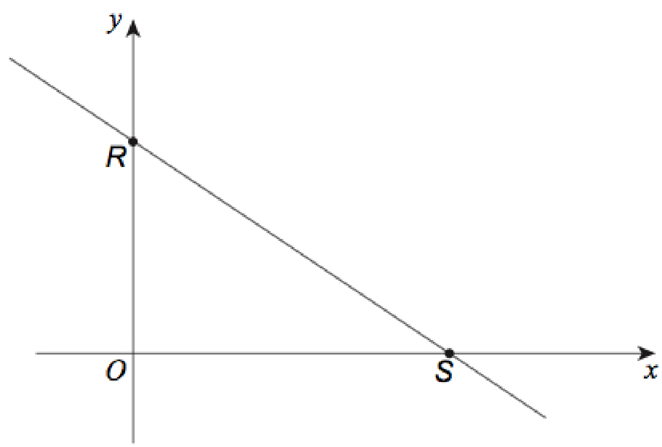
Dr Oliver Mathematics
AQA Further Maths Level 2
January 2013 Paper 2
2 hours

The total number of marks available is 105.
You must write down all the stages in your working.
You are permitted to use a scientific or graphical calculator in this paper.

1. A sketch of $2x + 3y = 12$ is shown.



(a) Work out the coordinates of R. (1)

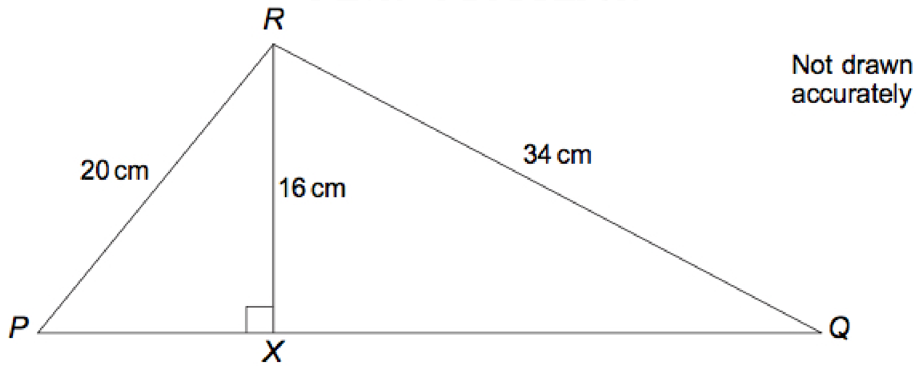
Solution
$\underline{\underline{R(0, 4)}}$.

(b) Work out the coordinates of the midpoint of RS. (2)

Solution
$S(6, 0)$ and

$$\left(\frac{0+6}{2}, \frac{4+0}{2}\right) = \underline{\underline{(3, 2)}}.$$

2. In triangle PQR, X is a point on PQ. (4)
RX is perpendicular to PQ.



Work out the ratio $PX : XQ$.
Give your answer in its simplest form.

Solution

$$\begin{aligned} PQ &= \sqrt{20^2 - 16^2} \\ &= \sqrt{400 - 256} \\ &= \sqrt{144} \\ &= 12 \end{aligned}$$

and

$$\begin{aligned} QR &= \sqrt{34^2 - 16^2} \\ &= \sqrt{1\,156 - 196} \\ &= \sqrt{900} \\ &= 30. \end{aligned}$$

Finally,

$$\begin{aligned} PX : XQ &= 12 : 30 \\ &= \underline{\underline{2 : 5}}. \end{aligned}$$

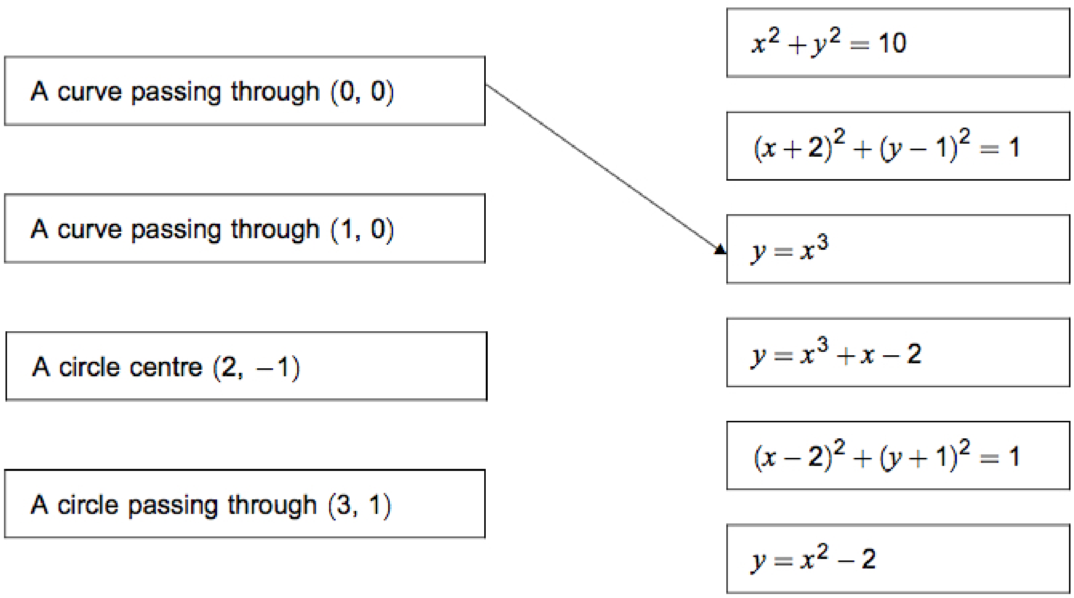
3. Solve (2)

$$5d - 3 > d + 17.$$

Solution

$$\begin{aligned} 5d - 3 > d + 17 &\Rightarrow 4d > 20 \\ &\Rightarrow \underline{\underline{d > 5}}. \end{aligned}$$

4. Match each statement with an equation. (3)
You will not use all of the equations.
One has been done for you.



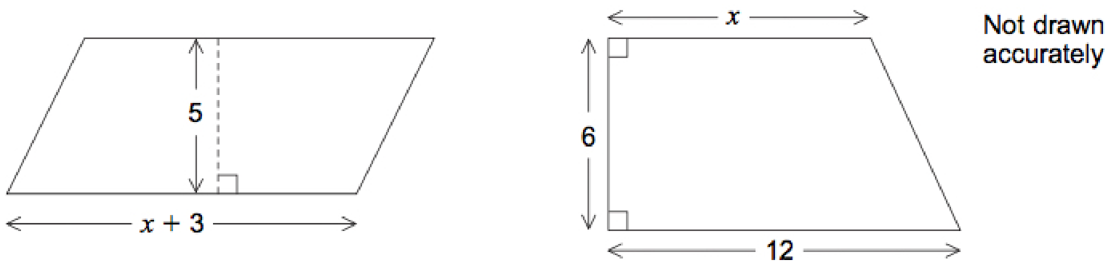
Solution

A curve passing through $(1, 0) \Leftrightarrow \underline{\underline{y = x^3 + x - 2}}$.

A circle centre $(2, -1) \Leftrightarrow \underline{\underline{(x - 2)^2 + (y + 1)^2 = 1}}$.

A curve passing through $(3, 0) \Leftrightarrow \underline{\underline{x^2 + y^2 = 10}}$.

5. A parallelogram and a trapezium are shown. (4)
All lengths are in centimetres.



The area of the parallelogram is equal to the area of the trapezium.
Work out the value of x.

Solution

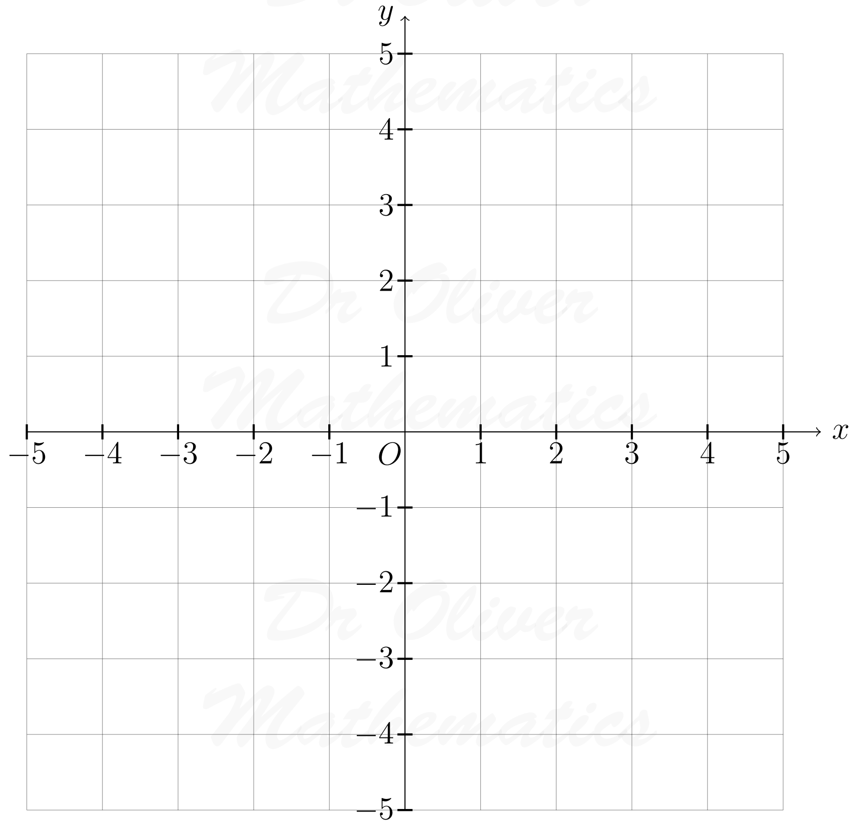
Since the area of the parallelogram is equal to the area of the trapezium,

$$\begin{aligned} 5(x+3) = \tfrac{1}{2} \times 6 \times (x+12) &\Rightarrow 5(x+3) = 3(x+12) \\ &\Rightarrow 5x + 15 = 3x + 36 \\ &\Rightarrow 2x = 21 \\ &\Rightarrow \underline{\underline{x = 10\tfrac{1}{2} \text{ cm}}}. \end{aligned}$$

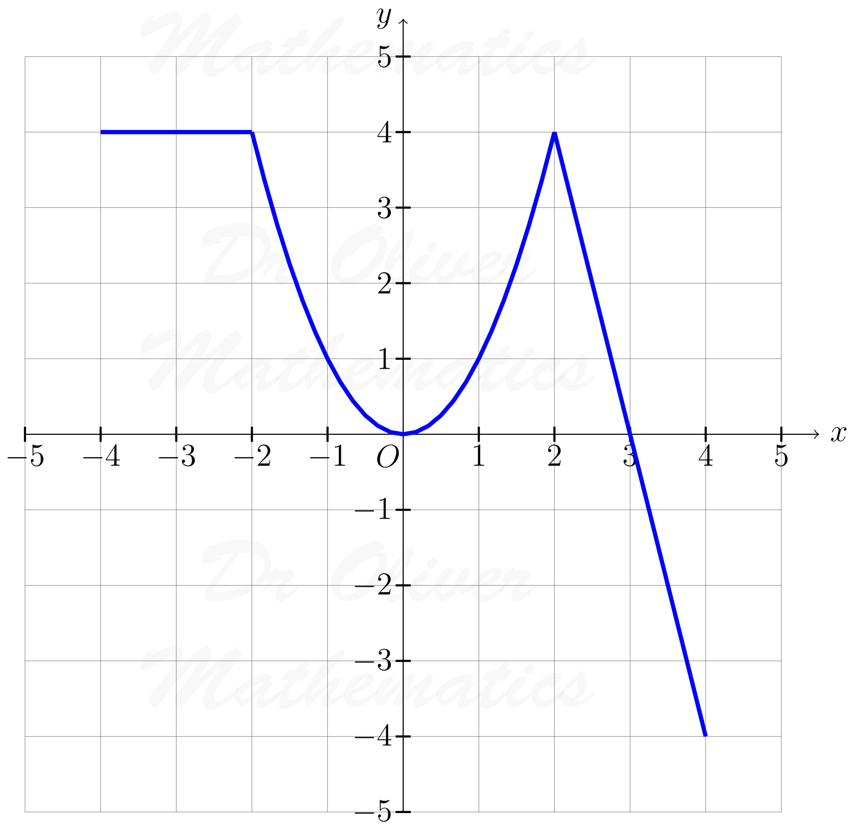
6. A function $f(x)$ is defined as

$$f(x) = \begin{cases} 4, & x < -2 \\ x^2, & -2 \leqslant x \leqslant 2 \\ 12 - 4x, & x > 2. \end{cases}$$

(a) Draw the graph of $y = f(x)$ for $-4 \leqslant x \leqslant 4$. (3)



Solution



(b) Use your graph to write down **how many** solutions there are to $\mathrm{f}(x) = 3$. (1)

Solution

$\underline{\underline{3}}$.

(c) Solve $\mathrm{f}(x) = -10$. (2)

Solution

Clearly, $x > 4$ (why?) and

$$\begin{aligned}\mathrm{f}(x) = -10 &\Rightarrow 12 - 4x = -10 \\ &\Rightarrow -4x = -22 \\ &\Rightarrow \underline{\underline{x = 5\tfrac{1}{2}}}.\end{aligned}$$

7. Here are the first four terms of a sequence: (2)

$$4a \qquad 9a \qquad 14a \qquad 19a$$

The nth term of the sequence is

$$\frac{10n-2}{3}.$$

Work out the value of a.

Solution

When $n = 1$,

$$\begin{aligned}\frac{10\times 1-2}{3} = 4a &\Rightarrow \tfrac{8}{3} = 4a \\ &\Rightarrow a = \tfrac{3}{2}\end{aligned}$$

which makes

$$\frac{1}{a} = \underline{\underline{\tfrac{2}{3}}}.$$

8. (a) Factorise fully (3)

$$5m^2 - 20p^2.$$

Solution

$$\begin{aligned}5m^2 - 20p^2 &= 5(m^2 - 4p^2) \\ &= 5[(m)^2 - (2p)^2] \\ &= \underline{\underline{5(m-2p)(m+2p)}}\end{aligned}$$

by the difference of two squares.

You are given that

$$p = 15 \text{ and } 5m^2 - 20p^2 = 0.$$

(b) Using your answer to part (a), or otherwise, work out the values of m. (2)

Solution

Well,

$$p = 15 \Rightarrow 20p^2 = 4\,500.$$

Finally,

$$\begin{aligned}5m^2 - 20p^2 = 0 &\Rightarrow 5m^2 = 4\,500\\ &\Rightarrow m^2 = 900\\ &\Rightarrow \underline{\underline{m = \pm 30}}.\end{aligned}$$

9. (a) Expand (1)

$$(x+m)(x+n).$$

Solution

$\times$	x	$+m$
x	x^2	$+mx$
$+n$	$+nx$	$+mn$

$$(x+m)(x+n) = \underline{\underline{x^2 + (m+n)x + mn}}.$$

$$x^2 + qx + r \equiv (x+m)(x+n).$$

(b) Use your answer to part (a) to write q and r in terms of m and n. (2)

Solution

$\underline{\underline{q = m + n}}$ and $\underline{\underline{r = mn}}$.

r is an odd integer.

(c) Use your answer to part (b) to explain why q is an even integer. (2)

Solution

r is an odd integer which must mean m and n are odd integers (why?). Now,

$$\text{odd integer} + \text{another odd integer} = \text{even integer}$$

and q is an $\underline{\underline{\text{even integer}}}$.

10.
$$S = \frac{a}{1-r}.$$

(a) Show that (3)

$$r = \frac{S-a}{S}.$$

Solution

$$\begin{aligned} S = \frac{a}{1-r} &\Rightarrow S(1-r) = a \\ &\Rightarrow S - rS = a \\ &\Rightarrow S - a = rS \\ &\Rightarrow \underline{\underline{r = \frac{S-a}{S}}}, \end{aligned}$$

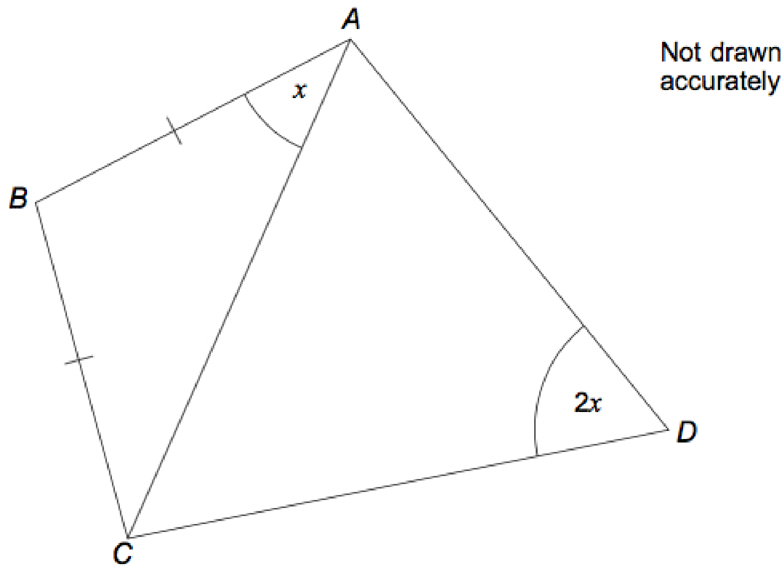
as required.

(b) Work out the value of r when $S = 10a$. (2)

Solution

$$\begin{aligned} S = 10a \Rightarrow r &= \frac{10a - a}{10a} \\ \Rightarrow r &= \frac{9a}{10a} \\ \Rightarrow \underline{\underline{r = \tfrac{9}{10}}}. \end{aligned}$$

11. In the diagram, $AB = BC$. (3)



Prove that $ABCD$ is a cyclic quadrilateral.
Give reasons for any statements you make.

Solution
$\angle BCA = x$ (base angles)
$\angle ABC = 180 - x - x = 180 - 2x$ (completing the triangle)
$\angle ABC + \angle ADC = 180^\circ$ (opposite angles of cyclic quadrilateral)
Which means $\angle BAD + \angle BCD = 180^\circ$ (opposite angles of cyclic quadrilateral)
Hence, $ABCD$ is a cyclic quadrilateral.

12.

$$f(x) = \sin x,\ 180^\circ \leqslant x \leqslant 360^\circ,$$
$$g(x) = \cos x,\ 0^\circ \leqslant x \leqslant \theta^\circ.$$

(a) Calculate the value of $f(210^\circ)$. (1)

Solution
$f(210^\circ) = -\frac{1}{2}$.

(b) Complete this inequality for the range of $f(x)$. (2)

Solution

$\underline{\underline{-1 \leqslant \mathrm{f}(x) \leqslant 0}}$.

You are given that $0 \leqslant \mathrm{g}(x) \leqslant 1$.

(c) Work out the value of θ. (1)

Solution

$\underline{\underline{\theta = 90}}$.

13. (a) Show that (2)

$$\frac{4}{x} + \frac{2}{x-1}$$

simplifies to

$$\frac{6x-4}{x(x-1)}.$$

Solution

$$\begin{aligned}\frac{4}{x} + \frac{2}{x-1} &= \frac{4(x-1)}{x(x-1)} + \frac{2x}{x(x-1)}\\ &= \frac{4(x-1)+2x}{x(x-1)}\\ &= \frac{(4x-4)+2x}{x(x-1)}\\ &= \underline{\underline{\frac{6x-4}{x(x-1)}}},\end{aligned}$$

as required.

(b) Hence, or otherwise, solve (5)

$$\frac{4}{x} + \frac{2}{x-1} = 3.$$

Give your solutions to 3 significant figures.

Solution

$$
\begin{aligned}
\frac{4}{x} + \frac{2}{x-1} = 3 \Rightarrow \frac{6x-4}{x(x-1)} &= 3 \\
\Rightarrow 6x - 4 &= 3x(x-1) \\
\Rightarrow 6x - 4 &= 3x^2 - 3x \\
\Rightarrow 3x^2 - 9x + 4 &= 0
\end{aligned}
$$

$a = 3$, $b = -9$, and $c = 4$

$$
\begin{aligned}
\Rightarrow x &= \frac{9 \pm \sqrt{(-9)^2 - 4 \times 3 \times 4}}{2 \times 3} \\
\Rightarrow x &= \frac{9 \pm \sqrt{33}}{6} \\
\Rightarrow x &= 0.542\,572\,892\,2,\ 2.457\,427\,108 \text{ (FCD)} \\
\Rightarrow x &= 0.543,\ 2.46 \text{ (3 sf)}.
\end{aligned}
$$

14. The value of x is 50% **more** than the value of t. (4)
The value of y is 10% **less** than the value of w.
$x = y$.
Work out $\frac{t}{w}$.
Give your answer as a decimal.

Solution

So,

$$x = 1.5t \Rightarrow t = \tfrac{2}{3}x$$

and

$$y = 0.9w \Rightarrow w = \tfrac{10}{9}y = \tfrac{10}{9}x.$$

Finally,

$$
\begin{aligned}
\frac{t}{w} &= \frac{\frac{2}{3}x}{\frac{10}{9}x} \\
&= \tfrac{3}{5}.
\end{aligned}
$$

15. Describe fully the **single** transformation represented by the matrix (3)

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Solution

It is a rotation, through 90° anticlockwise, about the origin.

16. (5)

$$y = (x^3 - 1)^2 + (\sqrt{x})^8.$$

Work out $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

$\times$	x^3	-1
x^3	x^6	$-x^3$
-1	$-x^3$	$+1$

$$\begin{aligned} y = (x^3 - 1)^2 + (\sqrt{x})^8 &\Rightarrow y = (x^3 - 1)^2 + (x^{\frac{1}{2}})^8 \\ &\Rightarrow y = (x^6 - 2x^3 + 1) + x^4 \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 6x^5 - 6x^2 + 4x^3. \end{aligned}$$

17. (2)

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

represents a reflection in the y-axis and

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

represents a reflection in the $y = x$.

Work out the matrix that represents a reflection in the y-axis followed by a reflection in the line $y = x$.

Solution

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}}.$$

18. Express (3)

$$1 - \tan\theta \sin\theta \cos\theta$$

in terms of $\cos\theta$.

Solution

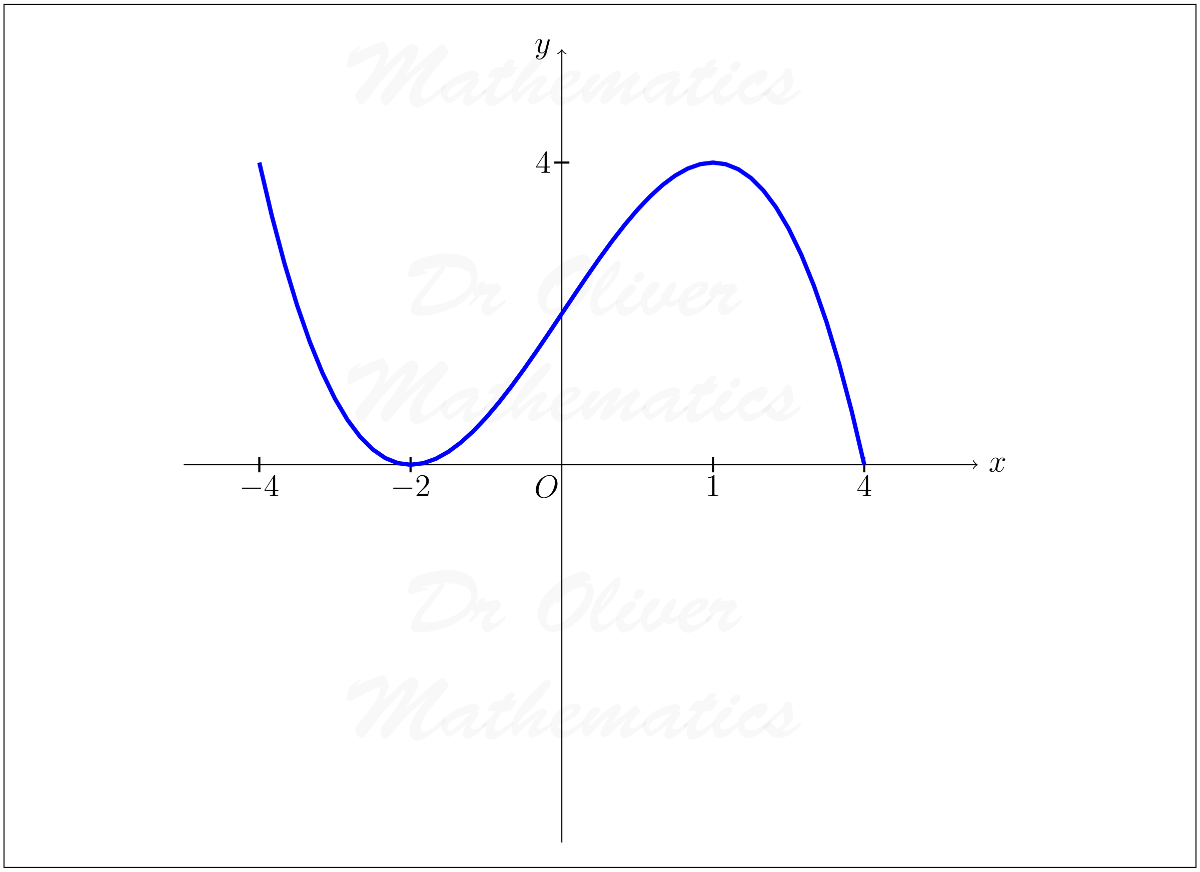
$$\begin{aligned} 1 - \tan\theta \sin\theta \cos\theta &\equiv 1 - \sin^2\theta \\ &\equiv \underline{\underline{\cos^2\theta}}. \end{aligned}$$

19. A cubic function $\mathrm{f}(x)$ has domain $-4 \leqslant x \leqslant 4$. (4)
The curve $y = \mathrm{f}(x)$

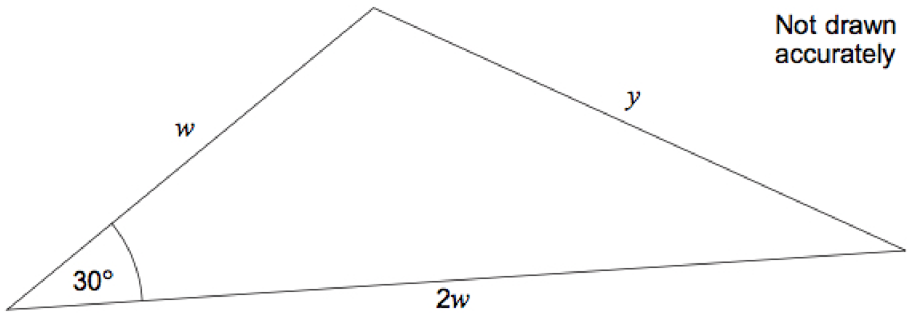
- has a minimum point at $(-2, 0)$,
- has a maximum point at $(1, 4)$, and
- meets the x-axis at $(4, 0)$.

Sketch the graph of $y = \mathrm{f}(x)$.
Label any points where the graph meets the x-axis.

Solution



20. The area of this triangle is 18 cm^2. (5)



Work out y.

Solution

We use area under a triangle and then the cosine rule.

$$\begin{aligned}\tfrac{1}{2}(w)(2w)\sin 30^\circ = 18 &\Rightarrow \tfrac{1}{2}w^2 = 18\\ &\Rightarrow w^2 = 36\\ &\Rightarrow w = 6\end{aligned}$$

and

$$\begin{aligned}y &= \sqrt{6^2 + 12^2 - 2(6)(12)\cos 30^\circ}\\ &= \sqrt{180 - 72\sqrt{3}}\\ &= 7.435\,882\,05 \text{ (FCD)}\\ &= \underline{\underline{7.44 \text{ (3 sf)}}}.\end{aligned}$$

21. Work out the equation of the normal to the curve (5)

$$y = x^2 + 4x + 5$$

at the point where $x = -3$

Solution

$$y = x^2 + 4x + 5 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2x + 4.$$

Now,

$$\begin{aligned}x = -3 &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -2\\ &\Rightarrow m_{\text{normal}} = \tfrac{1}{2}.\end{aligned}$$

Next,

$$x = -3 \Rightarrow y = (-3)^2 + 4(-3) + 5 = 2$$

and the equation of the normal to the curve is

$$\begin{aligned}y - 2 = \tfrac{1}{2}(x + 3) &\Rightarrow y - 2 = \tfrac{1}{2}x + \tfrac{3}{2}\\ &\Rightarrow \underline{\underline{y = \tfrac{1}{2}x + \tfrac{7}{2}}}.\end{aligned}$$

22. (5)

$$\mathrm{f}(x) = x^3 + ax^2 + bx + 24 \text{ for all values of } x.$$

Two of the factors of $f(x)$ are $(x-2)$ and $(x+3)$.
Work out the values of a and b.

Solution

$$\frac{24}{(-2)(3)} = -4$$

and we conclude that the last factor is $(x-4)$.

$\times$	x	-2
x	x^2	$-2x$
$+3$	$+3x$	-6

$$(x-2)(x+3) = x^2 + x - 6.$$

$\times$	x^2	$+x$	-6
x	x^3	$+x^2$	$-6x$
-4	$-4x^2$	$-4x$	$+24$

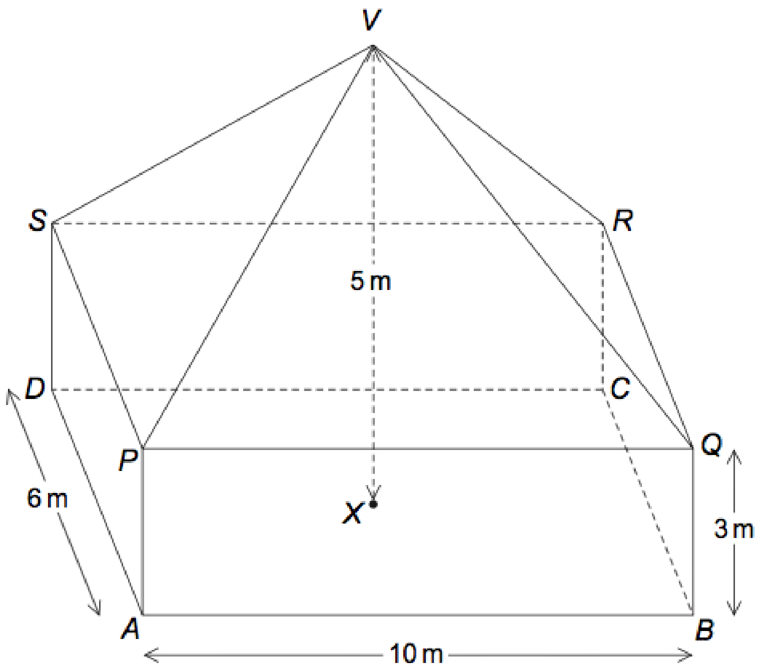
Finally,

$$a = 1 - 4 = \underline{\underline{-3}}$$

and

$$b = -4 - 6 = \underline{\underline{-10}}.$$

23. The diagram shows a cuboid $ABCDPQRS$ and a pyramid $PQRSV$.
V is directly above the centre, X, of $ABCD$.



The total height, VX, is 5 metres.

(a) Work out the angle between the line VA and the plane $ABCD$. (4)

Solution

Let Y be the mid-point of AB. Then

$$\begin{aligned} AX^2 = AY^2 + XY^2 &\Rightarrow AX^2 = 5^2 + 3^2 \\ &\Rightarrow AX^2 = 25 + 9 \\ &\Rightarrow AX^2 = 34 \\ &\Rightarrow AX = \sqrt{34} \end{aligned}$$

and

$$\begin{aligned} \tan(\text{angle}) = \frac{5}{\sqrt{34}} &\Rightarrow \text{angle} = 40.612\,855\,18 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{\text{angle} = 40.6^\circ \text{ (3 sf)}}}. \end{aligned}$$

(b) Work out the angle between the planes VQR and $PQRS$. (2)

Solution

$$\tan(\text{angle}) = \frac{5-3}{5} \Rightarrow \tan(\text{angle}) = \tfrac{2}{5}$$
$$\Rightarrow \text{angle} = 21.801\,409\,49 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{\text{angle} = 21.8^\circ \text{ (3 sf)}}}.$$

24. Solve (4)

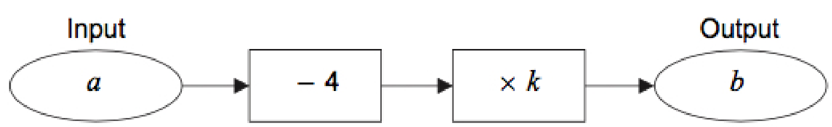
$$3\cos^2 y - 1 = 0$$

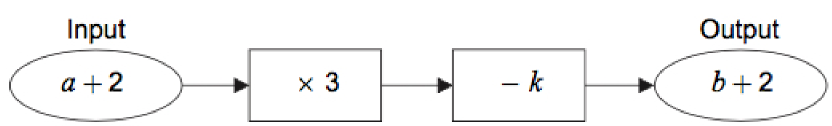
for $0^\circ \leqslant y \leqslant 180^\circ$.

Solution

$$3\cos^2 y - 1 = 0 \Rightarrow 3\cos^2 y = 1$$
$$\Rightarrow \cos^2 y = \tfrac{1}{3}$$
$$\Rightarrow \cos y = \pm\tfrac{\sqrt{3}}{3}.$$

25. Here are two number machines. (6)





Work out a in terms of k.

Solution

$$k(a-4) = b$$
$$3(a+2) - k = b + 2$$

Now,

$$k(a-4) = b \Rightarrow ak - 4k = b \quad (1)$$

and

$$\begin{aligned} 3(a+2) - k = b + 2 &\Rightarrow (3a+6) - k = b + 2 \\ &\Rightarrow 3a + 4 - k = b \quad (2). \end{aligned}$$

Do (1) = (2):

$$\begin{aligned} ak - 4k = 3a + 4 - k &\Rightarrow ak - 3k = 3a + 4 \\ &\Rightarrow ak - 3a = 3k + 4 \\ &\Rightarrow a(k-3) = 3k + 4 \\ &\Rightarrow \underline{\underline{a = \frac{3k+4}{k-3}}}. \end{aligned}$$