Dr Oliver Mathematics
Extended Mathematics Certificate
Sample Assessment Materials: Calculator
1 hour 15 minutes

The total number of marks available is 60.
You must write down all the stages in your working.

1. (a) Factorise (1)

$$x^2 - 25.$$

Solution

Well,

$$\left.\begin{array}{ll} \text{add to:} & 0 \\ \text{multiply to:} & -25 \end{array}\right\} +5,\ -5$$

so

$$x^2 - 25 = \underline{\underline{(x+5)(x-5)}}.$$

(b) Write (3)

$$(x-3)(x+7)(x+3)(x-6)$$

in the form

$$(x^2 - d)(ax^2 + bx + c),$$

where a, b, c, and d are integers.

Solution

$$(x-3)(x+7)(x+3)(x-6) = (x-3)(x+3)(x+7)(x-6)$$

×	x	-3
x	x^2	$-3x$
$+3$	$+3x$	-9

and

×	x	$+7$
x	x^2	$+7x$
-6	$-6x$	-42

$$= \underline{\underline{(x^2 - 9)(x^2 + x - 42)}};$$

hence, $\underline{\underline{a = 1}}$, $\underline{\underline{b = 1}}$, $\underline{\underline{c = -42}}$, and $\underline{\underline{d = -9}}$.

2. w, x, y, and z are four consecutive integers. (5)

   Prove algebraically, that for any set of four consecutive integers

   $$yz - wx$$

   is equal to the sum of the four consecutive integers.

Solution

Let them be w, $(w + 1)$, $(w + 2)$, and $(w + 3)$, respectively. Then

$$yz - wx = (w + 2)(w + 3) - w(w + 1)$$

$\times$	w	$+2$
w	w^2	$+2w$
$+3$	$+3w$	$+6$

$$\begin{aligned} &= (w^2 + 5w + 6) - (w^2 + w) \\ &= 4w + 6 \\ &= w + (w + 1) + (w + 2) + (w + 3) \\ &= \underline{\underline{w + x + y + z}}; \end{aligned}$$

hence, $(yz - wx)$is equal to the sum of the four consecutive integers.

3. Triangles ABC and PQR are similar. (4)

Triangle ABC is an isosceles triangle where

- one of the angles is 40°,
- one of the angles is obtuse, and
- two of the sides are each 10 cm.

Length $PQ = 1.5 \times$ length AB.

Work out the area of triangle PQR.
Give your answer correct to 3 significant figures.

Solution

Well, *two* of the angles are 40° and the other is

$$180 - 2 \times 40 = 100^\circ.$$

$$\begin{aligned} PQ &= 1.5 \times AB \\ &= 15 \text{ cm.} \end{aligned}$$

Finally,

$$\begin{aligned} \text{area} &= \tfrac{1}{2}ab\sin C \\ &= \tfrac{1}{2} \times 15 \times 15 \times \sin 100^\circ \\ &= 110.790\,872\,2 \text{ (FCD)} \\ &= \underline{\underline{111 \text{ cm}^2 \text{ (3 sf)}}}. \end{aligned}$$

4. The graph of

$$y = ab^{-x}$$

passes through the points $(0, 0.7)$ and $(3, 0.087\,5)$

(a) Find the value of a and the value of b. (4)

Solution

Well,

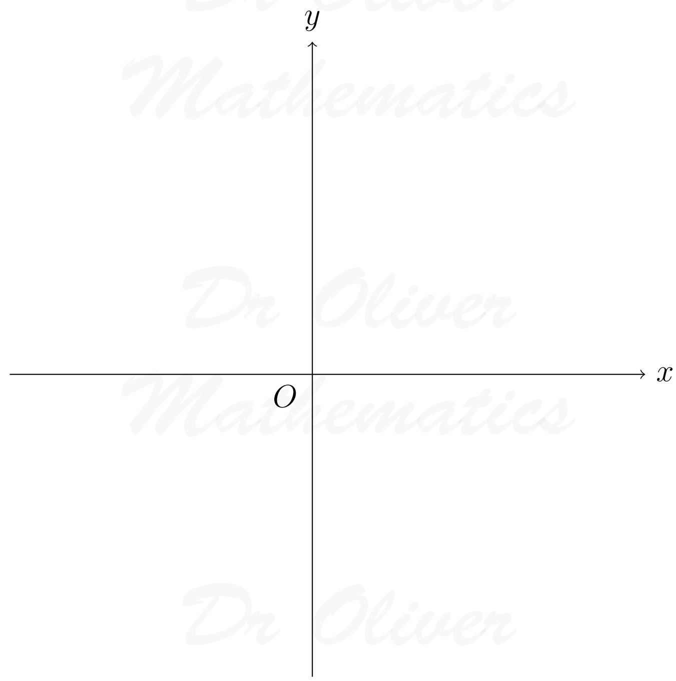
$$\begin{aligned} x = 0, y = 0.7 &\Rightarrow 0.7 = ab^0 \\ &\Rightarrow \underline{\underline{a = 0.7}} \end{aligned}$$

and

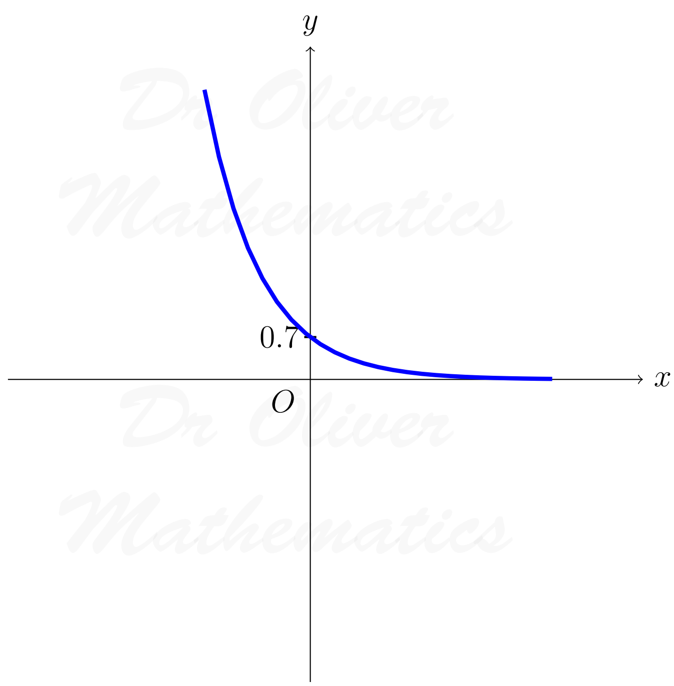
$$\begin{aligned} x = 3, y = 0.087\,5 &\Rightarrow 0.087\,5 = 0.7 \times b^{-3} \\ &\Rightarrow b^{-3} = \tfrac{1}{8} \\ &\Rightarrow b^3 = 8 \\ &\Rightarrow \underline{\underline{b = 2}}. \end{aligned}$$

(b) Hence, on the grid below, sketch the graph of (2)

$$y = ab^{-x}.$$

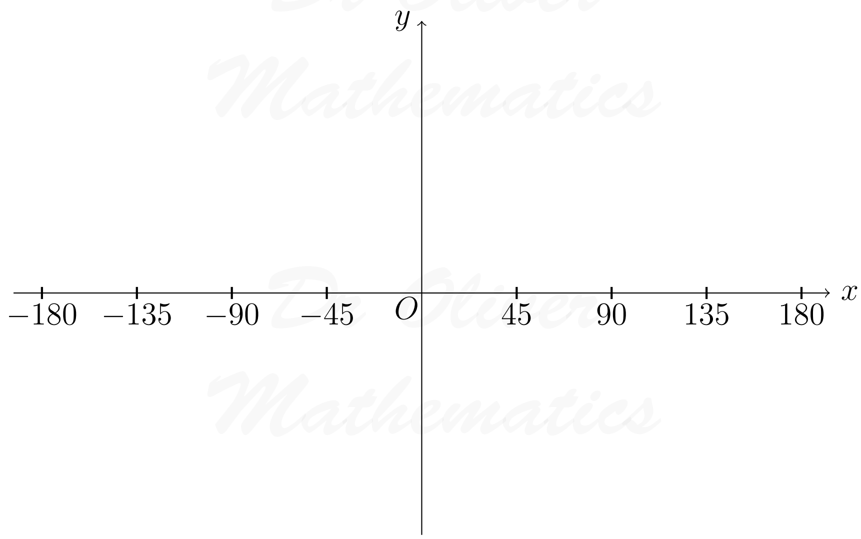


Solution

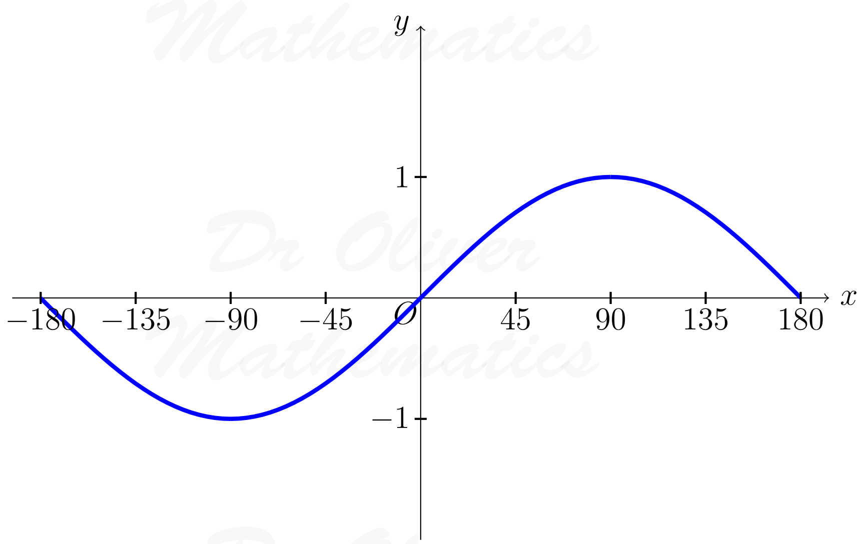


5. (a) Sketch the graph of (2)

$$y = \sin x^\circ \text{ for } -180 \leqslant x \leqslant 180.$$



Solution



(b) Solve (2)

$$2w^2 + 3w + 1 = 0.$$

Solution

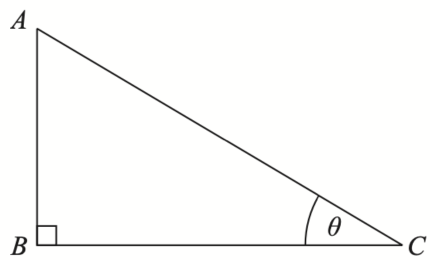
Well,

$$\left.\begin{array}{ll} \text{add to:} & +3 \\ \text{multiply to:} & (+2) \times (+1) = +2 \end{array}\right\} +1,\ +2$$

and so

$$\begin{aligned}2w^2+3w+1=0 &\Rightarrow 2w^2+2w+w+1=0\\ &\Rightarrow 2w(w+1)+1(w+1)=0\\ &\Rightarrow (2w+1)(w+1)=0\\ &\Rightarrow 2w+1=0 \text{ or } w+1=0\\ &\Rightarrow \underline{\underline{w=-\tfrac{1}{2} \text{ or } w=-1}}.\end{aligned}$$

ABC is a right-angled triangle.



(c) Use Pythagoras' theorem to show that (3)

$$\sin^2\theta^\circ+\cos^2\theta^\circ=1.$$

Solution

Well,

$$\begin{aligned}\sin^2 x+\cos^2 x &= (\sin x)^2+(\cos x)^2\\ &= \left(\frac{AB}{AC}\right)^2+\left(\frac{BC}{AC}\right)^2\\ &= \frac{AB^2}{AC^2}+\frac{BC^2}{AC^2}\\ &= \frac{AB^2+BC^2}{AC^2}\\ &= \frac{AC^2}{AC^2}\\ &= \underline{\underline{1}},\end{aligned}$$

as required.

Given that

$$\sin^2 x^\circ + \cos^2 x^\circ = 1$$

is true for all values of x,

(d) solve (4)

$$3 - 2\cos^2 x^\circ + 3\sin x^\circ = 0 \text{ for } -180 \leqslant x \leqslant 180.$$

Solution

Now,

$$\begin{aligned} & 3 - 2\cos^2 x^\circ + 3\sin x^\circ = 0 \\ \Rightarrow\ & 3 - 2(1 - \sin^2 x^\circ) + 3\sin x^\circ = 0 \\ \Rightarrow\ & 3 - 2 + 2\sin^2 x^\circ + 3\sin x^\circ = 0 \\ \Rightarrow\ & 2\sin^2 x^\circ + 3\sin x^\circ + 1 = 0 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & +3 \\ \text{multiply to:} & (+2) \times (+1) = +2 \end{array}\right\} +1,\ +2$$

e.g.,

$$\begin{aligned} \Rightarrow\ & 2\sin^2 x^\circ + 2\sin x^\circ + \sin x^\circ + 1 = 0 \\ \Rightarrow\ & 2\sin x^\circ(\sin x^\circ + 1) + 1(\sin x^\circ + 1) = 0 \\ \Rightarrow\ & (2\sin x^\circ + 1)(\sin x^\circ + 1) = 0 \\ \Rightarrow\ & 2\sin x^\circ + 1 = 0 \text{ or } \sin x^\circ + 1 = 0 \\ \Rightarrow\ & \sin x^\circ = -\tfrac{1}{2} \text{ or } \sin x^\circ = -1. \end{aligned}$$

$\underline{\sin x^\circ = -\tfrac{1}{2}}$:

$$\sin x^\circ = -\tfrac{1}{2} \Rightarrow \underline{\underline{x = -150 \text{ or } x = -30}}.$$

$\underline{\sin x^\circ = -1}$:

$$\sin x^\circ = -1 \Rightarrow \underline{\underline{x = -90}}.$$

6. (a) Use the factor theorem to show that $(x - 2)$ is a factor of (2)

$$x^3 - x^2 - 14x + 24.$$

Solution

Let

$$f(x) = x^3 - x^2 - 14x + 24.$$

Now,

$$\begin{aligned} f(2) &= 2^3 - 2^2 - 14(2) + 24 \\ &= 8 - 4 - 28 + 24 \\ &= 0. \end{aligned}$$

As there is no remainder, $\underline{\underline{(x-2)}}$ is a factor of the cubic.

Hence or otherwise, given that $x = 2y$,

(b) write the expression (4)

$$8y^3 - 4y^2 - 28y + 24$$

as a product of its linear factors.

Solution

Well,

$$\begin{aligned} 8y^3 - 4y^2 - 28y + 24 &= (2y)^3 - (2y)^2 - 14(2y) + 24 \\ &= x^3 - x^2 - 14x + 24. \end{aligned}$$

We use synthetic division:

2	1	−1	−14	24
	↓	1	2	−24
	1	1	−12	0

so

$$x^3 - x^2 - 14x + 24 = (x-2)(x^2 + x - 12)$$

$$\left.\begin{array}{ll} \text{add to:} & +1 \\ \text{multiply to:} & -12 \end{array}\right\} -3,\ +4$$

$$\begin{aligned} &= (x-2)(x-3)(x+4) \\ &= \underline{\underline{(2y-2)(2y-3)(2y+4)}}. \end{aligned}$$

7. Use the trapezium rule to find an estimate for the area of the region under the curve (4)

$$y = 2^x$$

and between $x = 1$, $x = 7$, and the x-axis.

Use 4 strips of equal width.
Give your answer correct to 3 significant figures.

Solution

Well,

$$\frac{7-1}{4} = 1.5$$

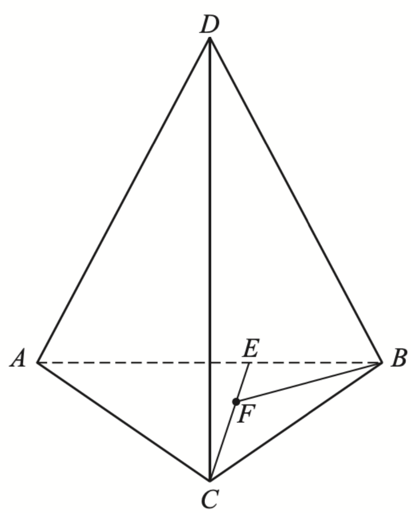
and

x	1	2.5	4	5.5	7
y	1	5.656...	16	45.254...	128

Now,

$$\begin{aligned}\text{area} &= \tfrac{1}{2} \times 1.5 \times [1 + (5.656\ldots + 16 + 45.254\ldots) + 1] \\ &= 197.117\,532\,4 \text{ (FCD)} \\ &= \underline{\underline{197 \text{ (3 sf)}}}.\end{aligned}$$

8. $ABCD$ is a triangular based pyramid. (6)



- E is a point on the line AB.
- F is a point on the line CE, such that $CF : FE = 3 : 2$.
- $BC = 7.2$ cm.
- $BF = 4.1$ cm.
- Angle $CBF = 49°$.
- Angle $CED = 109°$.
- Angle $CDE = 32°$.

Find the length of CD.
Give your answer correct to 3 significant figures.

Solution

Cosine rule:

$$\begin{aligned} & CF^2 = BC^2 + BF^2 - 2 \times BC \times BF \times \cos CBF \\ \Rightarrow\quad & CF^2 = 7.2^2 + 4.1^2 - 2 \times 7.2 \times 4.1 \times \cos 49^\circ \\ \Rightarrow\quad & CF^2 = 29.916\,274\,93 \text{ (FCD)} \\ \Rightarrow\quad & CF = 5.469\,577\,217 \text{ (FCD).} \end{aligned}$$

Now,

$$\begin{aligned} CE &= \frac{(3+2)}{3} \times CF \\ &= \tfrac{5}{3} \times 5.469\ldots \\ &= 9.115\,962\,028 \text{ (FCD).} \end{aligned}$$

Finally, the sine rule:

$$\begin{aligned} \frac{CD}{\sin CED} = \frac{CE}{\sin CDE} &\Rightarrow \frac{CD}{\sin 109^\circ} = \frac{9.115\ldots}{\sin 32^\circ} \\ &\Rightarrow CD = \frac{9.115\ldots \sin 109^\circ}{\sin 32^\circ} \\ &\Rightarrow CD = 16.265\,329\,48 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{CD = 16.3 \text{ cm (3 sf)}}}. \end{aligned}$$

9. Savio is buying base cupboards for a catering kitchen.

The cupboards come in two sizes, 600 mm wide and 900 mm wide.

Let x be the number of 600 mm cupboards and y be the number of 900 mm cupboards.

Two constraints are $x > 2$ and $0 < y \leqslant 9$.

(a) Explain in context what (2)

$$0 < y \leqslant 9$$

represents.

Solution

E.g., the number of 900 mm cupboards is greater than 0 and is less than 10.

A 600 mm cupboard costs £210.
A 900 mm cupboard costs £240.

Savio has a maximum budget of £3 600.

The total width of all the cupboards is 12 m or less.

(b) Use this information to show that (4)

$$7x + 8y \leqslant 120$$
$$2x + 3y \leqslant 40.$$

Solution

Now,

$$210 \times x + 240 \times y \leqslant 3\,600 \Rightarrow 210x + 240y \leqslant 3\,600$$

divide by 30:

$$\Rightarrow \underline{\underline{7x + 8y \leqslant 120}}.$$

Next,

$$12 \text{ m} \Rightarrow 1\,200 \text{ cm}$$
$$\Rightarrow 12\,000 \text{ mm}$$

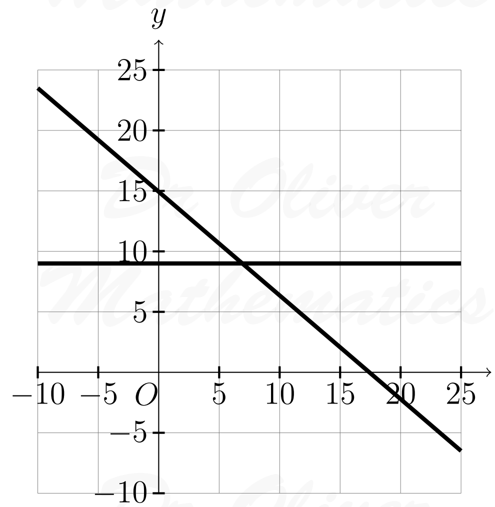
and

$$600 \times x + 900 \times y \leqslant 12\,000 \Rightarrow 600x + 900y \leqslant 12\,000$$

divide by 300:

$$\Rightarrow \underline{\underline{2x + 3y \leqslant 40}}.$$

(c) Draw a line on the grid and identify the feasible region. (1)
Label the feasible region **R**.



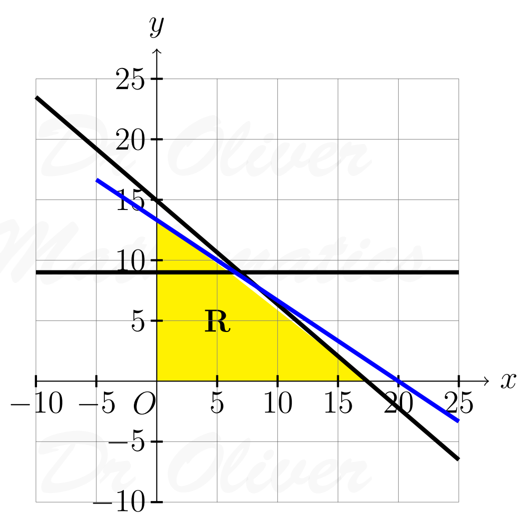
Solution

Well, the diagonal line represents

$$\begin{aligned} 7x + 8y \leqslant 120 &\Rightarrow 8y \leqslant -7x + 120 \\ &\Rightarrow y \leqslant -\tfrac{7}{8}x + 15. \end{aligned}$$

So

$$\begin{aligned} 2x + 3y \leqslant 40 &\Rightarrow 3y \leqslant -2x + 40 \\ &\Rightarrow y \leqslant -\tfrac{2}{3}x + \tfrac{40}{3}. \end{aligned}$$



Savio decides to buy 7 of the 600 mm cupboards and the maximum number of 900 mm cupboards possible.

(d) Work out the total amount of money Savio will spend buying these cupboards. (2)

Solution

Well,

$$\begin{aligned} x = 7 &\Rightarrow y \leqslant -\tfrac{7}{8}(7) + 15 \\ &\Rightarrow y \leqslant 8\tfrac{7}{8} \end{aligned}$$

and

$$\begin{aligned} x = 7 &\Rightarrow y \leqslant -\tfrac{2}{3}(7) + \tfrac{40}{3} \\ &\Rightarrow y \leqslant 8\tfrac{2}{3}; \end{aligned}$$

so, $y = 8$.

He will spend

$$7 \times 210 + 8 \times 240 = \underline{\underline{\pounds 3\,390}}.$$

10. A bag contains only red counters and yellow counters. (5)
There are more yellow counters than red counters.

A counter is taken at random from the bag, the colour noted, and then the counter is put back into the bag.
This process is repeated one more time.

The probability that exactly one of the two counters taken from the bag was red is 0.255.

Simon then takes one counter from the bag.

Find the probability that Simon takes a yellow counter from the bag.

Solution

Let x be the probability that one of the yellow is chosen. Then

$$\mathrm{P}(Y) = x \text{ and } \mathrm{P}(R) = 1 - x.$$

Now, the events are independent (why?) and so

$$\begin{aligned}
\text{P(exactly one of the two counters)} = 0.255 &\Rightarrow \mathrm{P}(R, Y) + \mathrm{P}(Y, R) = 0.255 \\
&\Rightarrow 2\,\mathrm{P}(R)\,\mathrm{P}(Y) = 0.255 \\
&\Rightarrow x(1-x) = \tfrac{51}{400} \\
&\Rightarrow 400x(1-x) = 51 \\
&\Rightarrow 400x - 400x^2 = 51 \\
&\Rightarrow 400x^2 - 400x + 51 = 0
\end{aligned}$$

e.g.,

$$\left.\begin{array}{ll} \text{add to:} & -400 \\ \text{multiply to:} & (+400) \times (+51) = +20\,400 \end{array}\right\} -340,\ -60$$

$$\begin{aligned}
&\Rightarrow 400x^2 - 340x - 60x + 51 = 0 \\
&\Rightarrow 20x(20x - 17) - 3(20x - 17) = 0 \\
&\Rightarrow (20x - 3)(20x - 17) = 0 \\
&\Rightarrow 20x - 3 = 0 \text{ or } 20x - 17 = 0 \\
&\Rightarrow x = 0.15 \text{ or } x = 0.85.
\end{aligned}$$

Hence, as there are more yellow counters than red counters, $\underline{\underline{x = 0.85}}$.