned} AD^2 = AE^2 + DE^2 &\Rightarrow (6.5x)^2 = (2.5x)^2 + DE^2 \\ &\Rightarrow 42.25x^2 = 6.25x^2 + DE^2 \\ &\Rightarrow DE^2 = 36x^2 \\ &\Rightarrow DE = 6x. \end{aligned}$$

Finally,

$$\begin{aligned} \text{area} &= \text{area of } ABC + \text{area of } ACD \\ &= \left(\tfrac{1}{2} \times 3x \times 4x\right) + \left(\tfrac{1}{2} \times 5x \times 6x\right) \\ &= 6x^2 + 15x^2 \\ &= \underline{\underline{21x^2 \text{ cm}^2}}. \end{aligned}$$

hence, $\underline{\underline{p = 21}}$.

20.
- A, B, C, and D are points on a circle.
- D, E, and F are points on a different circle, centre C.
- DCE, ADF, and BCF are straight lines.
- Angle $DEF = x$.



(a) Prove that (3)

angle $BAD = 2x$.

Solution
Well, $\angle EFC = x$ (base angles)
$\angle ECF = 180 - 2x$ (completing the triangle)
$\angle BCD = 180 - 2x$ (vertically opposite angles)
$\underline{\underline{\angle BAD = 2x}}$ (opposite angles in a cyclic quadrilateral).

(b) In the case when AB is parallel to DE, work out the size of angle x. (2)

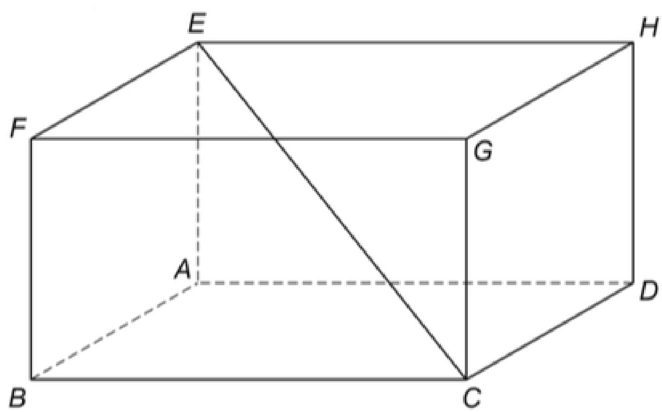
Solution
Now, $\angle BAD = \angle EDF = 2x$ (corresponding angles)
$\angle DFE = 90^\circ$ (why?) and

$$\begin{aligned}\angle EDF + \angle DFE + \angle FED = 180 &\Rightarrow 2x + 90 + x = 180\\ &\Rightarrow 3x = 90\\ &\Rightarrow \underline{\underline{x = 30}}.\end{aligned}$$

21. $ABCDEFGH$ is a cuboid. (4)

- $BC = 15$ cm.

- $CD = 12$ cm.
- $DH = 8$ cm.



Work out the size of the angle between the line CE and the plane $CDHG$.

Solution

Well, the space diagonal is

$$\begin{aligned} CE^2 = CD^2 + DH^2 + BC^2 &\Rightarrow CE^2 = 12^2 + 8^2 + 15^2 \\ &\Rightarrow CE^2 = 144 + 64 + 225 \\ &\Rightarrow CE^2 = 433 \\ &\Rightarrow CE = \sqrt{433}. \end{aligned}$$

Finally,

$$\begin{aligned} \sin = \frac{\text{opp}}{\text{hyp}} &\Rightarrow \sin(\text{angle}) = \frac{BC}{CE} \\ &\Rightarrow \sin(\text{angle}) = \frac{15}{\sqrt{433}} \\ &\Rightarrow \text{angle} = 46.125\,033\,09 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{\text{angle} = 46.1^\circ \text{ (3 sf)}}}. \end{aligned}$$

22. (a) Show that (3)

$$\frac{2\sin^2 x - 1 + \cos^2 x}{\sin x \cos x}$$

is equivalent to $\tan x$.

Solution

$$\begin{aligned}\frac{2\sin^2 x - 1 + \cos^2 x}{\sin x \cos x} &\equiv \frac{\sin^2 x - 1 + (\sin^2 x + \cos^2 x)}{\sin x \cos x} \\ &\equiv \frac{\sin^2 x - 1 + 1}{\sin x \cos x} \\ &\equiv \frac{\sin^2 x}{\sin x \cos x} \\ &\equiv \frac{\sin x}{\cos x} \\ &\equiv \underline{\underline{\tan x}},\end{aligned}$$

as required.

(b) Hence solve (2)

$$\frac{2\sin^2 x - 1 + \cos^2 x}{\sin x \cos x} = -1,$$

for $0^\circ \leqslant x \leqslant 360^\circ$.

Solution

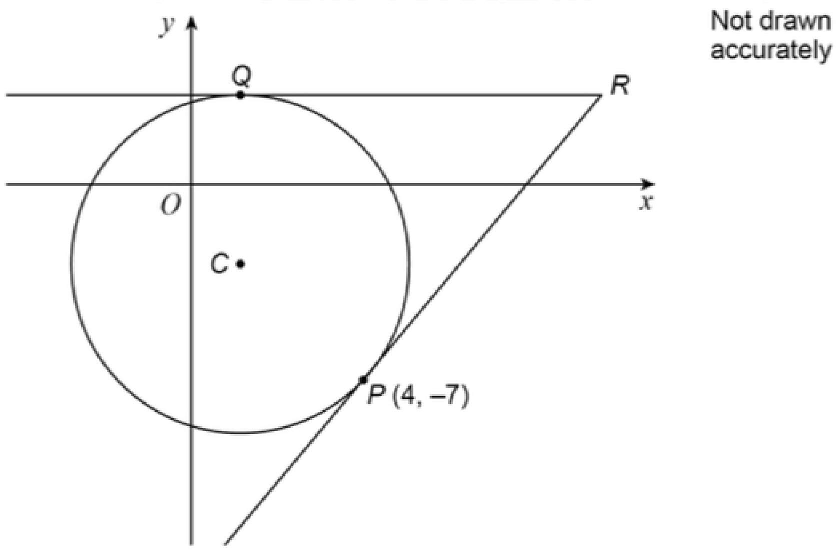
$$\begin{aligned}\frac{2\sin^2 x - 1 + \cos^2 x}{\sin x \cos x} = -1 &\Rightarrow \tan x = -1 \\ &\Rightarrow \underline{\underline{x = 135,\ 315}}.\end{aligned}$$

23. A circle has centre C and equation

$$(x-1)^2 + (y+3)^2 = 25.$$

$P(4, -7)$ and Q are points on the circle.

- The tangent at Q is parallel to the x-axis.
- The tangents at P and Q intersect at point R.



(a) Write down the coordinates of C. (1)

Solution
$\underline{\underline{C(1, -3)}}$.

(b) Show that the equation of the tangent at Q is

$$y = 2.$$

(1)

Solution

$$\begin{aligned} x = 1 &\Rightarrow (y+3)^2 = 25 \\ &\Rightarrow y + 3 = \pm 5 \\ &\Rightarrow y = -3 \pm 5; \end{aligned}$$

in other words, $y = -8$ or $y = 2$.

As $y > 0$, $\underline{\underline{y = 2}}$, as required.

(c) Work out the x-coordinate of R. (4)

Solution

Well,

$$m_{CP} = \frac{-7-(-3)}{4-1} = -\tfrac{4}{3}$$

and

$$m_{PR} = -\frac{1}{-\tfrac{4}{3}} = \tfrac{3}{4},$$

as $\angle CPR$ is a right-angle. Now, the equation of the line PR is

$$y-(-7) = \tfrac{3}{4}(x-4) \Rightarrow y+7 = \tfrac{3}{4}x - 3 \Rightarrow y = \tfrac{3}{4}x - 10.$$

Finally,

$$y = 2 \Rightarrow 2 = \tfrac{3}{4}x - 10 \Rightarrow \tfrac{3}{4}x = 12 \Rightarrow \underline{\underline{x = 16}}.$$

24. Show that the curve (4)

$$y = \tfrac{3}{5}x^5 + x^4$$

has **exactly two** stationary points.

Solution

$$y = \tfrac{3}{5}x^5 + x^4 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^4 + 4x^3$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow 3x^4 + 4x^3 = 0 \Rightarrow x^3(3x+4) = 0 \Rightarrow x^3 = 0 \text{ or } 3x+4 = 0 \Rightarrow x = 0 \text{ or } x = -\tfrac{4}{3};$$

hence, the curve has exactly two stationary points.

25. (3)

$$\mathrm{f}(x) = x^3 - 10x - c, \text{ where } c \text{ is a positive integer.}$$

$(x + c)$ is a factor of $f(x)$.

Use the factor theorem to work out the value of c.

Solution

Well, $(x + c) = [x - (-c)]$ and

$$\begin{aligned} f(-c) = 0 &\Rightarrow (-c)^3 - 10(-c) - c = 0 \\ &\Rightarrow -c^3 + 10c - c = 0 \\ &\Rightarrow c^3 - 9c = 0 \\ &\Rightarrow c(c^2 - 9) = 0 \\ &\Rightarrow c(c-3)(c+3) = 0 \\ &\Rightarrow c = 0,\ c = 3,\ \text{or } c = -3; \end{aligned}$$

now, c is a positive integer and so $\underline{\underline{c = 3}}$.

26. $f(x)$ is a function with domain all values of x. (4)

$$f(x) = x^2 + 6x - a, \text{ where } a \text{ is a constant.}$$

Work out the possible values of a.

Give your answer as an inequality.

Solution

$$f(x) = x^2 + 6x - a \Rightarrow f'(x) = 2x + 6$$

and

$$\begin{aligned} f'(x) = 0 &\Rightarrow 2x + 6 = 0 \\ &\Rightarrow 2x = -6 \\ &\Rightarrow x = -3. \end{aligned}$$

Finally,

$$\begin{aligned} x = -3 &\Rightarrow (-3)^2 + 6(-3) - a \geqslant 0 \\ &\Rightarrow 9 - 18 - a \geqslant 0 \\ &\Rightarrow \underline{\underline{a \leqslant -9}}. \end{aligned}$$

27. The curve $y = \mathrm{f}(x)$ has (3)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (x+2)^6 + (x+2)^4.$$

The curve has exactly one stationary point at P where $x = -2$.

Use the expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ to show that P is a point of inflection.

Solution

Pick, say, $x = -3$ and $x = -1$:

$$x = -3 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = (-3+2)^6 + (-3+2)^4 = 2$$

and

$$x = -1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = (-1+2)^6 + (-1+2)^4 = 2;$$

so, either side of P the derivative is positive.

Hence, P is a point of inflection.