Dr Oliver Mathematics
If And Only If

In this note, we will examine "if and only if".

What it means: **either** both statements are true **or** both are false. The result is that the truth of either one of the connected statements requires the truth of the other. We can denote it as follows:

(i) $P \Leftrightarrow Q$,

(ii) P is necessary and sufficient for Q,

(iii) P is equivalent to Q.

Essentially, the term "if and only if" is really a code word for equivalence. To prove a theorem of this form, you must prove that A and B are equivalent; that is, not only is B true whenever A is true, but A is true whenever B is true.

Example 1
Let $a \in \mathbb{Z}$. Show that a is odd if and only if $3a + 8$ is odd.

Solution
<u>a odd $\Rightarrow 3a + 8$ odd</u>:
Now, a is odd so $a = 2n + 1$ for some integer $n \in \mathbb{Z}$. Next,

$$\begin{aligned} 3a + 8 &= 3(2n + 1) + 8 \\ &= 6n + 11 \\ &= 2(3n + 5) + 1, \end{aligned}$$

which means that $3a + 8$ is odd.

<u>$3a + 8$ odd $\Rightarrow a$ odd</u>:
Now, we want to use the contrapositive:

$$a \text{ even} \Rightarrow 3a + 8 \text{ even}.$$

So a is even so $a = 2m$ for some integer $m \in \mathbb{Z}$. Next,

$$\begin{aligned} 3a + 8 &= 3(2m) + 8 \\ &= 6m + 8 \\ &= 2(3m + 4), \end{aligned}$$

which means that $3a + 8$ is even.

Hence, a is odd if and only if $3a + 8$ is odd. ■

Example 2
Let $b \in \mathbb{Z}$. Show that $35|b$ if and only if $5|b$ and $7|b$.

Solution
$35|b \Rightarrow 5|b$ and $7|b$:
Now, $35|b$ and so $b = 35n$ for some integer $n \in \mathbb{Z}$. Next,

$$b = 5(7n) \Rightarrow 5|b$$

and

$$b = 7(5n) \Rightarrow 7|b.$$

Hence, $35|b \Rightarrow 5|b$ and $7|b$.

$5|b, 7|b \Rightarrow 35|b$:
Now, $b = 5k$ and $b = 7l$ for some constants $k, l \in \mathbb{Z}$. Next,

$$\begin{aligned} l &= \frac{b}{7} \\ &= \frac{5k}{7} \end{aligned}$$

and so we have

$$7|k \Rightarrow k = 7m$$

for some constant $m \in \mathbb{Z}$. Finally,

$$\begin{aligned} b &= 5k \\ &= 5(7m) \\ &= 35m \end{aligned}$$

and so $35|b$.

Hence, $35|b$ if and only if $5|b$ and $7|b$. ■

Here are some examples for you to try.

1. Prove that a whole number is divisible by 9 if and only if the sum of the digits is divisible by 9.

Solution

Let $a = a_n a_{n-1} \ldots a_2 a_1 a_0$ where all the digits a_0, a_1, $\ldots$, a_{n-1}, and a_n are between 0 and 9. So

$$a = 10^n a_n + 10^{n-1} a_{n-1} + \ldots + 10^2 a_2 + 10 a_1 + a_0.$$

<u>a is divisible by 9 $\Rightarrow$ the sum of the digits is divisible by 9</u>:
a is divisible by 9 which means

$$(10^n a_n + 10^{n-1} a_{n-1} + \ldots + 10^2 a_2 + 10a_1 + a_0) - (\underbrace{99\ldots9}_{n \text{ nines}} a_n + \underbrace{99\ldots9}_{(n-1) \text{ nines}} a_{n-1} + \ldots + 99a_2 + 9a_1)$$

is divisible by 9. And that is

$$a_n + a_{n-1} + \ldots + a_2 + a_1 + a_0$$

is divisible by 9.

<u>The sum of the digits is divisible by 9 $\Rightarrow$ a is divisible by 9</u>:

$$a_n + a_{n-1} + \ldots + a_2 + a_1 + a_0$$

is divisible by 9 and so

$$(a_n + a_{n-1} + \ldots + a_2 + a_1 + a_0) + (\underbrace{99\ldots9}_{n \text{ nines}} a_n + \underbrace{99\ldots9}_{(n-1) \text{ nines}} a_{n-1} + \ldots + 99a_2 + 9a_1)$$

is divisible by 9. And that is the number itself:

$$a = 10^n a_n + 10^{n-1} a_{n-1} + \ldots + 10^2 a_2 + 10a_1 + a_0.$$

Hence, a number is divisible by 9 if and only if the sum of the digits is divisible by 9.

2. Suppose $x, y \geqslant 0$. Then $x = y$ if and only if $\dfrac{x+y}{2} = \sqrt{xy}$.

Solution

<u>$x = y \Rightarrow \frac{x+y}{2} = \sqrt{xy}$</u>:
If $x = y \geqslant 0$, then

$$\frac{x+y}{2} = \frac{2x}{2} = x$$

and

$$\sqrt{xy} = \sqrt{x^2} = x$$

since $x \geqslant 0$.

$\frac{x+y}{2} = \sqrt{xy} \Rightarrow x = y$:

$$\begin{aligned}\frac{x+y}{2} = \sqrt{xy} &\Rightarrow x + y = 2\sqrt{xy}\\ &\Rightarrow (x+y)^2 = (2\sqrt{xy})^2\\ &\Rightarrow x^2 + 2xy + y^2 = 4xy\\ &\Rightarrow x^2 - 2xy + y^2 = 0\\ &\Rightarrow (x-y)^2 = 0\\ &\Rightarrow x - y = 0\\ &\Rightarrow x = y.\end{aligned}$$

Hence, $x = y$ if and only if $\dfrac{x+y}{2} = \sqrt{xy}$.

3. Let n be a positive integer. Then n is even if and only if n^2 is even.

Solution

n is even $\Rightarrow n^2$ is even:
Suppose that n is even. This means that $n = 2m$ for some integer m. Now,

$$n^2 = (2m)^2 = 4m^2.$$

Since $4m^2$ is divisible by 2, we conclude that $n^2 = 4m^2$ is even.

n^2 is even $\Rightarrow n$ is even:
(Contrapositive) Suppose that n is odd. So $n = 2l + 1$ for some integer l. Now,

$$\begin{aligned}n^2 &= (2l+1)^2\\ &= 4l^2 + 4l + 1\\ &= 2(2l^2 + 2l) + 1\end{aligned}$$

which is an odd integer.

Hence, n is even if and only if n^2 is even.

4. Let x and y be two natural numbers. Then xy is odd if and only if x is odd and y is odd.

Solution

xy is odd $\Rightarrow x$ is odd and y is odd:
(Contrapositive) Suppose either or both of the number are even. With loss of generality, we will assume that x is even. So $x = 2n$ for some number $n \in \mathbb{N}$. Now,

$$xy = (2n)y = 2(ny)$$

which is even.

x is odd and y is odd $\Rightarrow xy$ is odd:
Suppose $x = 2a + 1$ and $y = 2b + 1$ for some numbers $a, b \in \mathbb{N}$. Then,

$$\begin{aligned} xy &= (2a + 1)(2b + 1) \\ &= 4ab + 2a + 2b + 1 \\ &= 2(2ab + a + b + 1), \end{aligned}$$

which is odd.

Hence, xy is odd if and only if x is odd and y is odd.