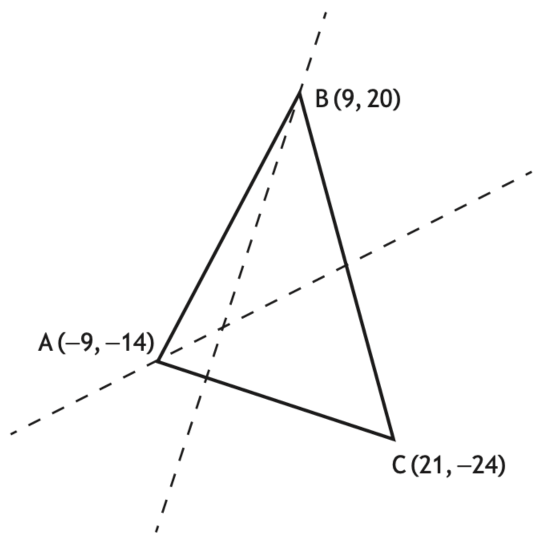
Dr Oliver Mathematics
Mathematics: Higher
2025 Paper 2: Calculator
1 hour 30 minutes

The total number of marks available is 65.
You must write down all the stages in your working.

1. Triangle ABC has vertices $A(-9, -14)$, $B(9, 20)$, and $C(21, -24)$.



(a) Find the equation of the altitude through B. (3)

Solution

Well,

$$\begin{aligned} m_{AC} &= \frac{-24-(-14)}{21-(-9)} \\ &= \frac{-10}{30} \\ &= -\tfrac{1}{3}, \end{aligned}$$

so that makes the perpendicular through AC $m_{\text{normal}} = 3$.

Finally, the equation of the altitude through B is

$$\begin{aligned} y - 20 = 3(x-9) &\Rightarrow y - 20 = 3x - 27 \\ &\Rightarrow \underline{\underline{y = 3x - 7}}. \end{aligned}$$

(b) Find the equation of the median through A. (3)

Solution

The midpoint — let's call it D — of BC is

$$\left(\frac{9+21}{2}, \frac{20+(-24)}{2}\right) = D(15, -2).$$

Now,

$$\begin{aligned} m_{AD} &= \frac{-2-(-14)}{15-(-9)} \\ &= \frac{12}{24} \\ &= \tfrac{1}{2} \end{aligned}$$

and the equation of the median through A is

$$\begin{aligned} y + 14 = \tfrac{1}{2}(x+9) &\Rightarrow y + 14 = \tfrac{1}{2}x + \tfrac{9}{2} \\ &\Rightarrow \underline{\underline{y = \tfrac{1}{2}x - \tfrac{19}{2}}}. \end{aligned}$$

(c) Determine the point of intersection of the altitude through B and the median through A. (2)

Solution

Now,

$$\begin{aligned} 3x - 7 = \tfrac{1}{2}x - \tfrac{19}{2} &\Rightarrow \tfrac{5}{2}x = -\tfrac{5}{2} \\ &\Rightarrow x = -1 \\ &\Rightarrow y = -10; \end{aligned}$$

hence, the point of intersection is $\underline{\underline{(-1, -10)}}$.

2. Express (3)

$$2x^2 + 16x + 5$$

in the form

$$p(x+q)^2 + r.$$

Solution

Well,

$$2x^2 + 16x + 5 = 2(x^2 + 8x) + 5$$

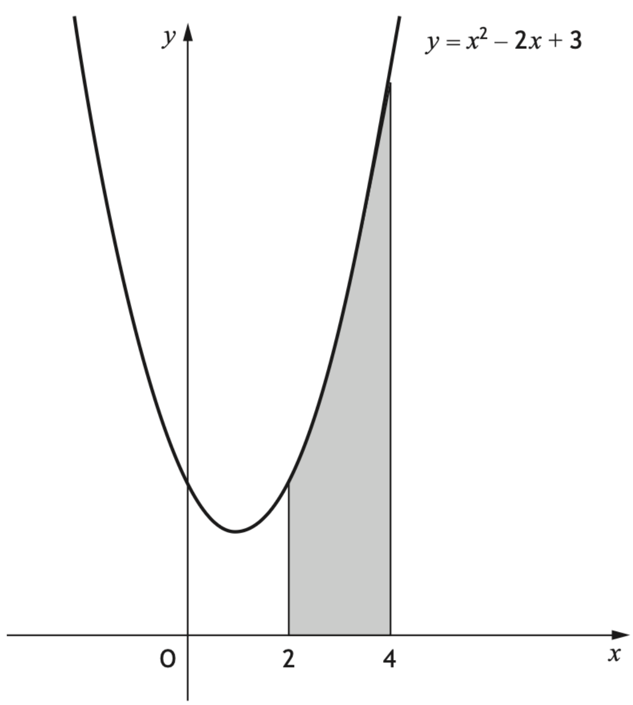
$$\begin{array}{ll} \text{coefficient of } x: & +8 \\ \text{half it:} & +4 \\ \text{square it:} & (+4)^2 = +16 \end{array}$$

$$\begin{aligned} &= 2[(x^2 + 8x + 16) - 16] + 5 \\ &= 2[(x + 4)^2 - 16] + 5 \\ &= 2(x + 4)^2 - 32 + 5 \\ &= \underline{\underline{2(x + 4)^2 - 27}}; \end{aligned}$$

hence, $\underline{\underline{p = 2}}$, $\underline{\underline{q = 4}}$, and $\underline{\underline{r = -27}}$.

3. The diagram shows the graph of (4)

$$y = x^2 - 2x + 3.$$



Calculate the shaded area.

Solution

Now,

$$\begin{aligned}\int_2^4 (x^2 - 2x + 3)\,\mathrm{d}x &= \left[\tfrac{1}{3}x^3 - x^2 + 3x\right]_{x=2}^{4}\\ &= \left(\tfrac{64}{3} - 16 + 12\right) - \left(\tfrac{8}{3} - 4 + 6\right)\\ &= \underline{\underline{12\tfrac{2}{3}}}.\end{aligned}$$

4. A function, g, is defined by (3)

$$\mathrm{g}(x) = (x-4)^3, \text{ where } x \in \mathbb{R}.$$

Find the inverse function, $\mathrm{g}^{-1}(x)$.

Solution

Well,

$$\begin{aligned}y = (x-4)^3 &\Rightarrow \sqrt[3]{y} = x - 4\\ &\Rightarrow \sqrt[3]{y} + 4 = x,\end{aligned}$$

and so

$$\underline{\underline{\mathrm{g}^{-1}(x) = \sqrt[3]{x} + 4}}.$$

5. (a) Show that the points $A(-3, 2, -1)$, $B(6, -1, 5)$, and $C(12, -3, 9)$ are collinear. (3)

Solution

Well,

$$\begin{aligned}
\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\
&= -\overrightarrow{OA} + \overrightarrow{OA} \\
&= -\begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 6 \\ -1 \\ 5 \end{pmatrix} \\
&= \begin{pmatrix} 9 \\ -3 \\ 6 \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\overrightarrow{BC} &= \overrightarrow{BO} + \overrightarrow{OC} \\
&= -\overrightarrow{OB} + \overrightarrow{OC} \\
&= -\begin{pmatrix} 6 \\ -1 \\ 5 \end{pmatrix} + \begin{pmatrix} 12 \\ -3 \\ 9 \end{pmatrix} \\
&= \begin{pmatrix} 6 \\ -2 \\ 4 \end{pmatrix} \\
&= \tfrac{2}{3}\begin{pmatrix} 9 \\ -3 \\ 6 \end{pmatrix} \\
&= \tfrac{2}{3}\overrightarrow{AB};
\end{aligned}$$

as they have a point in common, A, B, and C are collinear.

(b) State the ratio in which B divides AC. (1)

Solution

Now,

$$AB : BC = 1 : \tfrac{2}{3} = \underline{\underline{3 : 2}}.$$

6. (a) Express (4)

$$5\cos x - 9\sin x$$

in the form

$$k\cos(x + a),$$

where $k > 0$ and $0 < a < 2\pi$.

Solution

Well,

$$\begin{aligned} k\cos(x+a) &\equiv k(\cos x \cos a - \sin x \sin a) \\ &\equiv k\cos a \cos x - k \sin a \sin x, \end{aligned}$$

so

$$k\cos a = 5 \text{ and } k\sin a = 9.$$

Now,

$$\begin{aligned} k &= \sqrt{k^2} \\ &= \sqrt{k^2(\cos^2 a + \sin^2 a)} \\ &= \sqrt{(k\cos a)^2 + (k\sin a)^2} \\ &= \sqrt{5^2 + 9^2} \\ &= \sqrt{25 + 81} \\ &= \underline{\underline{\sqrt{106}}} \end{aligned}$$

and

$$\begin{aligned} \tan a = \frac{k\sin a}{k\cos a} &\Rightarrow \tan a = \tfrac{9}{5} \\ &\Rightarrow a = 1.063\,697\,822 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{a = 1.06 \text{ (3 sf)}}}. \end{aligned}$$

(b) Hence solve (3)

$$5\cos x - 9\sin x = 7, \text{ for } 0 \leqslant x < 2\pi.$$

Solution

Well,

$$\begin{aligned}5\cos x - 9\sin x = 7 &\Rightarrow \sqrt{106}\cos(x + 1.063\ldots) = 7\\ &\Rightarrow \cos(x + 1.063\ldots) = \tfrac{7}{\sqrt{106}}\\ &\Rightarrow x + 1.063\ldots = 5.460\,015\,379,\ 7.106\,355\,236 \text{ (FCD)}\\ &\Rightarrow x = 4.396\,317\,556,\ 6.042\,657\,413 \text{ (FCD)}\\ &\Rightarrow \underline{\underline{x = 4.40,\ 6.04 \text{ (3 sf)}}}.\end{aligned}$$

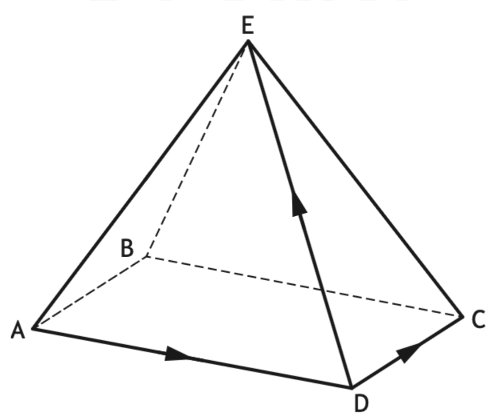
7. Find (2)

$$\int (3x+2)^7 \,\mathrm{d}x$$

Solution

$$\int (3x+2)^7 \,\mathrm{d}x = \underline{\underline{\tfrac{1}{24}(3x+2)^8 + c}}.$$

8. $ABCDE$ is a rectangular-based pyramid as shown. (2)



- $\overrightarrow{AD} = 6\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$.
- $\overrightarrow{DC} = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$.
- $\overrightarrow{DE} = -4\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$.

Express $\overrightarrow{BE}$ in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

Solution

Well,

$$\begin{aligned}\overrightarrow{BE} &= \overrightarrow{BA} + \overrightarrow{AD} + \overrightarrow{DE} \\ &= \overrightarrow{CD} + \overrightarrow{AD} + \overrightarrow{DE} \\ &= -\overrightarrow{DC} + \overrightarrow{AD} + \overrightarrow{DE} \\ &= -(2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) + (6\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) + (-4\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) \\ &= \underline{\underline{5\mathbf{j} + 4\mathbf{k}}}.\end{aligned}$$

9. A sequence satisfies the recurrence relation

$$u_{n+1} = mu_n + 4,$$

where m is a constant.

(a) The sequence approaches a limit of 10 as $n \to \infty$. (2)
Determine the value of m.

Solution

Well,

$$\begin{aligned}10 = 10m + 4 \Rightarrow 10m &= 6 \\ \Rightarrow \underline{\underline{m = \tfrac{3}{5}}}&.\end{aligned}$$

(b) Given that $u_1 = 19$, calculate the value of u_0. (1)

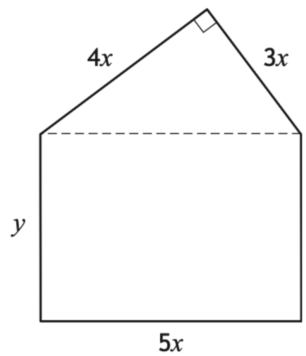
Solution

Now,

$$\begin{aligned}u_1 = \tfrac{3}{5}u_0 + 4 \Rightarrow 19 &= \tfrac{3}{5}u_0 + 4 \\ \Rightarrow 15 &= \tfrac{3}{5}u_0 \\ \Rightarrow \underline{\underline{u_0 = 25}}&.\end{aligned}$$

10. A hotel owner is designing signs showing the room numbers.

- Each sign is a rectangle with a right-angled triangle above it.
- The length and breadth of the rectangle are $5x$ centimetres and y centimetres respectively.
- The shorter sides of the triangle are $3x$ centimetres and $4x$ centimetres.



The area of the sign is 150 square centimetres.

(a) Show that the perimeter, P cm, of the sign is given by (3)

$$P = 9.6x + \frac{60}{x}.$$

Solution

Well,

$$\begin{aligned} \text{area} = \text{triangle} + \text{rectangle} &\Rightarrow 150 = [\tfrac{1}{2}(4x)(3x)] + 5xy \\ &\Rightarrow 150 = 6x^2 + 5xy \\ &\Rightarrow 5xy = 150 - 6x^2 \\ &\Rightarrow y = \frac{150 - 6x^2}{5x} \end{aligned}$$

and

$$\begin{aligned} P &= y + 4x + 3x + y + 5x \\ &= 2y + 12x \\ &= \frac{2(150 - 6x^2)}{5x} + 12x \\ &= \frac{300}{5x} - \tfrac{12}{5}x + 12x \\ &= \underline{\underline{9.6x + \frac{60}{x}}}, \end{aligned}$$

as required.

Each sign will be lit using a lighting strip placed around its perimeter.

The hotel owner requires the perimeter, P, of the sign to be as small as possible.

(b) Find the minimum value of P. (6)

Solution

Now,

$$\begin{aligned} P = 9.6x + \frac{60}{x} &\Rightarrow P = 9.6x + 60x^{-1} \\ &\Rightarrow \frac{\mathrm{d}P}{\mathrm{d}x} = 9.6 - 60x^{-2} \\ &\Rightarrow \frac{\mathrm{d}^2P}{\mathrm{d}x^2} = 120x^{-3}, \end{aligned}$$

and

$$\begin{aligned} \frac{\mathrm{d}P}{\mathrm{d}x} = 0 &\Rightarrow 9.6 - 60x^{-2} = 0 \\ &\Rightarrow 9.6 = 60x^{-2} \\ &\Rightarrow x^2 = 6.25 \\ &\Rightarrow x = 2.5 \\ &\Rightarrow \underline{\underline{P = 48 \text{ cm}}}. \end{aligned}$$

Is the value a minimum? Well,

$$x = 2.5 \Rightarrow \frac{\mathrm{d}^2P}{\mathrm{d}x^2} = 7.68 > 0;$$

hence, it is a minimum.

11. Solve (4)

$$3\sin 2x^\circ + 4\cos x^\circ = 0, \text{ for } x \leqslant x < 360.$$

Solution

Well,

$$\begin{aligned} 3\sin 2x^\circ + 4\cos x^\circ = 0 &\Rightarrow 3(2\sin x^\circ \cos x^\circ) + 4\cos x^\circ = 0 \\ &\Rightarrow 6\sin x^\circ \cos x^\circ + 4\cos x^\circ = 0 \\ &\Rightarrow 2\cos x^\circ(3\sin x^\circ + 2) = 0 \\ &\Rightarrow \cos x^\circ = 0 \text{ or } 3\sin x^\circ + 2 = 0 \\ &\Rightarrow \cos x^\circ = 0 \text{ or } \sin x^\circ = -\tfrac{2}{3}. \end{aligned}$$

$\underline{\cos x^\circ = 0}$:

$$\cos x^\circ = 0 \Rightarrow \underline{\underline{x = 90,\ 270}}.$$

$\underline{\sin x^\circ = -\frac{2}{3}}$:

$$\begin{aligned} \sin x^\circ = -\tfrac{2}{3} &\Rightarrow x = 221.810\,314\,9,\ 318.189\,685\,1 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 222,\ 318 \text{ (3 sf)}}}. \end{aligned}$$

12. Functions f and g are defined on the set of real numbers by:

- $\mathrm{f}(x) = x^5 + 3$ and
- $\mathrm{g}(x) = 1 - x^3$.

(a) Find an expression for $\mathrm{h}(x)$, where $\mathrm{h}(x) = \mathrm{f}(\mathrm{g}(x))$. (2)

Solution

Well,

$$\begin{aligned} \mathrm{h}(x) &= \mathrm{f}(\mathrm{g}(x)) \\ &= \mathrm{f}(1 - x^3) \\ &= \underline{\underline{(1 - x^3)^5 + 3}}. \end{aligned}$$

(b) Find $\mathrm{h}'(x)$. (2)

Solution

Now,

$$\begin{aligned} \mathrm{h}'(x) &= 5(1 - x^3)^4 \times (-3x^2) \\ &= \underline{\underline{-15x^2(1 - x^3)^4}}. \end{aligned}$$

13. A radioactive substance, which has been collected, decays over time.

The mass of the radioactive substance remaining is modelled by

$$M = 150\mathrm{e}^{-0.005\,4t},$$

where M is the mass, in micrograms, t years after the radioactive substance was collected.

(a) Determine the initial mass of the radioactive substance. (1)

Solution

Well,

$$t = 0 \Rightarrow \underline{\underline{M = 150 \text{ micrograms}}}.$$

(b) Calculate the time taken for the mass of the radioactive substance to decay to 120 micrograms. (4)

Solution

Now,

$$\begin{aligned} 150\mathrm{e}^{-0.005\,4t} = 120 &\Rightarrow \mathrm{e}^{-0.005\,4t} = \tfrac{4}{5} \\ &\Rightarrow \mathrm{e}^{0.005\,4t} = \tfrac{5}{4} \\ &\Rightarrow 0.005\,4t = \ln \tfrac{5}{4} \\ &\Rightarrow t = \tfrac{1}{0.005\,4} \ln \tfrac{5}{4} \\ &\Rightarrow t = 41.322\,879\,87 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{t = 41.3 \text{ years (3 sf)}}}. \end{aligned}$$

14. Circle C_1 has equation

$$(x + 5)^2 + (y - 6)^2 = 9.$$

(a) State the centre and radius of C_1. (2)

Solution

The centre is $\underline{\underline{(-5, 6)}}$ and the radius is $\underline{\underline{3}}$.

Circle C_2 has equation

$$x^2 + y^2 - 14x + 6y + 54 = 0.$$

(b) State the centre and radius of C_2. (2)

Solution

Now,

$$\begin{array}{ll} \text{coefficient of } x: & -14 \\ \text{half it:} & -7 \\ \text{square it:} & (-7)^2 = +49 \end{array}$$

and

$$\begin{array}{ll} \text{coefficient of } y: & +6 \\ \text{half it:} & +3 \\ \text{square it:} & (+3)^2 = +9 \end{array}$$

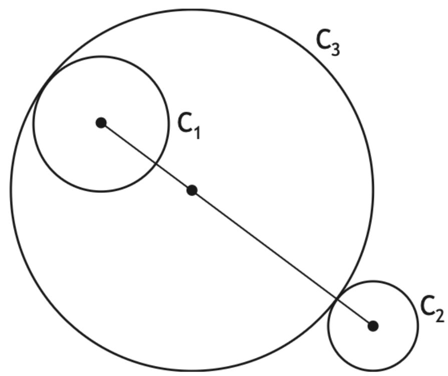
Next,

$$\begin{aligned} & x^2 + y^2 - 14x + 6y + 54 = 0 \\ \Rightarrow\quad & x^2 - 14x + y^2 + 6y = -54 \\ \Rightarrow\quad & (x^2 - 14x + 49) + (y^2 + 6y + 9) = -54 + 49 + 9 \\ \Rightarrow\quad & (x - 7)^2 + (y + 3)^2 = 4. \end{aligned}$$

so, the centre is $\underline{\underline{(7, -3)}}$ and the radius is $\underline{\underline{2}}$.

Circles C_1, C_2, and C_3 are touching as shown in the diagram.

The centre of circle C_3 lies on the line joining the centres of C_1 and C_2.



(c) Determine the equation of C_3. (3)

Solution

The distance between the centres of C_1 and C_2 is

$$\begin{aligned}\sqrt{[7-(-5)]^2+(-3-6)^2} &= \sqrt{12^2+9^2}\\ &= 15.\end{aligned}$$

Now,

$$\begin{aligned}\text{diameter} &= 15+3-2\\ &= 16\end{aligned}$$

and that means the radius of C_3 is 8.

Next, the ratio is

$$C_1C_3 : C_3C_2 = 5 : 10.$$

So the means the centre of C_3 is

$$(-5+\tfrac{1}{3}\times 12, 6+\tfrac{1}{3}\times(-9)) = (-1,3).$$

Finally, the equation of C_3 is

$$\underline{\underline{(x+1)^2+(y-3)^2=64}}.$$