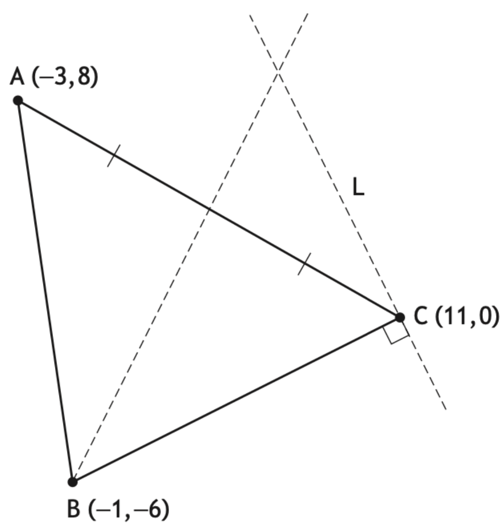
Dr Oliver Mathematics
Mathematics: Higher
2024 Paper 2: Calculator
1 hour 30 minutes

The total number of marks available is 65.
You must write down all the stages in your working.

1. Triangle ABC has vertices $A(3, 8)$, $B(1, 6)$, and $C(11, 0)$.



   (a) Find the equation of the median through B. (3)

Solution

Well, the midpoint of AC — D, say — is

$$\left(\frac{-3+11}{2}, \frac{8+0}{2}\right) = D(4, 4).$$

Now,

$$\begin{aligned} m_{BD} &= \frac{4-(-6)}{4-(-1)} \\ &= \frac{10}{5} \\ &= 2. \end{aligned}$$

Finally, the equation of the median through B is

$$y - 4 = 2(x - 4) \Rightarrow y - 4 = 2x - 8$$
$$\Rightarrow \underline{\underline{y = 2x - 4}}.$$

(b) Find the equation of L, the line perpendicular to BC passing through C. (3)

Solution
Now,

$$m_{BC} = \frac{0 - (-6)}{11 - (-1)} = \frac{6}{12} = \tfrac{1}{2}$$

and

$$m_{\text{normal}} = -2.$$

Finally, the equation of L is

$$y - 0 = -2(x - 11) \Rightarrow \underline{\underline{y = -2x + 22}}.$$

(c) Determine the coordinates of the point of intersection of the median through B and the line L. (2)

Solution
Well,

$$2x - 4 = -2x + 22 \Rightarrow 4x = 26$$
$$\Rightarrow x = 6\tfrac{1}{2}$$
$$\Rightarrow y = 9;$$

hence, $\underline{\underline{(6\tfrac{1}{2}, 9)}}$.

2. A curve has equation (5)

$$y = \frac{8}{x^3},\ x > 0.$$

Find the equation of the tangent to this curve at the point where $x = 2$.

Solution

Well,

$$x = 2 \Rightarrow y = 1$$

so the point is $(2, 1)$.

Now,

$$\begin{aligned} y = \frac{8}{x^3} &\Rightarrow y = 8x^{-3} \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -24x^{-4}. \end{aligned}$$

Next,

$$x = 2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\tfrac{3}{2}.$$

Finally, the equation of the tangent is

$$\begin{aligned} y - 1 = -\tfrac{3}{2}(x - 2) &\Rightarrow y - 1 = -\tfrac{3}{2}x + 3 \\ &\Rightarrow \underline{\underline{y = -\tfrac{3}{2}x + 4}}. \end{aligned}$$

3. The coordinates of points D, E, and F are given by $D(2, -3, 4)$, $E(1, 1, -2)$, and $F(3, 2, 1)$.

   (a) Express $\overrightarrow{ED}$ and $\overrightarrow{EF}$ in component form. (2)

Solution

$$\begin{aligned} \overrightarrow{ED} &= \begin{pmatrix} 2 - 1 \\ -3 - 1 \\ 4 - (-2) \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 1 \\ -4 \\ 6 \end{pmatrix}}} \end{aligned}$$

and

$$\overrightarrow{EF} = \begin{pmatrix} 3-1 \\ 2-1 \\ 1-(-2) \end{pmatrix} = \underline{\underline{\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}}}.$$

(b) (i) Calculate (1)

$$\overrightarrow{ED}.\overrightarrow{EF}.$$

Solution

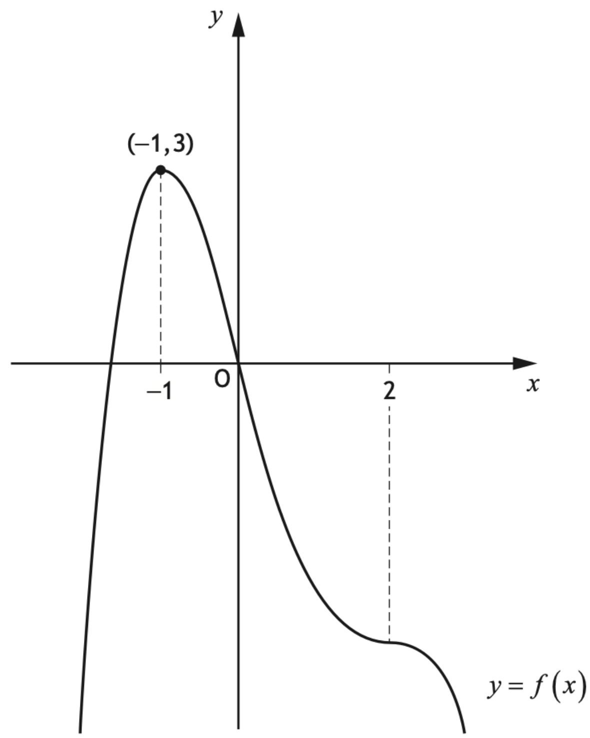
$$\begin{aligned}\overrightarrow{ED}.\overrightarrow{EF} &= 2 + (-4) + 18 \\ &= \underline{\underline{16}}.\end{aligned}$$

(ii) Hence, or otherwise, calculate the size of angle DEF. (4)

Solution

$$\begin{aligned} & \overrightarrow{ED}.\overrightarrow{EF} = |\overrightarrow{ED}||\overrightarrow{EF}| \cos DEF \\ \Rightarrow \quad & 16 = \sqrt{1^2 + (-4)^2 + 6^2}\sqrt{2^2 + 1^2 + 3^2} \cos DEF \\ \Rightarrow \quad & 16 = \sqrt{53}\sqrt{14} \cos DEF \\ \Rightarrow \quad & \cos DEF = \frac{16}{\sqrt{53}\sqrt{14}} \\ \Rightarrow \quad & \angle DEF = 54.028\,803\,06 \text{ (FCD)} \\ \Rightarrow \quad & \underline{\underline{\angle DEF = 54.0^\circ \text{ (3 sf)}}}. \end{aligned}$$

4. The diagram shows the graph of a quartic function $y = \mathrm{f}(x)$.



- A maximum turning point occurs at $(-1, 3)$.
- The graph of $y = \mathrm{f}(x)$ also has a point of inflection at $x = 2$..

(a) Determine the coordinates of the maximum turning point on the graph of (2)

$$y = \mathrm{f}(x - 4) + 2.$$

> **Solution**
> $\underline{\underline{(3, 5)}}$.

(b) Sketch the graph of (3)

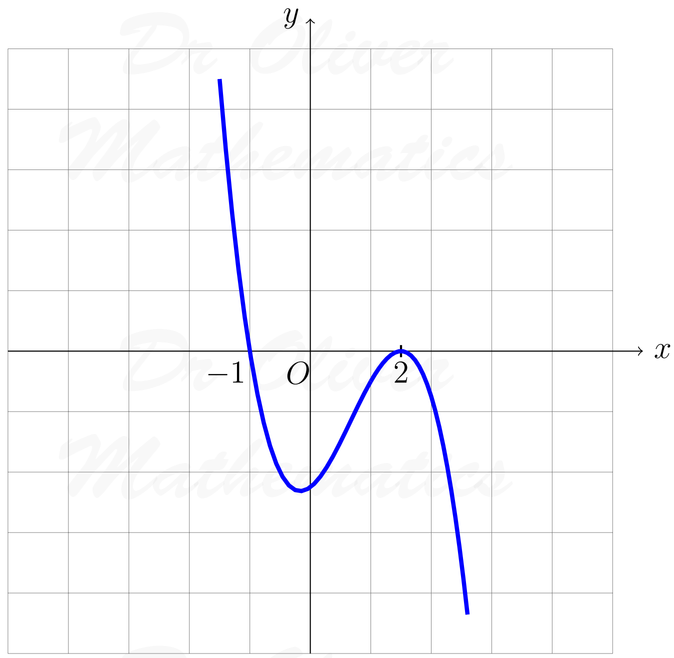
$$y = \mathrm{f}'(x).$$

> **Solution**
> Well, if the function $y = \mathrm{f}(x)$ with the coefficient of x^4 being positive, it would be like $\diagdown\diagup\diagdown\diagup$. But it doesn't! And that means the coefficient of x^4 is negative.

Now, the $y = \mathrm{f}(x)$ has a maximum $x = -1$ and a point of inflexion at $x = 2$, so

$$\mathrm{f}'(x) = -(x+1)(x-2)^2.$$

So the graph looks like this:



5. Evaluate (3)

$$\int_0^{\frac{1}{7}\pi} \sin 5x \,\mathrm{d}x.$$

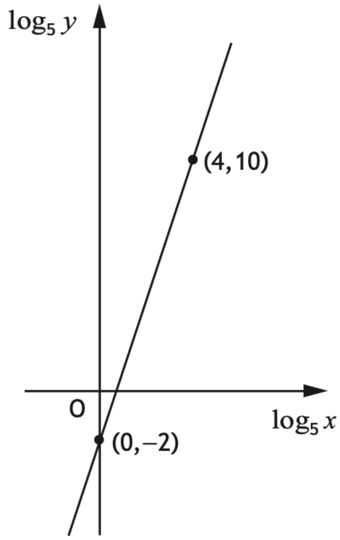
Solution

$$\begin{aligned}\int_0^{\frac{1}{7}\pi} \sin 5x \,\mathrm{d}x &= \left[-\tfrac{1}{5}\cos 5x\right]_{x=0}^{\frac{1}{7}\pi} \\ &= -\tfrac{1}{5}\cos\tfrac{5}{7}\pi - (-\tfrac{1}{5}) \\ &= \underline{\underline{\tfrac{1}{5}(1-\cos\tfrac{5}{7}\pi)}}.\end{aligned}$$

6. Two variables, x and y, are connected by the equation (5)

$$y = ax^b.$$

The graph of $\log_5 y$ against $\log_5 x$ is a straight line as shown.



Find the values of a and b.

Solution

Now,

$$\begin{aligned} m &= \frac{10-(-2)}{4-0} \\ &= \frac{12}{4} \\ &= 3 \end{aligned}$$

and the equation is

$$\begin{aligned} \log_5 y - 10 = 3(\log_5 x - 4) &\Rightarrow \log_5 y - 10 = 3\log_5 x - 12 \\ &\Rightarrow \log_5 y = 3\log_5 x - 2 \\ &\Rightarrow \log_5 y = \log_5 x^3 - 2 \\ &\Rightarrow \log_5 y - \log_5 x^3 = -2 \\ &\Rightarrow \log_5\left(\frac{y}{x^3}\right) = -2 \\ &\Rightarrow \frac{y}{x^3} = 5^{-2} \\ &\Rightarrow \frac{y}{x^3} = \tfrac{1}{25} \\ &\Rightarrow y = \tfrac{1}{25}x^3; \end{aligned}$$

hence, $\underline{\underline{a = \frac{1}{25}}}$ and $\underline{\underline{b = 3}}$.

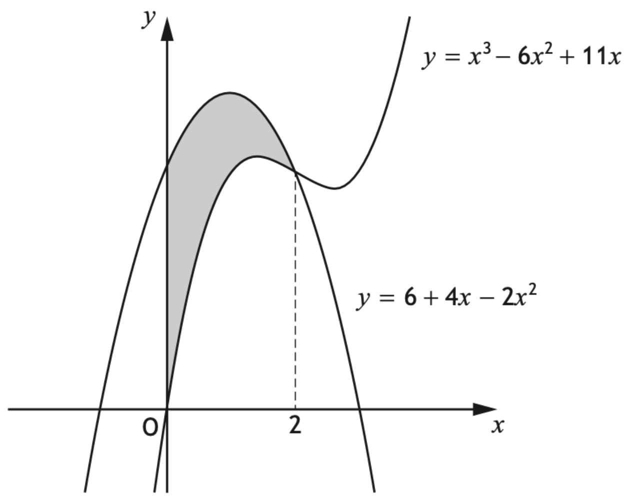
7. The diagram shows the curve with equation (5)

$$y = x^3 - 6x^2 + 11x$$

intersecting the curve with equation

$$y = 6 + 4x - 2x^2$$

at $x = 2$.



Calculate the shaded area.

Solution

$$\begin{aligned} & \int_0^2 [(6 + 4x - 2x^2) - (x^3 - 6x^2 + 11x)]\,\mathrm{d}x \\ = & \int_0^2 (6 - 7x + 4x^2 - x^3)\,\mathrm{d}x \\ = & \left[6x - \tfrac{7}{2}x^2 + \tfrac{4}{3}x^3 - \tfrac{1}{4}x^4\right]_{x=0}^2 \\ = & \left(12 - 14 + 10\tfrac{2}{3} - 4\right) - (0 - 0 + 0 - 0) \\ = & \underline{\underline{4\tfrac{2}{3}}}. \end{aligned}$$

8. Functions f and g are defined on $\mathbb{R}$, the set of real numbers, by:

- $f(x) = 2x^2 - 18$.
- $g(x) = x + 1$.

(a) Find an expression for $f(g(x))$. (2)

Solution

$$\begin{aligned} f(g(x)) &= f(x+1) \\ &= \underline{\underline{2(x+1)^2 - 18}}. \end{aligned}$$

(b) Find the values of x for which (2)

$$\frac{1}{f(g(x))}$$

is undefined.

Solution

Well,

$$\begin{aligned} \frac{1}{f(g(x))} \text{ is undefined} &\Rightarrow f(g(x)) = 0 \\ &\Rightarrow 2(x+1)^2 - 18 = 0 \\ &\Rightarrow 2(x+1)^2 = 18 \\ &\Rightarrow (x+1)^2 = 9 \\ &\Rightarrow x + 1 = -3 \text{ or } x + 1 = 3 \\ &\Rightarrow \underline{\underline{x = -4 \text{ or } x = 2}}. \end{aligned}$$

9. (a) Determine the coordinates of the stationary points on the curve with equation (4)

$$y = \tfrac{1}{3}x^3 - x^2 - 3x + 1.$$

Solution

Now,

$$y = \tfrac{1}{3}x^3 - x^2 - 3x + 1 \Rightarrow \frac{dy}{dx} = x^2 - 2x - 3$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow x^2 - 2x - 3 = 0$$

$$\left.\begin{array}{ll}\text{add to:} & -2 \\ \text{multiply to:} & -3\end{array}\right\} -3, +1$$

$$\begin{aligned}&\Rightarrow (x-3)(x+1) = 0\\ &\Rightarrow x = 3 \text{ or } x = -1\\ &\Rightarrow y = -8 \text{ or } y = 2\tfrac{2}{3};\end{aligned}$$

hence, the coordinates of the stationary points are $\underline{\underline{(3, -8)}}$ and $\underline{\underline{(-1, 2\frac{2}{3})}}$

(b) Hence, determine the greatest and least values of y in the interval $-1 \leqslant x \leqslant 6$. (2)

Solution
Well,

$$x = 6 \Rightarrow y = 19.$$

So, the greatest value is $\underline{\underline{19}}$ and the least value is $\underline{\underline{-8}}$.

10. The circle C_1 has equation

$$x^2 + y^2 + 18x - 2y - 8 = 0.$$

(a) Find the centre and radius of C_1. (2)

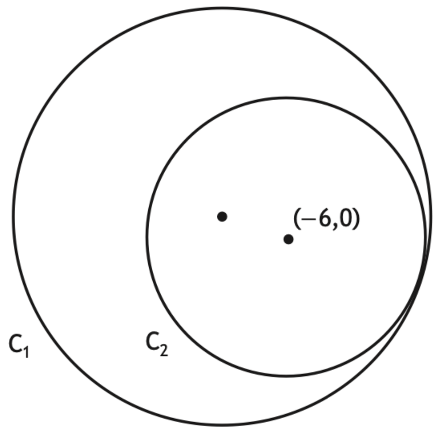
Solution

$$\begin{aligned}& x^2 + y^2 + 18x - 2y - 8 = 0\\ \Rightarrow\quad & x^2 + 18x + y^2 - 2y = 8\\ \Rightarrow\quad & (x^2 + 18x + 81) + (y^2 - 2y + 1) = 8 + 81 + 1\\ \Rightarrow\quad & (x+9)^2 + (y-1)^2 = 90;\end{aligned}$$

hence, the centre is $\underline{\underline{(-9, 1)}}$ and the radius is

$$\sqrt{90} = \underline{\underline{3\sqrt{10}}}.$$

A second circle, C_2, touches C_1 internally.



The centre of C_2 is $(6, 0)$.

(b) Determine the equation of C_2. (2)

Solution

Well,

$$\begin{aligned}\text{distance between the centres} &= \sqrt{(-9-(-6))^2 + (1-0)^2} \\ &= \sqrt{(-3)^2 + (1)^2} \\ &= \sqrt{9+1} \\ &= \sqrt{10}\end{aligned}$$

and

$$\begin{aligned}\text{radius of } C_2 &= 3\sqrt{10} - \sqrt{10} \\ &= 2\sqrt{10} \\ &= \sqrt{40}.\end{aligned}$$

Hence, the equation of C_2 is

$$\underline{\underline{(x+6)^2 + y^2 = 40.}}$$

11. The number of electric vehicles worldwide can be modelled by

$$N = 6.8\mathrm{e}^{kt},$$

where:

- N is the estimated number of vehicles in millions,
- t is the number of years since the end of 2020, and
- k is a constant.

(a) Use the model to estimate the number of electric vehicles worldwide at the end of 2020. (1)

Solution

Well,

$$\begin{aligned} t = 0 &\Rightarrow N = 6.8\mathrm{e}^0 \\ &\Rightarrow N = 6.8; \end{aligned}$$

hence, there are 6.8 million cars now.

At the end of 2030, it is estimated there will be 125 million electric vehicles worldwide.

(b) Determine the value of k. (4)

Solution

Now,

$$\begin{aligned} t = 3 &\Rightarrow 125 = 6.8\mathrm{e}^{10k} \\ &\Rightarrow \mathrm{e}^{10k} = 18\tfrac{13}{34} \\ &\Rightarrow 10k = \ln(18\tfrac{13}{34}) \\ &\Rightarrow k = \tfrac{1}{10}\ln(18\tfrac{13}{34}) \\ &\Rightarrow k = 0.291\,139\,112\,5 \text{ (FCD)} \\ &\Rightarrow k = 0.291 \text{ (3 sf)}. \end{aligned}$$

12. Solve the equation (5)

$$2\sin 2x^\circ - \sin^2 x^\circ = 0,\ 0 \leqslant x < 360.$$

Solution

$$
\begin{aligned}
2\sin 2x^\circ - \sin^2 x^\circ = 0 &\Rightarrow 2(2\sin x^\circ \cos x^\circ) - \sin^2 x^\circ = 0\\
&\Rightarrow 4\sin x^\circ \cos x^\circ - \sin^2 x^\circ = 0\\
&\Rightarrow \sin x^\circ(4\cos x^\circ - \sin x^\circ) = 0\\
&\Rightarrow \sin x^\circ = 0 \text{ or } 4\cos x^\circ - \sin x^\circ) = 0\\
&\Rightarrow \sin x^\circ = 0 \text{ or } 4\cos x^\circ = \sin x^\circ\\
&\Rightarrow \sin x^\circ = 0 \text{ or } \tan x^\circ = 4.
\end{aligned}
$$

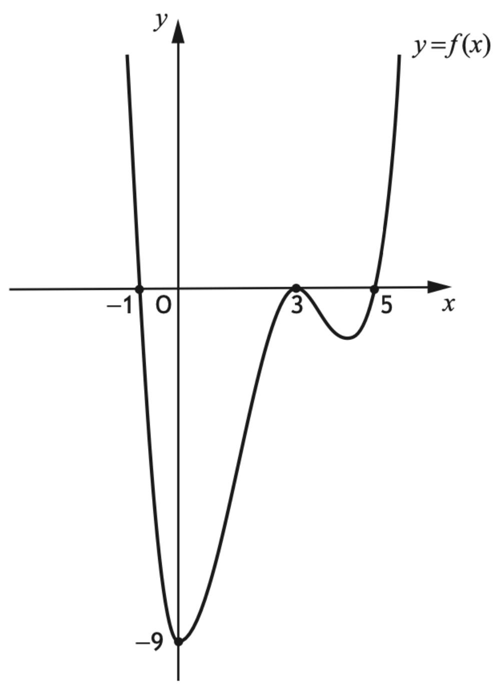
$\underline{\sin x^\circ = 0}$:

$$\sin x^\circ = 0 \Rightarrow \underline{\underline{x = 0,\ 180}}.$$

$\underline{\tan x^\circ = 4}$:

$$
\begin{aligned}
\tan x^\circ = 4 &\Rightarrow x = 75.963\,756\,53,\ 255.963\,756\,53 \text{ (FCD)}\\
&\Rightarrow \underline{\underline{x = 76.0,\ 256 \text{ (3 sf)}}}.
\end{aligned}
$$

13. The diagram shows the graph of $y = \mathrm{f}(x)$, where $\mathrm{f}(x)$ is a quartic function. (3)



Express $\mathrm{f}(x)$ in the form

$$\mathrm{f}(x) = k(x+a)^2(x+b)(x+c).$$

Solution

Well,

$$f(x) = k(x-3)^2(x+1)(x-5).$$

Now,

$$\begin{aligned} x = 0, f(x) = -9 &\Rightarrow k(0-3)^2(0+1)(0-5) = -9 \\ &\Rightarrow -45k = -9 \\ &\Rightarrow k = \tfrac{1}{5}. \end{aligned}$$

Hence,

$$\underline{\underline{f(x) = \tfrac{1}{5}(x-3)^2(x+1)(x-5)}}.$$