Dr Oliver Mathematics
OCR FMSQ Additional Mathematics
2019 Paper
2 hours

The total number of marks available is 100.
You must write down all the stages in your working.
You are permitted to use a scientific or graphical calculator in this paper.
Final answers should be given correct to three significant figures where appropriate.

1. A committee consists of five people. (2)

   The roles of Chairman, Secretary and Treasurer are to be allocated at random from the committee with no one person taking more than one role.

   In how many ways can this allocation of roles be made?

Solution

There are 5 choices for the Chairman, 4 choices for the Secretary, and 3 choices for the Treasurer:

$$5 \times 4 \times 3 = \underline{\underline{60 \text{ ways}}}.$$

2. (a) Solve the inequality (3)

$$x^2 - x - 12 \leqslant 0.$$

Solution

$$\left.\begin{array}{ll} \text{add to:} & -1 \\ \text{multiply to:} & -12 \end{array}\right\} -4,\ +3$$

So,

$$\begin{aligned} x^2 - x - 12 = 0 &\Rightarrow (x-4)(x+3) = 0 \\ &\Rightarrow x - 4 = 0 \text{ or } x + 3 = 0 \\ &\Rightarrow x = 4 \text{ or } x = -3. \end{aligned}$$

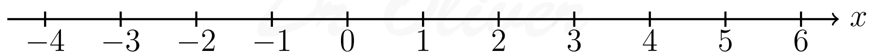
We need a 'table of signs':

	$x < -3$	$x = -3$	$-3 < x < 4$	$x = 4$	$x > 4$
$x + 3$	$-$	0	$+$	$+$	$+$
$x - 4$	$-$	$-$	$-$	0	$+$
$(x+3)(x-4)$	$+$	0	$-$	0	$+$

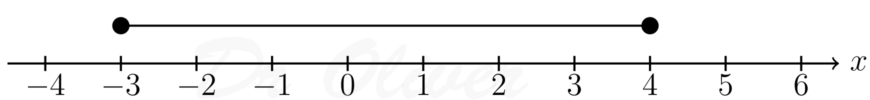
Hence,

$$\underline{\underline{-3 \leqslant x \leqslant 4}}.$$

(b) Illustrate your answer to part (a) on the number line provided. (2)



Solution



3. Find the equation of the normal to the curve (6)

$$y = x^3 - 2x^2 + 2x + 4$$

at the point $(2, 8)$.

Solution

Well,

$$y = x^3 - 2x^2 + 2x + 4 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 4x + 2$$

and

$$x = 2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3(2^2) - 4(2) + 2 = 6.$$

Now,

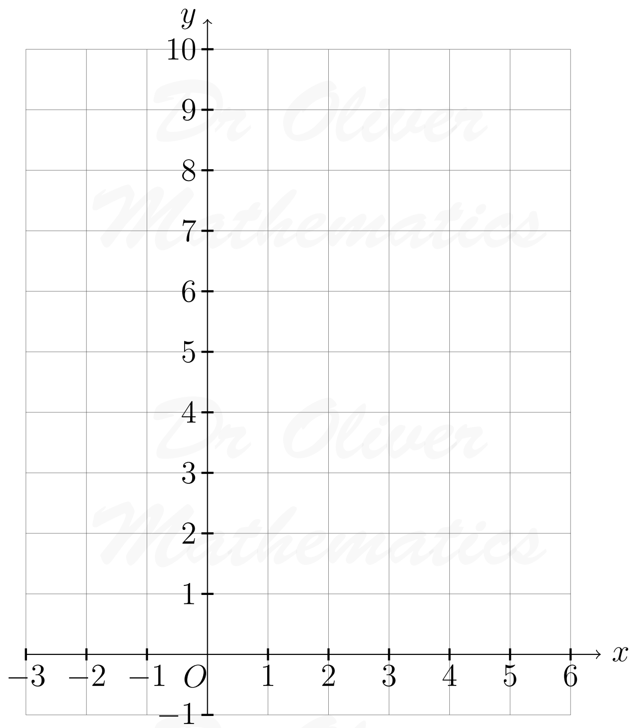
$$m_{\text{tangent}} = 6 \Rightarrow m_{\text{normal}} = -\tfrac{1}{6}.$$

Finally, the equation of the normal is

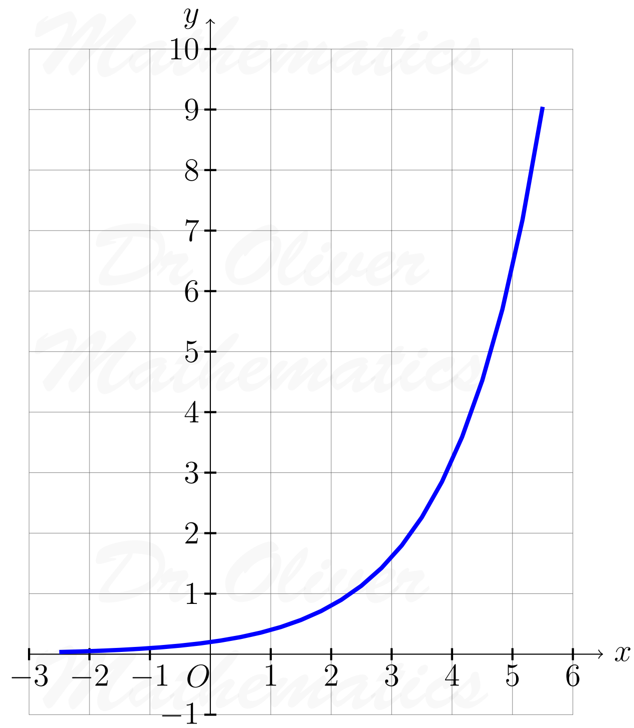
$$y - 8 = -\tfrac{1}{6}(x - 2) \Rightarrow y - 8 = -\tfrac{1}{6}x + \tfrac{1}{3}$$
$$\Rightarrow \underline{\underline{y = -\tfrac{1}{6}x + \tfrac{25}{3}}}.$$

4. (a) On the grid provided, sketch the curve (2)

$$y = \tfrac{1}{5} \times 2^x.$$



Solution



(b) Solve algebraically the equation (3)

$$\tfrac{1}{5} \times 2^x = 3,$$

giving your answer correct to 3 significant figures.

Solution

$$\begin{aligned} \tfrac{1}{5} \times 2^x = 3 &\Rightarrow 2^x = 15 \\ &\Rightarrow \log_{10} 2^x = \log_{10} 15 \\ &\Rightarrow x \log_{10} 2 = \log_{10} 15 \\ &\Rightarrow \underline{\underline{x = \frac{\log_{10} 15}{\log_{10} 2} \text{ or } x = \log_2 15.}} \end{aligned}$$

5. Solve the equation (5)

$$\log_{10} x + \log_{10}(x+2) = 3 \log_{10} 2.$$

You must show detailed reasoning.

Solution

$$\begin{aligned}\log_{10} x + \log_{10}(x+2) = 3\log_{10} 2 &\Rightarrow \log_{10}[x(x+2)] = \log_{10} 2^3 \\ &\Rightarrow \log_{10}[x(x+2)] = \log_{10} 8 \\ &\Rightarrow x(x+2) = 8 \\ &\Rightarrow x^2 + 2x = 8 \\ &\Rightarrow x^2 + 2x - 8 = 0\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & +2 \\ \text{multiply to:} & -8\end{array}\right\} + 4, -2$$

$$\begin{aligned}&\Rightarrow (x+4)(x-2) = 0 \\ &\Rightarrow x + 4 = 0 \text{ or } x - 2 = 0 \\ &\Rightarrow x = -4 \text{ or } x = 2.\end{aligned}$$

But

$$x + 2 > 0 \Rightarrow x > -2.$$

So $\underline{\underline{x = 2}}$.

6. Angle θ is such that

$$\tan\theta = 1.5.$$

(a) Find the two values of θ in the range $0^\circ \leqslant \theta \leqslant 360^\circ$. (2)

Solution

$$\begin{aligned}\tan\theta = 1.5 &\Rightarrow \theta = 56.309\,932\,47,\ 236.309\,932\,47 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{\theta = 56.3,\ 236.3 \text{ (1 dp)}}}.\end{aligned}$$

(b) Find the exact values of $\sin\theta$. (3)
You must show detailed reasoning.

Solution

$$\begin{aligned}\tan^2\theta + 1 = \sec^2\theta &\Rightarrow 1.5^2 + 1 = \sec^2\theta \\ &\Rightarrow 2.25 + 1 = \sec^2\theta \\ &\Rightarrow 3.25 = \sec^2\theta \\ &\Rightarrow \cos^2\theta = \tfrac{4}{13} \\ &\Rightarrow \sin^2\theta + \tfrac{4}{13} = 1 \\ &\Rightarrow \sin^2\theta = \tfrac{9}{13} \\ &\Rightarrow \underline{\underline{\sin\theta = \pm\tfrac{3\sqrt{13}}{13}}}.\end{aligned}$$

7. The equation (5)

$$x^3 - 3x + k = 0,$$

where k is a constant, has a root $x = 2$.

Find the numerical value(s) of the other roots of this equation.
You must show detailed reasoning.

Solution

We use synthetic division:

$$\begin{array}{c|cccc} 2 & 1 & 0 & -3 & k \\ & \downarrow & 2 & 4 & 2 \\ \hline & 1 & 2 & 1 & k+2 \end{array}$$

Now, $k + 2$ is equal to zero (why?). So $k = -2$. Next,

$$x^3 - 3x - 2 = 0 \Rightarrow (x-2)(x^2 + 2x + 1) = 0$$

$$\left.\begin{array}{ll} \text{add to:} & +2 \\ \text{multiply to:} & 1 \end{array}\right\} +1,\ +1$$

$$\Rightarrow (x-2)(x+1)^2 = 0;$$

hence, the other root is $\underline{\underline{-1 \text{ (repeated)}}}$.

8. Each of five students has a fair coin.
They play a game in which each student tosses their coin and when the result of their

toss is a head then that student is eliminated from the game.
The game continues with the remaining students tossing their coin again.
As before, any student who tosses a head is eliminated.
The game continues until all the students have been eliminated or there is a single winner.

Calculate the probability that

(a) all students are eliminated after their first toss of the coin, (2)

Solution

$$(\tfrac{1}{2})^5 = \underline{\underline{\tfrac{1}{32}}}.$$

(b) exactly two students are eliminated after their first toss and exactly two after their second toss, leaving one winner. (4)

Solution

$$\begin{aligned}\mathrm{P} &= \binom{5}{2}(\tfrac{1}{2})^5 \times \binom{3}{2}(\tfrac{1}{2})^3 \\ &= (10 \times \tfrac{1}{32}) \times (3 \times \tfrac{1}{8}) \\ &= \tfrac{5}{16} \times \tfrac{3}{8} \\ &= \underline{\underline{\tfrac{15}{128}}}.\end{aligned}$$

9. The equation

$$x^3 + 2x^2 - x - 1 = 0$$

has two negative roots, α and β, and one positive root, γ.

(a) By considering a change of sign, show that γ lies in the interval $[0, 1]$. (2)

Solution

Let

$$\mathrm{f}(x) = x^3 + 2x^2 - x - 1.$$

Then

$$\begin{aligned}\mathrm{f}(0) &= 0 + 0 - 0 - 1 = -1 \\ f(1) &= 1 + 2 - 1 - 1 = 1.\end{aligned}$$

Now, $\mathrm{f}(x)$ is continuous and we have a change in sign.
Hence, γ lies in the $\underline{\underline{\text{interval}}}$ $[0, 1]$

(b) Show that (2)

$$x = \sqrt{\frac{x+1}{x+2}}$$

is a rearrangement of the equation.

Solution

$$\begin{aligned} x^3 + 2x^2 - x - 1 = 0 &\Rightarrow x^3 + 2x^2 = x + 1 \\ &\Rightarrow x^2(x+2) = x + 1 \\ &\Rightarrow x^2 = \frac{x+1}{x+2} \\ &\Rightarrow \underline{\underline{x = \sqrt{\frac{x+1}{x+2}}}}, \end{aligned}$$

as required.

(c) Using the iterative formula (3)

$$x_{r+1} = \sqrt{\frac{x_r+1}{x_r+2}}$$

with $x_0 = 0.8$, find γ correct to 3 decimal places, showing the result of each iteration.

Solution

$$\begin{aligned} x_1 &= 0.801\,783\,725\,7 \text{ (FCD)} \\ x_2 &= 0.801\,925\,504\,1 \text{ (FCD)} \\ x_3 &= 0.801\,936\,764\,4 \text{ (FCD)}; \end{aligned}$$

so, $\underline{\underline{\gamma = 0.802 \text{ (3 dp)}}}$.

10. You are given that the line

$$y = 2x + k$$

cuts the circle

$$x^2 + y^2 = 5$$

in two points, A and B.

(a) Show that the x-coordinates of A and B satisfy the equation (2)

$$5x^2 + 4kx + (k^2 - 5) = 0.$$

Solution

$$x^2 + y^2 = 5 \Rightarrow x^2 + (2x+k)^2 = 5$$

$\times$	$2x$	$+k$
$2x$	$4x^2$	$+2kx$
$+k$	$+2kx$	$+k^2$

$$\begin{aligned} &\Rightarrow x^2 + (4x^2 + 4kx + k^2) = 5 \\ &\Rightarrow \underline{\underline{5x^2 + 4kx + (k^2 - 5) = 0}}, \end{aligned}$$

as required.

(b) Hence find the values of k for which the line is a tangent to the circle. (2)

Solution

We must have $b^2 - 4ac = 0$:

$$\begin{aligned} (4k)^2 - 4(5)(k^2 - 5) = 0 &\Rightarrow 16k^2 - 20(k^2 - 5) = 0 \\ &\Rightarrow 16k^2 - 20k^2 + 100 = 0 \\ &\Rightarrow 4k^2 = 100 \\ &\Rightarrow k^2 = 25 \\ &\Rightarrow \underline{\underline{k = \pm 5}}. \end{aligned}$$

11. John makes wooden toys in his workshop at home.
He classifies the toys as small or large.

It takes 5 hours to make a small toy and 8 hours to make a large toy.
He works for a maximum of 60 hours each week.

Let x be the number of small toys and y be the number of large toys he makes each week.

(a) Write down an inequality giving the time constraint. (1)

> **Solution**
>
> $\underline{\underline{5x + 8y = 60}}$.

John knows from experience that

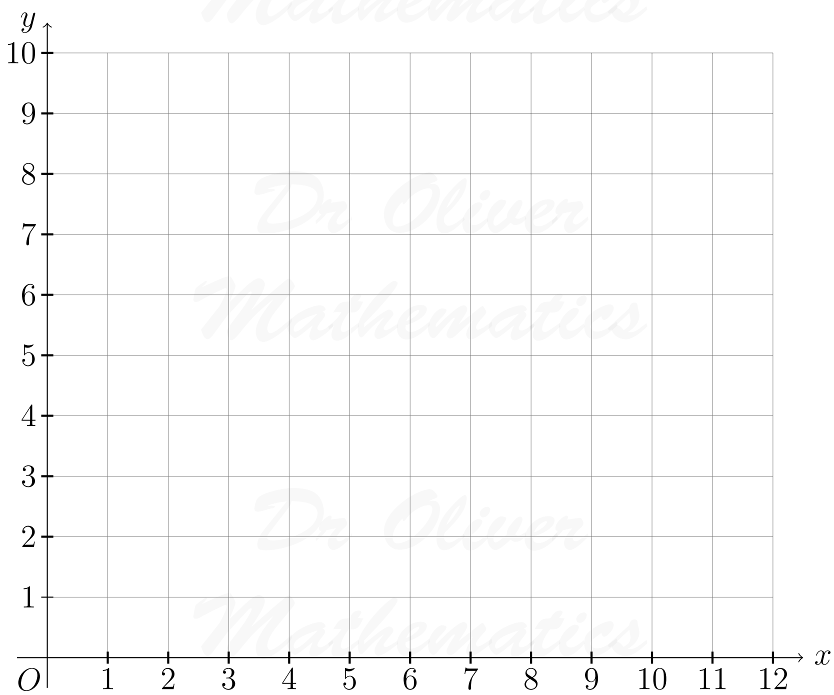
- he needs to make at least 3 large toys each week and
- the number of large toys should be no more than double the number of small toys.

He never leaves any toys unfinished at the end of the week.

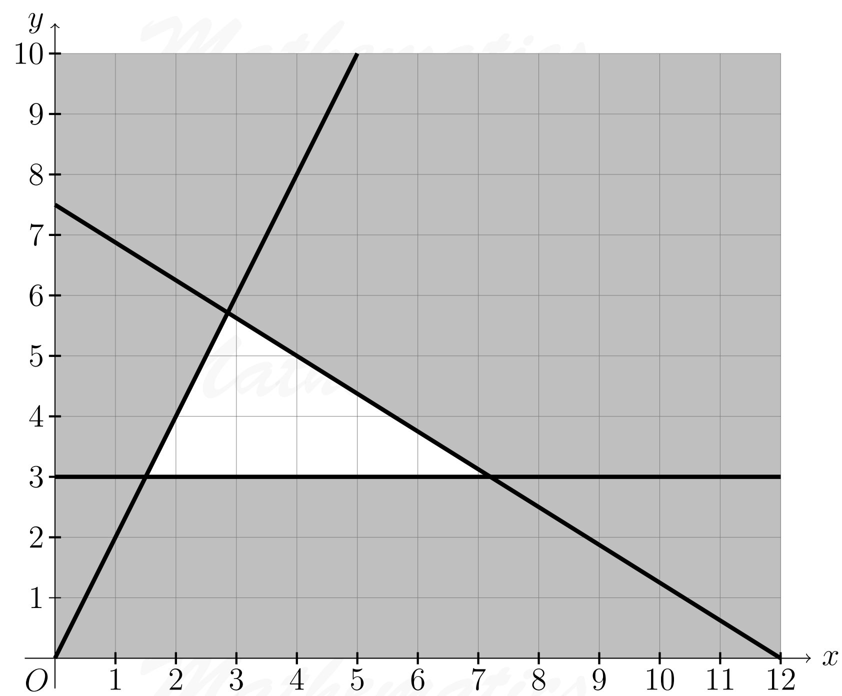
(b) From this information, write down two more inequalities in x and y. (2)

> **Solution**
>
> $\underline{\underline{y \geqslant 3}}$ and $\underline{\underline{y \leqslant 2x}}$.

(c) On the grid provided, illustrate these three inequalities.
Shade the region that is not required. (4)



> **Solution**



(d) Find the maximum number of toys that John can make in a week and the number of hours he would take to make them. (2)

Solution

He can make 10 toys (seven small, three large), working

$$5(7) + 8(3) = \underline{\underline{59 \text{ hours}}}.$$

The price for which John sells his wooden toys is such that the profit made is £28 for each small toy and £60 for each large toy.

(e) Assuming that at the end of each week he sells all the toys, find the number of each type of toy he should make to maximise his profit and calculate the profit in this case. (3)

Solution

Draw $28x + 60y = c$ on to the grid, for various values of c:

$$x = 7, y = 3 \Rightarrow P = 28(7) + 60(3) = 376,$$
$$x = 5, y = 4 \Rightarrow P = 28(5) + 60(4) = 380,$$
$$x = 4, y = 5 \Rightarrow P = 28(4) + 60(5) = 412;$$

hence, his maximum profit is $\underline{\underline{£412}}$.

12. The curve C_1 has equation

$$y = 10x - x^2 + k$$

and passes through the point $(5, 10)$.

(a) Show that $k = -15$. (2)

Solution

$$\begin{aligned} x = 5, y = 10 \Rightarrow 10 &= 10(5) - 5^2 + k \\ \Rightarrow 10 &= 50 - 25 + k \\ \Rightarrow \underline{\underline{k = -15}}, & \end{aligned}$$

as required.

(b) Show that there is a maximum value at the point $(5, 10)$. (3)

Solution

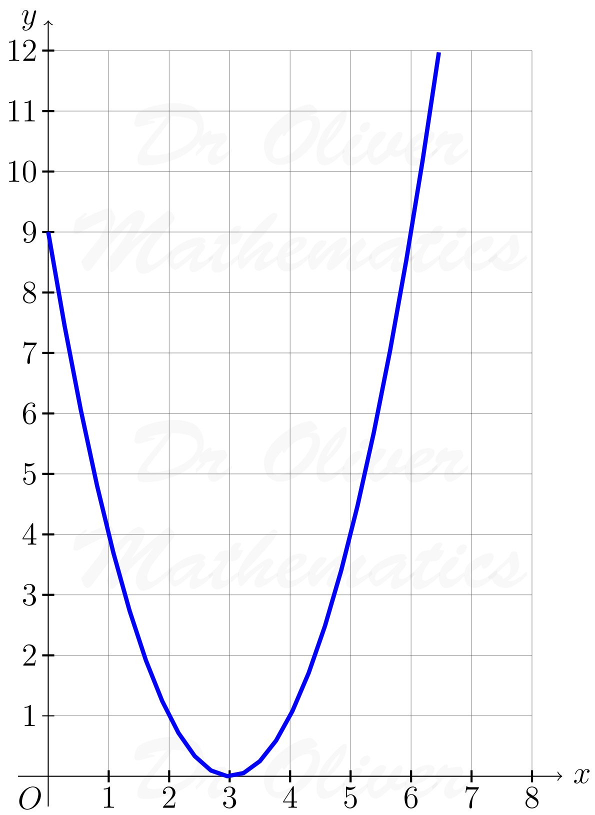
$$\begin{aligned} y &= 10x - x^2 - 15 \\ &= -(x^2 - 10x) - 15 \\ &= -[(x^2 - 10x + 25) - 25] - 15 \\ &= -(x^2 - 10x + 25) + 25 - 15 \\ &= -(x - 5)^2 + 10. \end{aligned}$$

Now, we see clearly that the maximum value for y is 10 and it occurs when x is 5.
Hence, there is a $\underline{\underline{\text{maximum value}}}$ at the point $\underline{\underline{(5, 10)}}$.

The curve C_2 with equation

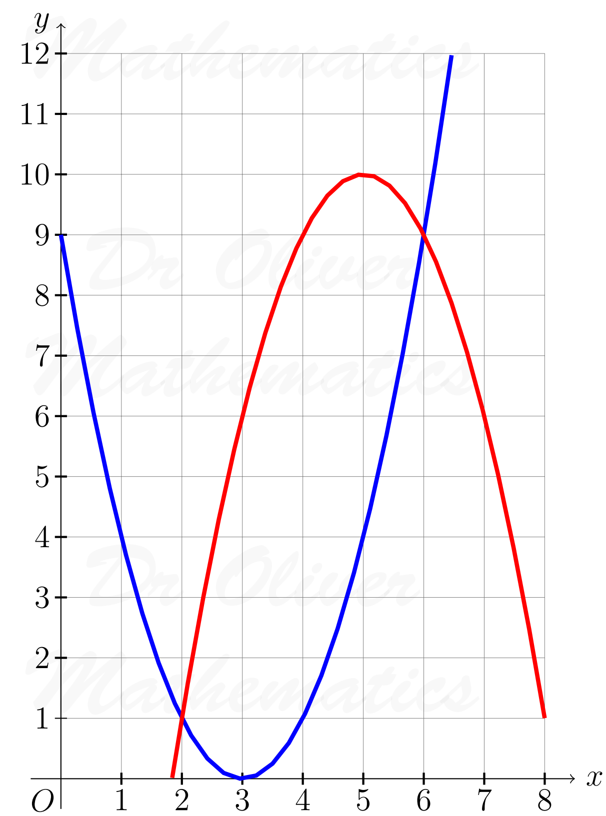
$$y = (x-3)^2$$

has been plotted.



(c) On the same grid, sketch the curve C_1. (2)

Solution



(d) Find the coordinates of the points of intersection of the curves C_1 and C_2. (2)

Solution

It *looks* they cross at $x = 2$ and $x = 6$ but it that really the case?

$\times$	x	-3
x	x^2	$-3x$
-3	$-3x$	$+9$

Now,

$$\begin{aligned} 10x - x^2 - 15 = (x-3)^2 &\Rightarrow 10x - x^2 - 15 = x^2 - 6x + 9 \\ &\Rightarrow 0 = 2x^2 - 16x + 24 \\ &\Rightarrow 0 = 2(x^2 - 8x + 12) \end{aligned}$$

$$\left.\begin{array}{lr}\text{add to:} & -8 \\ \text{multiply to:} & +12\end{array}\right\} -6,\ -2$$

$$\begin{aligned}&\Rightarrow 0 = 2(x-6)(x-2)\\ &\Rightarrow x-6=0 \text{ or } x-2=0\\ &\Rightarrow x=6 \text{ or } x=2;\end{aligned}$$

it remains to find the y-coordinate: using $y = (x-3)^2$,

$$x = 2 \Rightarrow y = (2-3)^2 = 1$$

and

$$x = 2 \Rightarrow y = (6-3)^2 = 9;$$

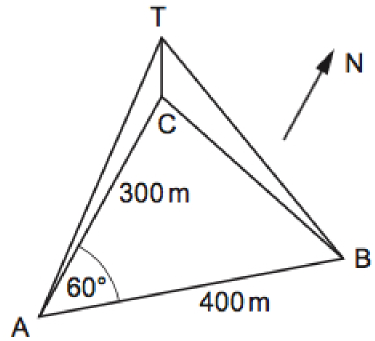
hence, they cross at $\underline{\underline{(2, 1)}}$ and $\underline{\underline{(6, 9)}}$.

(e) Find the area between the curves C_1 and C_2. (5)

Solution

$$\begin{aligned}\text{Area} &= \text{top curve} - \text{bottom curve}\\ &= \int_1^6 (10x - x^2 - 15)\,\mathrm{d}x - \int_1^6 (x-3)^2\,\mathrm{d}x\\ &= \int_1^6 \left[(10x - x^2 - 15) - (x^2 - 6x + 9)\right]\mathrm{d}x\\ &= \int_1^6 (-24 + 16x - 2x^2)\,\mathrm{d}x\\ &= \left[-24x + 8x^2 - \tfrac{2}{3}x^3\right]_{x=1}^{6}\\ &= (-144 + 288 - 144) - \left(-24 + 8 - \tfrac{2}{3}\right)\\ &= \underline{\underline{16\tfrac{2}{3}}}.\end{aligned}$$

13. A straight road runs on a bearing of 060° from a point A to a point B, 400 m from A.
A vertical mast, CT, stands at a point C, 300 m due north of A.
From the point A the angle of elevation of the top of the mast, T, is 7°.
The triangle ABC is on horizontal ground.



(a) Find the height of the mast. (2)

Solution

$$\begin{aligned}\tan = \frac{\text{opp}}{\text{adj}} &\Rightarrow \tan 7^\circ = \frac{h}{300}\\ &\Rightarrow h = 300\tan 7^\circ\\ &\Rightarrow h = 36.835\,368\,27 \text{ (FCD)}\\ &\Rightarrow \underline{\underline{h = 36.8 \text{ m (3 sf)}}}.\end{aligned}$$

(b) Find the angle of elevation of the top of the mast from point B. (5)

Solution

We use the cosine rule:

$$\begin{aligned}a^2 = b^2 + c^2 - 2bc\cos A &\Rightarrow BC^2 = 300^2 + 400^2 - 2(300)(400)(\cos 60^\circ)\\ &\Rightarrow BC^2 = 90\,000^2 + 160\,000^2 - 120\,000\\ &\Rightarrow BC^2 = 130\,000\\ &\Rightarrow BC = 360.555\,127\,5 \text{ (FCD)}.\end{aligned}$$

Finally,

$$\begin{aligned}\text{elevation} &= \tan^{-1}\left(\frac{h}{BC}\right)\\ &= \tan^{-1}\left(\frac{36.835\ldots}{360.555\ldots}\right)\\ &= 5.833\,266\,444 \text{ (FCD)}\\ &= \underline{\underline{5.83^\circ \text{ (3 sf)}}}.\end{aligned}$$

(c) Find the bearing of the base of the mast from point B. (5)

Solution

We use the sine rule:

$$\frac{\sin B}{AC} = \frac{\sin A}{BC} \Rightarrow \frac{\sin B}{300} = \frac{\sin 60^\circ}{360.555\ldots}$$
$$\Rightarrow \sin B = \frac{300 \sin 60^\circ}{360.555\ldots}$$
$$\Rightarrow B = 46.102\,113\,75 \text{ (FCD)}.$$

Finally,

$$\begin{aligned} \text{bearing} &= 180 + 60 + 46.102\ldots \\ &= 286.102\,113\,75 \text{ (FCD)} \\ &= \underline{\underline{286^\circ \text{ (3 sf)}}}. \end{aligned}$$

14. Speed bumps are designed to encourage drivers to drive slowly.
On a particular road, the bumps put onto the road are designed to give minimum discomfort and damage at a speed of 9 ms^{-1}.
Paul is driving along the road at a speed of 14 ms^{-1} when he sees the warning sign, he is 50 m before the first bump.
He immediately slows down with uniform deceleration so that when he reaches the first bump he is travelling at a speed of 9 ms^{-1}.

(a) Calculate the uniform deceleration **and** the time taken for Paul to reach the first bump. (3)

Solution

$s = 50$, $u = 14$, $v = 9$, $a = ?$, and $t = ?$: we use $v^2 = u^2 + 2as$:

$$\begin{aligned} 9^2 = 14^2 + 2 \times a \times 50 &\Rightarrow 81 = 196 + 100a \\ &\Rightarrow 100a = -115 \\ &\Rightarrow a = -1.15; \end{aligned}$$

so, Paul decelerates at $\underline{\underline{1.15 \text{ ms}^{-2}}}$ and the time taken is

$$\begin{aligned} v = u + at &\Rightarrow 9 = 14 - 1.15t \\ &\Rightarrow 1.15t = 5 \\ &\Rightarrow \underline{\underline{t = 4\tfrac{8}{23} \text{ s or } 4.35 \text{ s (3 sf)}}}. \end{aligned}$$

Immediately after the bump he accelerates such that at t seconds after leaving the bump his speed, v ms^{-1}, is given by

$$v = \tfrac{1}{100}(15t^2 - t^3) + 9.$$

(b) Show that he reaches his original speed of 14 ms^{-1} in 10 seconds. (1)

Solution
Well,

$$\begin{aligned} t = 10 \Rightarrow v &= \tfrac{1}{100}[15(10^2) - 10^3] + 9 \\ \Rightarrow v &= \tfrac{1}{100}(1\,500 - 1\,000) + 9 \\ \Rightarrow v &= \tfrac{1}{100}(500) + 9 \\ \Rightarrow v &= 5 + 9 \\ \Rightarrow \underline{\underline{v}} &\underline{\underline{= 14}}, \end{aligned}$$

as required.

(c) Find the distance travelled from the speed bump by the time he reaches this speed. (4)

Solution
Now,

$$v = \tfrac{1}{100}(15t^2 - t^3) + 9 \Rightarrow s = \tfrac{1}{100}(5t^3 - \tfrac{1}{4}t^4) + 9t + c,$$

for some constant c. Finally,

$$\begin{aligned} \text{distance travelled} &= s(10) - s(0) \\ &= \left[\tfrac{1}{100}(5\,000 - 2\,500) + 90 + c\right] - c \\ &= \underline{\underline{115 \text{ m}}}. \end{aligned}$$

(d) Find the maximum acceleration in this period. (3)

Solution

$$\begin{aligned} v = \tfrac{1}{100}(15t^2 - t^3) + 9 &\Rightarrow v = \tfrac{3}{20}t^2 - \tfrac{1}{100}t^3 + 9 \\ &\Rightarrow a = \tfrac{3}{10}t - \tfrac{3}{100}t^2 \\ &\Rightarrow a = \tfrac{3}{100}(10t - t^2) \\ &\Rightarrow a = -\tfrac{3}{100}(t^2 - 10t) \\ &\Rightarrow a = -\tfrac{3}{100}[(t^2 - 10t + 25) - 25] \\ &\Rightarrow a = -\tfrac{3}{100}(t-5)^2 + \tfrac{3}{4}; \end{aligned}$$

hence, the maximum acceleration is $\underline{\underline{0.75 \text{ ms}^{-2}}}$.

(e) If all drivers decelerate and accelerate in the same way as Paul, suggest a maximum distance between bumps to ensure that drivers do not exceed a speed of 14 ms^{-1} when driving down the road. (1)

Solution

$$\begin{aligned} \text{Maximum distance} &= \text{speeding up} + \text{slowing down} \\ &= 115 + 50 \\ &= \underline{\underline{165 \text{ m}}}. \end{aligned}$$