Dr Oliver Mathematics
Mathematics: Higher
2015 Paper 1: Non-Calculator
1 hour 10 minutes

The total number of marks available is 60.
You must write down all the stages in your working.

1. Vectors (2)

$$\mathbf{u} = 8\mathbf{i}+2\mathbf{j}-\mathbf{k} \text{ and } \mathbf{v} = -3\mathbf{i}+t\mathbf{j}-6\mathbf{k}$$

are perpendicular.

Determine the value of t.

Solution

$$\begin{aligned} \mathbf{u}.\mathbf{v} = 0 &\Rightarrow (8\mathbf{i}+2\mathbf{j}-\mathbf{k}).(-3\mathbf{i}+t\mathbf{j}-6\mathbf{k}) = 0 \\ &\Rightarrow -24 + 2t + 6 = 0 \\ &\Rightarrow 2t = 18 \\ &\Rightarrow \underline{\underline{t = 9}}. \end{aligned}$$

2. Find the equation of the tangent to the curve (4)

$$y = 2x^3 + 3$$

at the point where $x = -2$.

Solution

$$y = 2x^3 + 3 \Rightarrow \frac{dy}{dx} = 6x^2$$

and

$$x = -2 \Rightarrow \frac{dy}{dx} = 6 \cdot (-2)^2 = 24.$$

Now,

$$x = -2 \Rightarrow y = 2(-2)^3 + 3 = -16 + 3 = -13$$

and, finally, the equation of the tangent is

$$y + 13 = 24(x + 2) \Rightarrow y + 13 = 24x + 48$$
$$\Rightarrow \underline{\underline{y = 24x + 35}}.$$

3. Show that $(x + 3)$ is a factor of (4)

$$x^3 - 3x^2 - 10x + 24$$

and hence factorise

$$x^3 - 3x^2 - 10x + 24$$

fully.

Solution

-3	1	-3	-10	24
	$\downarrow$	-3	18	-24
	1	-6	8	0

Hence, because there is no remainder, $(x + 3)$ is a $\underline{\underline{\text{factor}}}$ of $x^3 - 3x^2 - 10x + 24$ and

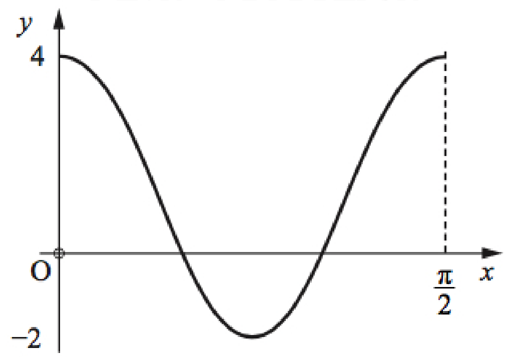
$$x^3 - 3x^2 - 10x + 24 = (x + 3)(x^2 - 6x + 8)$$

$$\left.\begin{array}{ll}\text{add to:} & -6 \\ \text{multiply to:} & +8\end{array}\right\} -2, -4$$

$$= \underline{\underline{(x + 3)(x - 2)(x - 4)}}.$$

4. The diagram shows part of the graph of the function (3)

$$y = p \cos qx + r.$$



Write down the values of p, q, and r.

Solution

$p = 3$, $q = 4$, and $r = 1$.

5. A function g is defined on $\mathbb{R}$, the set of real numbers, by

$$g(x) = 6 - 2x.$$

(a) Determine an expression for $g^{-1}(x)$. (2)

Solution

$$y = 6 - 2x \Rightarrow 2x = 6 - y$$
$$\Rightarrow x = \frac{6-y}{2}$$

and

$$g^{-1}(x) = \frac{6-x}{2}.$$

(b) Write down an expression for $g(g^{-1}(x))$. (1)

Solution

$g(g^{-1}(x)) = x$.

6. Evaluate (3)

$$\log_6 12 + \tfrac{1}{3}\log_6 27.$$

Solution

$$\begin{aligned}\log_6 12 + \tfrac{1}{3}\log_6 27 &= \log_6 12 + \log_6 (3^3)^{\frac{1}{3}} \\ &= \log_6 12 + \log_6 3 \\ &= \log_6 (12 \times 3) \\ &= \log_6 36 \\ &= \log_6 6^2 \\ &= 2\log_6 6 \\ &= \underline{\underline{2}}.\end{aligned}$$

7. A function f is defined on a suitable domain by (4)

$$\mathrm{f}(x) = \sqrt{x}\left(3x - \frac{2}{x\sqrt{x}}\right).$$

Find $\mathrm{f}'(4)$.

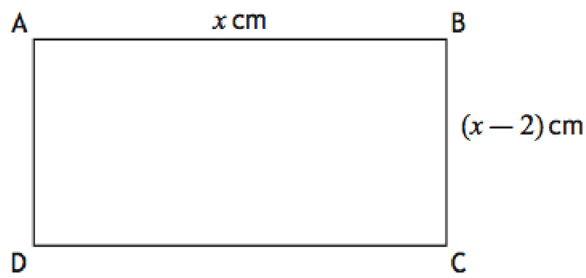
Solution

$$\begin{aligned}\mathrm{f}(x) = \sqrt{x}\left(3x - \frac{2}{x\sqrt{x}}\right) &\Rightarrow \mathrm{f}(x) = 3x^{\frac{3}{2}} - 2x^{-1} \\ &\Rightarrow \mathrm{f}'(x) = \tfrac{9}{2}x^{\frac{1}{2}} + 2x^{-2}\end{aligned}$$

and

$$\begin{aligned}\mathrm{f}'(4) &= \tfrac{9}{2}(4)^{\frac{1}{2}} + 2(4)^{-2} \\ &= \tfrac{9}{2}(2) + 2(\tfrac{1}{16}) \\ &= \underline{\underline{9\tfrac{1}{8}}}.\end{aligned}$$

8. $ABCD$ is a rectangle with sides of lengths x centimetres and $(x-2)$ centimetres, as shown. (4)



If the area of $ABCD$ is less than 15 cm^2, determine the range of possible values of x.

Solution

$$\begin{aligned}\text{Area of } ABCD < 15 &\Rightarrow x(x-2) < 15 \\ &\Rightarrow x^2 - 2x < 15 \\ &\Rightarrow x^2 - 2x - 15 < 0\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & -2 \\ \text{multiply to:} & -15\end{array}\right\} -5, +3$$

$$\begin{aligned}&\Rightarrow (x-5)(x+3) < 0 \\ &\Rightarrow -3 < x < 5;\end{aligned}$$

as $x > 2$ (why?), $\underline{\underline{2 < x < 5}}$.

9. A, B, and C are points such that AB is parallel to the line with equation (3)

$$y + \sqrt{3}x$$

and BC makes an angle of $150°$ with the positive direction of the x-axis.

Are the points A, B, and C collinear?

Solution

$$y + \sqrt{3}x \Rightarrow y = -\sqrt{3}x$$

and

$$\tan^{-1}(-\sqrt{3}) = 120°;$$

hence, the points A, B, and C $\underline{\underline{\text{are not collinear}}}$.

10. Given that

$$\tan 2x = \tfrac{3}{4},\ 0 < x < \tfrac{1}{4}\pi,$$

find the exact value of

(a) $\cos 2x$, (1)

Solution

$$\begin{aligned}\sec^2 2x = \tan^2 2x + 1 &\Rightarrow \sec^2 2x = (\tfrac{3}{4})^2 + 1\\ &\Rightarrow \sec^2 2x = \tfrac{9}{16} + 1\\ &\Rightarrow \sec^2 2x = \tfrac{25}{16}\\ &\Rightarrow \sec 2x = \tfrac{5}{4}\\ &\Rightarrow \underline{\underline{\cos 2x = \tfrac{4}{5}}}.\end{aligned}$$

(b) $\cos x$. (2)

Solution

$$\begin{aligned}\cos 2x = 2\cos^2 x - 1 &\Rightarrow \tfrac{4}{5} = 2\cos^2 x - 1\\ &\Rightarrow 2\cos^2 x = \tfrac{9}{5}\\ &\Rightarrow \cos^2 x = \tfrac{9}{10}\\ &\Rightarrow \underline{\underline{\cos x = \tfrac{3}{\sqrt{10}} \text{ or } \tfrac{3\sqrt{10}}{10}}}.\end{aligned}$$

11. $T(-2, -5)$ lies on the circumference of the circle with equation

$$(x + 8)^2 + (y + 2)^2 = 45.$$

(a) Find the equation of the tangent to the circle passing through T. (4)

Solution

$$\begin{aligned}2(x + 8) + 2(y + 2)\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 2(y + 2)\frac{\mathrm{d}y}{\mathrm{d}x} = -2(x + 8)\\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x + 8}{y + 2}.\end{aligned}$$

Now,

$$x = -2, y = -5 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{-2 + 8}{-5 + 2} = -\frac{6}{-3} = 2$$

and the equation of the tangent is

$$\begin{aligned} y + 5 = 2(x + 2) &\Rightarrow y + 5 = 2x + 4 \\ &\Rightarrow \underline{\underline{y = 2x - 1}}. \end{aligned}$$

This tangent is also a tangent to a parabola with equation

$$y = -2x^2 + px + 1 - p,$$

where $p > 3$.

(b) Determine the value of p. (6)

Solution

$$2x - 1 = -2x^2 + px + 1 - p \Rightarrow 2x^2 + (2 - p)x + (p - 2) = 0.$$

Now, '$b^2 - 4ac = 0$':

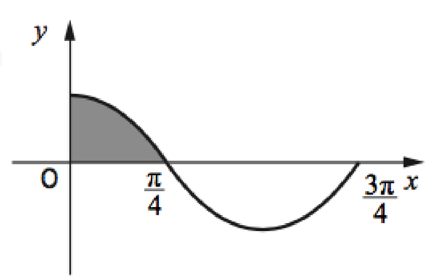
$$\begin{aligned} (2 - p)^2 - 4 \times 2 \times (p - 2) = 0 &\Rightarrow (2 - p)[(2 - p) - 4 \times 2 \times (-1)] = 0 \\ &\Rightarrow (2 - p)(2 - p + 8) = 0 \\ &\Rightarrow (2 - p)(10 - p) = 0 \\ &\Rightarrow p = 2 \text{ or } p = 10; \end{aligned}$$

hence, $p > 3$ so $\underline{\underline{p = 10}}$.

12. The diagram shows part of the graph of (2)

$$y = a \cos bx.$$

The shaded area is $\frac{1}{2}$ unit^2.



What is the value of

$$\int_0^{\frac{3}{4}\pi} (a \cos bx) \, \mathrm{d}x?$$

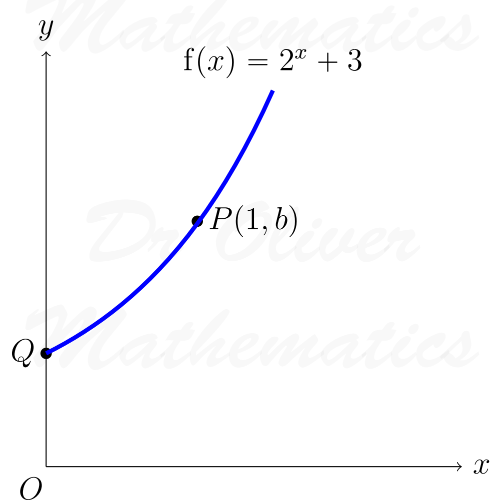
Solution

$$\int_0^{\frac{3}{4}\pi} (a \cos bx) \, \mathrm{d}x = \underline{\underline{-\tfrac{1}{2}}}.$$

13. The function

$$\mathrm{f}(x) = 2^x + 3$$

is defined on $\mathbb{R}$,the set of real numbers.
The graph with equation $y = \mathrm{f}(x)$ passes through the point $P(1, b)$ and cuts the y-axis at Q as shown in the diagram.



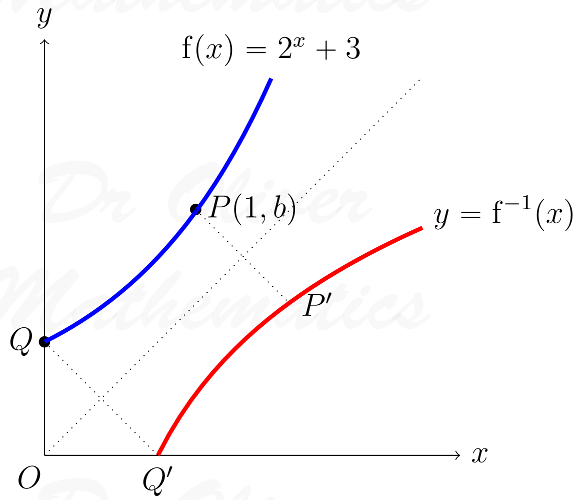
(a) What is the value of b? (1)

Solution

$$b = 1 \Rightarrow y = 2^1 + 3 = \underline{\underline{5}}.$$

(b) (i) Copy the above diagram. (1)
On the same diagram, sketch the graph with equation $y = \mathrm{f}^{-1}(x)$.

Solution



(ii) Write down the coordinates of the images of P and Q. (3)

Solution
The images of P and Q are $\underline{\underline{(5, 1)}}$ and $\underline{\underline{(4, 0)}}$ respectively.

$R(3, 11)$ also lies on the graph with equation $y = \mathrm{f}(x)$.

(c) Find the coordinates of the image of R on the graph with equation $y = 4 - \mathrm{f}(x + 1)$. (2)

Solution

$$x + 1 = 3 \Rightarrow x = 2$$

and

$$y = 4 - \mathrm{f}(x + 1) = 4 - \mathrm{f}(3) = 4 - 11 = -7;$$

hence, the image of R is $\underline{\underline{(-2, -7)}}$.

14. The circle with equation (2)

$$x^2 + y^2 - 12x - 10y + k = 0$$

meets the coordinate axes at exactly three points.

What is the value of k?

Solution

$$\begin{aligned} & x^2+y^2-12x-10y+k=0 \\ \Rightarrow\quad & x^2-12x+y^2-10y=-k \\ \Rightarrow\quad & (x^2-12x+36)+(y^2-10y+25)=-k+36+25 \\ \Rightarrow\quad & (x-6)^2+(y-5)^2=61-k. \end{aligned}$$

Now, $(0,0)$ is on the circle (why?):

$$\begin{aligned} (0-6)^2+(0-5)^2=61-k &\Rightarrow 36+25=61-k \\ &\Rightarrow \underline{\underline{k=0}}. \end{aligned}$$

15. The rate of change of the temperature, T°C of a mug of coffee is given by (6)

$$\frac{\mathrm{d}T}{\mathrm{d}t}=\tfrac{1}{25}t-k,\ 0\leqslant t\leqslant 50,$$

where

- t is the elapsed time, in minutes, after the coffee is poured into the mug,
- k is a constant,
- initially, the temperature of the coffee is 100°C, and
- 10 minutes later the temperature has fallen to 82°C.

Express T in terms of t.

Solution

$$\frac{\mathrm{d}T}{\mathrm{d}t}=\tfrac{1}{25}t-k\Rightarrow T=\tfrac{1}{50}t^2-kt+c,$$

for some constant c. Now,

$$t=0,T=100\Rightarrow 100=0-0+c\Rightarrow c=100$$

and

$$\begin{aligned} t=10,T=82 &\Rightarrow 82=2-10k+100 \\ &\Rightarrow 10k=20 \\ &\Rightarrow k=2. \end{aligned}$$

Hence,

$$\underline{\underline{T=\tfrac{1}{50}t^2-2t+100.}}$$