Dr Oliver Mathematics
Applied Mathematics: Mechanics or Statistics
Section B
2009 Paper
1 hour

The total number of marks available is 32.
You must write down all the stages in your working.

1. Obtain the binomial expansion of (4)

$$\left(b - \frac{2}{b}\right)^5$$

and simplify the expression.

Solution

$$\begin{aligned} & \left(b - \frac{2}{b}\right)^5 \\ = \;& b^5 + \binom{5}{1} b^4 \left(-\frac{2}{b}\right) + \binom{5}{2} b^3 \left(-\frac{2}{b}\right)^2 + \binom{5}{3} b^2 \left(-\frac{2}{b}\right)^3 + \binom{5}{4} b \left(-\frac{2}{b}\right)^4 + \left(-\frac{2}{b}\right)^5 \\ = \;& \underline{\underline{b^5 - 10b^3 + 40b - \frac{80}{b} + \frac{80}{b^3} - \frac{32}{b^5}}}. \end{aligned}$$

2. Obtain (4)

$$\int_0^{\frac{1}{3}\pi} \cos^5 x \sin x \, \mathrm{d}x$$

by using the substitution $u = \cos x$ or otherwise.

Solution

We will use the suggested substitution:

$$\begin{aligned} u = \cos x &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x \\ &\Rightarrow \mathrm{d}u = -\sin x \, \mathrm{d}x \end{aligned}$$

and

$$x = 0 \Rightarrow u = 1$$
$$x = \tfrac{1}{3}\pi \Rightarrow u = \tfrac{1}{2}.$$

Now,

$$\begin{aligned}
\int_0^{\frac{1}{3}\pi} \cos^5 x \sin x \, \mathrm{d}x &= -\int_0^{\frac{1}{3}\pi} \cos^5 x(-\sin x \, \mathrm{d}x) \\
&= -\int_1^{\frac{1}{2}} u^5 \, \mathrm{d}u \\
&= -\left[\tfrac{1}{6}u^6\right]_{u=1}^{\frac{1}{2}} \\
&= -\left(\tfrac{1}{384} - \tfrac{1}{6}\right) \\
&= \underline{\underline{\tfrac{21}{128}}}.
\end{aligned}$$

3. A particle moves along a curve in the x-y plane. The curve is defined by the parametric equations

$$x = t^2 + 1,\ y = 1 - 3t^3,$$

where t is the time elapsed since the start.

(a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t. (3)

Solution

$$x = t^2 + 1 \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = 2t$$
$$y = 1 - 3t^3 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = -9t^2.$$

and

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\
&= \frac{-9t^2}{2t} \\
&= \underline{\underline{-\tfrac{9}{2}t}}.
\end{aligned}$$

(b) Hence obtain an equation of the tangent to the curve when $t = 2$. (2)

Solution
Well,

$$t = 2 \Rightarrow x = 5, y = -23, \frac{\mathrm{d}y}{\mathrm{d}x} = -9$$

and an equation of the tangent is

$$\begin{aligned} y + 23 = -9(x - 5) &\Rightarrow y + 23 = -9x + 45 \\ &\Rightarrow \underline{\underline{y = -9x + 22}}. \end{aligned}$$

4. Determine k such that the matrix (4)

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & k-2 & -1 \\ 1 & 2 & k \end{pmatrix}$$

does not have an inverse.

Solution

$$\begin{aligned} & \det \begin{pmatrix} 1 & 1 & 0 \\ 0 & k-2 & -1 \\ 1 & 2 & k \end{pmatrix} = 0 \\ \Rightarrow\ & 1[k(k-2) + 2] - 1[0 + 1] + 0 = 0 \\ \Rightarrow\ & k^2 - 2k + 2 - 1 = 0 \\ \Rightarrow\ & k^2 - 2k + 1 = 0 \\ \Rightarrow\ & (k-1)^2 = 0 \\ \Rightarrow\ & k - 1 = 0 \\ \Rightarrow\ & \underline{\underline{k = 1}}. \end{aligned}$$

5. An industrial scientist finds that the differential equation

$$t\frac{\mathrm{d}x}{\mathrm{d}t} - 2x = 3t^2$$

models a production process.

(a) Find the general solution of the differential equation. (5)

Solution

$$
\begin{aligned}
t\frac{\mathrm{d}x}{\mathrm{d}t} - 2x = 3t^2 &\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} - \frac{2}{t}x = 3t \\
&\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} + \left(-\frac{2}{t}\right)x = 3t
\end{aligned}
$$

$$
\begin{aligned}
\text{IF} &= \mathrm{e}^{\int -\frac{2}{t}\,\mathrm{d}t} \\
&= \mathrm{e}^{-2\ln t} \\
&= \mathrm{e}^{\ln t^{-2}} \\
&= t^{-2}
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow t^{-2}\frac{\mathrm{d}x}{\mathrm{d}t} - 2t^{-3}x = 3t^{-1} \\
&\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(t^{-2}x) = 3t^{-1} \\
&\Rightarrow t^{-2}x = \int 3t^{-1}\,\mathrm{d}t \\
&\Rightarrow t^{-2}x = 3\ln t + c \\
&\Rightarrow \underline{\underline{x = t^2(3\ln t + c)}}.
\end{aligned}
$$

(b) Hence find the particular solution given $x = 1$ when $t = 1$. (1)

Solution

$$x = 1, t = 1 \Rightarrow 1 = 1(0 + c)$$

and, hence,

$$\underline{\underline{x = t^2(3\ln t + 1)}}.$$

6. (a) Given (2)

$$\mathrm{f}(x) = x\tan 2x$$

for $-\frac{1}{4}\pi < x < \frac{1}{4}\pi$, obtain an expression for $\mathrm{f}'(x)$.

Solution

$$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$
$$v = \tan 2x \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 2\sec^2 2x$$

and

$$\mathrm{f}(x) = x\tan 2x \Rightarrow \mathrm{f}'(x) = (x)(2\sec^2 2x) + (1)(\tan 2x)$$
$$\Rightarrow \underline{\underline{\mathrm{f}'(x) = 2x\sec^2 2x + \tan 2x}}.$$

(b) Show that (3)

$$\mathrm{f}''(x) = 4\sec^2 2x(1 + 2x\tan 2x).$$

Solution

$$u = 2x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2$$
$$v = \sec^2 2x \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = (2\sec 2x)(\sec 2x\tan 2x)(2) = 4\sec^2 2x\tan 2x$$

and

$$\begin{aligned} & \mathrm{f}'(x) = 2x\sec^2 2x + \tan 2x \\ \Rightarrow\ & \mathrm{f}''(x) = (2x)(4\sec^2 2x\tan 2x) + (2)(\sec^2 2x) + 2\sec^2 2x \\ \Rightarrow\ & \mathrm{f}''(x) = 8x\sec^2 2x\tan 2x + 4\sec^2 2x \\ \Rightarrow\ & \underline{\underline{\mathrm{f}''(x) = 4\sec^2 2x(1 + 2x\tan 2x)}}, \end{aligned}$$

as required.

(c) Hence find the exact value of (4)

$$\int_0^{\frac{1}{6}\pi} \frac{1 + 2x\tan 2x}{\cos^2 2x}\,\mathrm{d}x.$$

Solution

$$
\begin{aligned}
\int_0^{\frac{1}{6}\pi} \frac{1+2x\tan 2x}{\cos^2 2x}\,\mathrm{d}x &= \int_0^{\frac{1}{6}\pi} \sec^2 2x(1+2x\tan 2x)\,\mathrm{d}x \\
&= \tfrac{1}{4}\int_0^{\frac{1}{6}\pi} 4\sec^2 2x(1+2x\tan 2x)\,\mathrm{d}x \\
&= \tfrac{1}{4}\left[2x\sec^2 2x + \tan 2x\right]_{x=0}^{\frac{1}{6}\pi} \\
&= \tfrac{1}{4}\left[\left(\tfrac{4}{3}\pi + \sqrt{3}\right) - (0+0)\right] \\
&= \underline{\underline{\tfrac{1}{12}\left(4\pi + 3\sqrt{3}\right)}}.
\end{aligned}
$$