Dr Oliver Mathematics
Further Pure Mathematics
Induction
Past Examination Questions

This booklet consists of 30 questions across a variety of examination topics.
The total number of marks available is 227.

1. $$\mathrm{f}(n) \equiv 24 \times 2^{4n} + 3^{4n},\ n \in \mathbb{N}.$$

   (a) Write down $\mathrm{f}(n+1) - \mathrm{f}(n)$. (5)

   (b) Prove, by induction, that $f(n)$ is divisible by 5 for all $n \in \mathbb{N}$. (4)

2. Prove, by induction, that (6)

$$\sum_{r=1}^{n} r(r+3) = \tfrac{1}{3}n(n+1)(n+5)$$

   for all $n \in \mathbb{N}$.

3. Prove, by induction, that $(\cos\theta + \mathrm{i}\sin\theta)^n \equiv \cos n\theta + \mathrm{i}\sin n\theta$, $n \in \mathbb{N}$.

4. Prove that the expression (8)

$$7^n + 4^n + 1$$

   is divisible by 6 for all $n \in \mathbb{N}$.

5. Prove that the expression (8)

$$4^n + 6n - 1$$

   is divisible by 9 for all $n \in \mathbb{Z}^+$.

6. Prove that the expression (8)

$$3^{4n-1} + 2^{4n-1} + 5$$

   is divisible by 10 for all positive integers n.

7. (a) Express $\dfrac{6x+10}{x+3}$ in the form $p + \dfrac{q}{x+3}$, where p and q are integers to be found. (1)

   The sequence of real numbers u_1, u_2, u_3, … is such that $u_1 = 5.2$ and $u_{n+1} = \dfrac{6u_n + 10}{u_n + 3}$.

   (b) Prove by induction that $u_n > 5$ for $n \in \mathbb{Z}^+$. (4)

8. Prove, by induction, that, for $n \in \mathbb{Z}^+$, (5)

$$\sum_{r=1}^{n} r2^r = 2\left[1 + (n-1)2^n\right].$$

9. (5)

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Prove by induction, that for all positive integers n,

$$\mathbf{A}^n = \begin{pmatrix} 1 & n & \frac{1}{2}(n^2+3n) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}.$$

10. Prove by induction, for $n \in \mathbb{Z}^+$, $\sum_{r=1}^{n}(2r-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$. (5)

11. De Moivre's theorem states that (5)

$$(\cos\theta + \mathrm{i}\sin\theta)^n \equiv \cos n\theta + \mathrm{i}\sin n\theta,\ n \in \mathbb{R}.$$

Use induction to prove de Moivre's theorem for $n \in \mathbb{Z}^+$.

12. Prove by induction, for $n \in \mathbb{Z}^+$, (5)

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}.$$

13. A series of positive integers $u_1, u_2, u_3, \ldots$ is defined by (5)

$$u_1 = 6 \text{ and } u_{n+1} = 6u_n - 5, \text{ for } n \geqslant 1.$$

Prove by induction that $u_n = 5 \times 6^{n-1} + 1$, for $n \geqslant 1$.

14. Prove by induction, for $n \in \mathbb{Z}^+$,

(a) $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^n = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}$, (5)

(b) $(4n+3)5^n - 3$ is divisible by 16. (5)

15. Prove by induction, for $n \in \mathbb{Z}^+$,

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 & 0 \\ n & 1 & 0 \\ n(n+2) & 2n & 1 \end{pmatrix}$, (5)

(b) $(2^{3n+1} + 5)$ is divisible by 7. (5)

16. Prove by induction, for $n \in \mathbb{Z}^+$,

(a) $\mathrm{f}(n) \equiv 5^n + 8n + 3$ is divisible by 4, (7)

(b) $\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^n = \begin{pmatrix} 2n+1 & -2n \\ 2n & 1-2n \end{pmatrix}$. (7)

17. A sequence of numbers is defined by (4)

$$u_1 = 2,$$
$$u_{n+1} = 5u_n - 4,\ n \geqslant 1.$$

Prove by induction that, $n \in \mathbb{Z}^+$, $u_n = 5^{n-1} + 1$.

18.

$$\mathrm{f}(n) \equiv 2^n + 6^n.$$

(a) Show that $\mathrm{f}(k+1) = 6\,\mathrm{f}(k) - 4 \times 2^k$. (3)

(b) Hence, or otherwise, prove by induction that, $n \in \mathbb{Z}^+$, $\mathrm{f}(n)$ is divisible by 8. (4)

19. A series of positive integers $u_1, u_2, u_3, \ldots$ is defined by (5)

$$u_1 = 2 \text{ and } u_{n+1} = 4u_n + 2, \text{ for } n \geqslant 1.$$

Prove by induction that $u_n = \frac{2}{3}(4^n - 1)$.

20. (a) $\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^n = \begin{pmatrix} 3^n & 0 \\ 3(3^n - 1) & 1 \end{pmatrix}$. (6)

(b) $f(n) \equiv 7^{2n-1} + 5$ is divisible by 12. (6)

21. A sequence can be described by the recurrence formula

$$u_1 = 1 \text{ and } u_{n+1} = 2u_n + 1, \text{ for } n \geqslant 1.$$

(a) Find u_2 and u_3. (2)

(b) Prove by induction that $u_n = 2^n - 1$. (5)

22. Prove by induction, for $n \in \mathbb{Z}^+$, (6)

$$\mathrm{f}(n) \equiv 2^{2n-1} + 3^{2n-1} \text{ is divisible by 5.}$$

23. (a) Prove by induction that, for $n \in \mathbb{Z}^+$, (6)

$$\sum_{r=1}^{n} r(r+3) = \tfrac{1}{3}n(n+1)(n+5).$$

(b) A sequence can be described by the recurrence formula (5)

$$u_1 = 1 \text{ and } u_{n+1} = u_n + n(3n+1), \text{ for } n \geqslant 1.$$

Prove by induction that

$$u_n = n^2(n-1) + 1.$$

24. (a) A sequence can be described by the recurrence formula (5)

$$u_1 = 8 \text{ and } u_{n+1} = 4u_n - 9n, \text{ for } n \geqslant 1.$$

Prove by induction that, for $n \in \mathbb{Z}^+$

$$u_n = 4^n + 3n + 1.$$

(b) Prove by induction that, for $m \in \mathbb{Z}^+$, (5)

$$\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^m = \begin{pmatrix} 2m+1 & -4m \\ m & 1-2m \end{pmatrix}.$$

25. Prove by induction that, $n \in \mathbb{Z}^+$, (6)

$$\sum_{r=1}^{n} r(2r-1) = \tfrac{1}{6}n(n+1)(4n-1).$$

26. Prove by induction that, $n \in \mathbb{Z}^+$,

$$\mathrm{f}(n) \equiv 8^n - 2^n$$

is divisible by 6.

27. (a) Prove by induction that, for $n \in \mathbb{Z}^+$, (5)

$$\sum_{r=1}^{n} (r+1)2^{r-1} = n2^n.$$

(b) A sequence can be described by the recurrence formula (7)

$$\begin{aligned} u_1 &= 0, \\ u_2 &= 32, \\ u_{n+2} &= 6u_{n+1} - 8u_n,\ n \geqslant 1. \end{aligned}$$

Prove by induction that, for $n \in \mathbb{Z}^+$,

$$u_n = 4^{n+1} - 2^{n+3}.$$

28. (a) Prove by induction that, $n \in \mathbb{Z}^+$, (6)

$$\begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^n - 1) & 5^n \end{pmatrix}.$$

(b) Prove by induction that, $n \in \mathbb{Z}^+$, (6)

$$\sum_{r=1}^{n} (2r-1)^2 = \tfrac{1}{3}n(4n^2 - 1).$$

29. (a) Prove by induction that, $n \in \mathbb{Z}^+$, (5)

$$\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(n+1)^2}.$$

(b) A sequence of positive rational numbers is defined by (5)

$$\begin{aligned} u_1 &= 3, \\ u_{n+1} &= \tfrac{1}{3}u_n + \tfrac{8}{9}, n \in \mathbb{Z}^+. \end{aligned}$$

Prove by induction that, $n \in \mathbb{Z}^+$,

$$u_n = 5 \times \left(\tfrac{1}{3}\right)^n + \tfrac{4}{3}.$$

30. (a) A sequence of numbers is defined by (6)

$$\begin{aligned} u_1 &= 6, \\ u_2 &= 27, \\ u_{n+2} &= 6u_{n+1} - 9u_n,\ n \geqslant 1. \end{aligned}$$

Prove by induction that, $n \in \mathbb{Z}^+$,

$$u_n = 3^n(n+1).$$

(b) Prove by induction that, $n \in \mathbb{Z}^+$, (6)

$$\mathrm{f}(n) = 3^{3n-2} + 2^{3n+1}$$

is divisible by 19.