

Dr Oliver Mathematics

Double Counting

In this note, we will explore double counting.

If the same set is counted in two different ways, you get the same answer.

Example 1

Prove, by double counting, that

$$r \binom{n}{r} = n \binom{n-1}{r-1}.$$

Solution

Consider a community of n people from which you have to select a committee of r members with a designated leader.

LHS: Choose the r members first in $\binom{n}{r}$ ways and then choose a leader from them in r ways. This gives a total of $r \binom{n}{r}$ ways.

RHS: First choose the leader, in n ways and then from the remaining $(n-1)$ people choose $(r-1)$ to complete the committee. This gives a total of $n \binom{n-1}{r-1}$ ways.

LHS = RHS: Well, we see that

$$r \binom{n}{r} = n \binom{n-1}{r-1}. \quad \blacksquare$$

Example 2

At a party of 11 people, every person claims that they shook hands with exactly 5 other people. Show that someone is not telling the truth.

Solution

Firstly, since everyone participated in 5 handshakes, there are

$$11 \times 5 = 55.$$

Secondly, since every handshake involves only 2 people, there are

$$n \times 2 = 2n.$$

Then

$$2n = 55 \Rightarrow n = 27\frac{1}{2},$$

which is a contradiction.

Thus, someone is not telling the truth. ■

Here are some examples for you to try.

1. Prove, by double counting, that

$$\binom{n}{r} = \binom{n}{n-r}.$$

2. Prove, by double counting, that

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

3. Prove, by double counting, that

$$\binom{n}{s} \binom{s}{r} = \binom{n}{r} \binom{n-r}{s-r}.$$

4. Prove, by double counting, that

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \dots + n\binom{n}{n} = n2^{n-1}.$$

5. (Vandermonde's Identity) Prove, by double counting, that

$$\binom{m+w}{k} = \sum_{i=0}^k \binom{m}{i} \binom{w}{k-i}.$$

6. (Christmas Stocking Identity or Hockey-Stick Identity) Prove, by double counting, that

$$\binom{m+n+1}{n+1} = \sum_{k=0}^m \binom{n+k}{n}.$$

7. Prove, by double counting, that

$$n^2 = 2\binom{n}{2} + \binom{n}{1}.$$

8. On a 8×8 chessboard, how many squares are there, of all possible sizes?

9. 15 students join a summer course. Every day, 3 students are on duty after school to clean the classroom. After the course, it was found that every pair of students has been on duty together exactly once. How many days does the course last for?