Dr Oliver Mathematics
Further Mathematics
Further Differentiation and Integration
Past Examination Questions

This booklet consists of 54 questions across a variety of examination topics.
The total number of marks available is 427.

1. Given that $f(x) \equiv \dfrac{1}{\sqrt{x^2 + 4x - 12}}$,

   (a) find $\int f(x)\,dx$. (4)

   Solution

$$\begin{aligned}\int f(x)\,dx &= \int \frac{1}{\sqrt{x^2+4x-12}}\,dx\\ &= \int \frac{1}{\sqrt{(x^2+4x+4)-16}}\,dx\\ &= \int \frac{1}{\sqrt{(x+2)^2-4^2}}\,dx\\ &= \underline{\underline{\operatorname{arcosh}\left(\frac{x+2}{4}\right)+c}}.\end{aligned}$$

   (b) Hence find the exact value of (3)

$$\int_6^{10} f(x)\,dx,$$

   giving your answer as a single logarithm.

   Solution

$$\begin{aligned}\int_6^{10} f(x)\,dx &= \left[\operatorname{arcosh}\left(\frac{x+2}{4}\right)\right]_{x=6}^{10}\\ &= \operatorname{arcosh} 3 - \operatorname{arcosh} 2\\ &= \ln(3+\sqrt{8}) - \ln(2+\sqrt{3})\\ &= \underline{\underline{\ln\left(\frac{3+\sqrt{8}}{2+\sqrt{3}}\right)}}.\end{aligned}$$

2. (a) Using the substitution $u = \mathrm{e}^x$, find (6)

$$\int \operatorname{sech} x \, \mathrm{d}x.$$

Solution

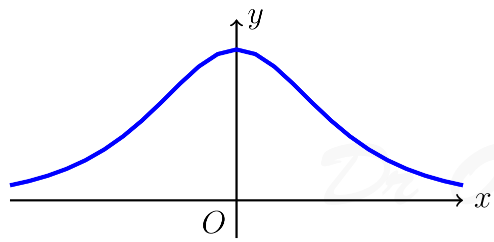
$$\begin{aligned}
\int \operatorname{sech} x \, \mathrm{d}x &= \int \frac{2}{\mathrm{e}^x + \mathrm{e}^{-x}} \, \mathrm{d}x \\
&= \int \frac{2\mathrm{e}^x}{\mathrm{e}^{2x} + 1} \, \mathrm{d}x \\
&= 2 \int \frac{\mathrm{e}^x}{(\mathrm{e}^x)^2 + 1} \, \mathrm{d}x
\end{aligned}$$

$$u = \mathrm{e}^x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x \Rightarrow \mathrm{d}u = \mathrm{e}^x \, \mathrm{d}x$$

$$\begin{aligned}
&= 2 \int \frac{1}{u^2 + 1} \, \mathrm{d}u \\
&= 2 \arctan u + c \\
&= \underline{\underline{2 \arctan(\mathrm{e}^x) + c}}.
\end{aligned}$$

(b) Sketch the curve $y = \operatorname{sech} x$. (1)

Solution



The finite region R is bounded by the curve with equation $y = \operatorname{sech} x$, the lines $x = 2$ and $x = -2$, and the x-axis.

(c) Using your result from part (a), find the area of R, giving your answer to 3 decimal places. (3)

Solution

$$\begin{aligned}\int_{-2}^{2} \operatorname{sech} x \, dx &= [2\arctan(e^x)]_{x=-2}^{2} \\ &= 2\arctan(e^2) - 2\arctan(e^{-2}) \\ &= 2.603\,520\,672 \text{ (FCD)} \\ &= \underline{\underline{2.604 \text{ (3 dp)}}}.\end{aligned}$$

3. (a) Simplify $(e^x + e^{-x})^2 - (e^x - e^{-x})^2$ and hence deduce that (4)

$$\cosh^2 x - \sinh^2 x = 1.$$

Solution

$$\begin{aligned}(e^x + e^{-x})^2 - (e^x - e^{-x})^2 &= (e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x}) \\ &= 4\end{aligned}$$

and

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \tfrac{1}{4}[(e^x + e^{-x})^2 - (e^x - e^{-x})^2] \\ &= \underline{\underline{1}}.\end{aligned}$$

(b) Given that $y = \operatorname{arsinh} x$, show that (4)

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 + 1}}.$$

Solution

$$\begin{aligned}y = \operatorname{arsinh} x \Rightarrow \sinh y &= x \\ \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y} &= \cosh y \\ \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y} &= \sqrt{\sinh^2 y + 1} \\ \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y} &= \sqrt{x^2+1} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \underline{\underline{\frac{1}{\sqrt{x^2+1}}}}.\end{aligned}$$

(c) Find $\int \operatorname{arsinh} x \, \mathrm{d}x$. (5)

Solution

$$u = \operatorname{arsinh} x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{\sqrt{x^2+1}} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = 1 \Rightarrow v = x$$

$$\begin{aligned}\int \operatorname{arsinh} x \, \mathrm{d}x &= \int (\operatorname{arsinh} x \times 1) \, \mathrm{d}x \\ &= x \operatorname{arsinh} x - \int \frac{x}{\sqrt{x^2+1}} \, \mathrm{d}x \\ &= \underline{\underline{x \operatorname{arsinh} x - \sqrt{x^2+1} + c}}.\end{aligned}$$

4. (a) Express $4x^2 + 4x + 26$ in the form $(px+q)^2 + r$, where p, q, and r are constants. (3)

Solution

$$\begin{aligned}4x^2 + 4x + 26 &= (4x^2 + 4x + 1) + 25 \\ &= \underline{\underline{(2x+1)^2 + 25}}.\end{aligned}$$

(b) Hence determine (3)

$$\int \frac{1}{\sqrt{4x^2+4x+26}} \, \mathrm{d}x.$$

Solution

$$\begin{aligned}\int \frac{1}{\sqrt{4x^2+4x+26}}\,\mathrm{d}x &= \int \frac{1}{\sqrt{(2x+1)^2+5^5}}\,\mathrm{d}x \\ &= \underline{\underline{\tfrac{1}{2}\operatorname{arsinh}\left(\frac{2x+1}{5}\right)+c}}.\end{aligned}$$

5. Find $\int x \operatorname{sech}^2 x \,\mathrm{d}x$.

Solution

$$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1 \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \operatorname{sech}^2 x \Rightarrow v = \tanh x$$

and

$$\begin{aligned}\int x\operatorname{sech}^2 x\,\mathrm{d}x &= x\tanh x - \int \tanh x\,\mathrm{d}x \\ &= \underline{\underline{x\tanh x - \ln(\cosh x) + c}}.\end{aligned}$$

6.

$$4x^2 + 4x + 5 \equiv (px+q)^2 + r.$$

(a) Find the values of p, q, and r. (3)

Solution

$$\begin{aligned}4x^2+4x+5 &\equiv (4x^2+4x+1)+4 \\ &\equiv \underline{\underline{(2x+1)^2+4}}.\end{aligned}$$

(b) Hence, or otherwise, find (4)

$$\int \frac{1}{4x^2+4x+5}\,\mathrm{d}x.$$

Solution

$$\int \frac{1}{4x^2+4x+5}\,\mathrm{d}x = \int \frac{1}{(2x+1)^2+2^2}\,\mathrm{d}x$$
$$= \underline{\underline{\frac{1}{4}\arctan\left(\frac{2x+1}{2}\right)+c.}}$$

(c) Show that (7)

$$\int \frac{2}{\sqrt{4x^2+4x+5}}\,\mathrm{d}x = \ln\left[(2x+1)+\sqrt{4x^2+4x+5}\right]+k,$$

where k is an arbitrary constant.

Solution

$$\int \frac{2}{\sqrt{4x^2+4x+5}}\,\mathrm{d}x = \int \frac{2}{\sqrt{(2x+1)^2+2^2}}\,\mathrm{d}x$$
$$= \operatorname{arsinh}\left(\frac{2x+1}{2}\right)+c$$
$$= \ln\left[\frac{2x+1}{2}+\sqrt{\left(\frac{2x+1}{2}\right)^2+1}\right]+c$$
$$= \ln\left[\frac{2x+1}{2}+\sqrt{\frac{4x^2+4x+1}{4}+1}\right]+c$$
$$= \ln\left[\frac{2x+1}{2}+\sqrt{\frac{4x^2+4x+5}{4}}\right]+c$$
$$= \ln\left[\tfrac{1}{2}(2x+1)+\tfrac{1}{2}\sqrt{4x^2+4x+5}\right]+c$$
$$= \ln\left[(2x+1)+\sqrt{4x^2+4x+5}\right]-\ln 2+c$$
$$= \underline{\underline{\ln\left[(2x+1)+\sqrt{4x^2+4x+5}\right]+k,}}$$

as required.

7.

$$4x^2+4x+5 \equiv (ax+b)^2+c, a>0.$$

(a) Find the values of a, b, and c. (3)

Solution

$$\begin{aligned}4x^2+4x+17 &\equiv (4x^2+4x+1)+16\\ &\equiv \underline{\underline{(2x+1)^2+16}}.\end{aligned}$$

(b) Find the exact value of (4)

$$\int_{-0.5}^{1.5} \frac{1}{4x^2+4x+17}\,\mathrm{d}x.$$

Solution

$$\begin{aligned}\int_{-0.5}^{1.5} \frac{1}{4x^2+4x+17}\,\mathrm{d}x &= \int_{-0.5}^{1.5} \frac{1}{(2x+1)^2+4^2}\,\mathrm{d}x\\ &= \left[\frac{1}{8}\arctan\left(\frac{2x+1}{4}\right)\right]_{x=-0.5}^{1.5}\\ &= \frac{1}{8}\arctan 1 - 0\\ &= \underline{\underline{\frac{\pi}{32}}}.\end{aligned}$$

8. Figure 1 shows the curve with parametric equations

$$x = a\cos^3\theta, y = a\sin^3\theta, 0 \leqslant \theta \leqslant 2\pi.$$

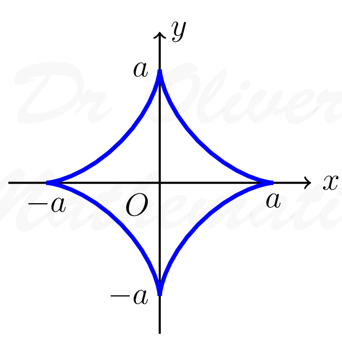


Figure 1: $x = a\cos^3\theta, y = a\sin^3\theta$

(a) Find the total length of this curve. (7)

Solution

$$\begin{aligned}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2} &= \sqrt{(-3a\cos^2\theta\sin\theta)^2+(3a\sin^2\theta\cos\theta)^2}\\ &= \sqrt{9a^2\cos^4\theta\sin^2\theta+9a^2\sin^4\theta\cos^2\theta}\\ &= \sqrt{9a^2\cos^2\theta\sin^2\theta(\cos^2\theta+\sin^2\theta)}\\ &= \sqrt{9a^2\cos^2\theta\sin^2\theta}\\ &= 3a\cos\theta\sin\theta.\end{aligned}$$

Now,

$$x=0\Rightarrow\theta=\frac{\pi}{2}\text{ and }x=a\Rightarrow\theta=0$$

and

$$\begin{aligned}\text{total length} &= 4\int_0^{\frac{\pi}{2}} 3a\cos\theta\sin\theta\,\mathrm{d}\theta\\ &= 6a\left[\sin^2\theta\right]_{\theta=0}^{\frac{\pi}{2}}\\ &= 6a(1-0)\\ &= \underline{\underline{6a}}.\end{aligned}$$

The curve is rotated through π radians about the x-axis.

(b) Find the area of the surface generated. (5)

Solution

$$\begin{aligned}\text{Area} &= 2\times 2\pi\int_0^{\frac{\pi}{2}}(a\sin^3\theta)(3a\cos\theta\sin\theta)\,\mathrm{d}\theta\\ &= 12a^2\pi\int_0^{\frac{\pi}{2}}\cos\theta\sin^4\theta\,\mathrm{d}\theta\\ &= \frac{12a^2\pi}{5}\left[\sin^5\theta\right]_{\theta=0}^{\frac{\pi}{2}}\\ &= \frac{12a^2\pi}{5}(1-0)\\ &= \underline{\underline{\frac{12a^2\pi}{5}}}.\end{aligned}$$

9. (a) Find (5)

$$\int \frac{1+x}{\sqrt{1-4x^2}}\,\mathrm{d}x.$$

Solution

$$\begin{aligned}\int \frac{1+x}{\sqrt{1-4x^2}}\,\mathrm{d}x &= \int \frac{1}{\sqrt{1-4x^2}}\,\mathrm{d}x + \int \frac{x}{\sqrt{1-4x^2}}\,\mathrm{d}x \\ &= \int \frac{1}{\sqrt{1-(2x)^2}}\,\mathrm{d}x + \int \frac{x}{\sqrt{1-4x^2}}\,\mathrm{d}x \\ &= \underline{\underline{\tfrac{1}{2}\arcsin(2x) - \tfrac{1}{4}\sqrt{1-4x^2} + c}}.\end{aligned}$$

(b) Find, to 3 decimal places, the value of (2)

$$\int_0^{0.3} \frac{1+x}{\sqrt{1-4x^2}}\,\mathrm{d}x.$$

Solution

$$\begin{aligned}\int_0^{0.3} \frac{1+x}{\sqrt{1-4x^2}}\,\mathrm{d}x &= \left[\tfrac{1}{2}\arcsin(2x) - \tfrac{1}{4}\sqrt{1-4x^2}\right]_{x=0}^{0.3} \\ &= \left(\tfrac{1}{2}\arcsin 0.6 - \tfrac{1}{5}\right) - \left(0 - \tfrac{1}{4}\right) \\ &= 0.371\,750\,554\,4 \text{ (FCD)} \\ &= \underline{\underline{0.372 \text{ (3 dp)}}}.\end{aligned}$$

10. Figure 2 shows the curve with parametric equations (7)

$$x = a\cos^3 t, y = a\sin^3 t, 0 \leqslant t \leqslant \tfrac{\pi}{2},$$

where a is a positive constant.

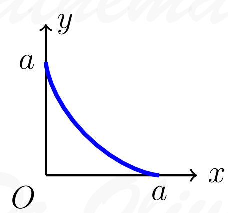


Figure 2: $x = a\cos^3 t, y = a\sin^3 t$

The curve is rotated through 2π radians about the x-axis. Find the area of the surface generated.

Solution

$$\begin{aligned}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} &= \sqrt{(-3a\cos^2 t\sin t)^2+(3a\sin^2 t\cos t)^2}\\ &= \sqrt{9a^2\cos^4 t\sin^2 t+9a^2\sin^4 t\cos^2 t}\\ &= \sqrt{9a^2\cos^2 t\sin^2 t(\cos^2 t+\sin^2 t)}\\ &= \sqrt{9a^2\cos^2 t\sin^2 t}\\ &= 3a\cos t\sin t.\end{aligned}$$

Now,

$$x=0\Rightarrow\theta=\frac{\pi}{2}\text{ and }x=a\Rightarrow\theta=0$$

and

$$\begin{aligned}\text{area} &= 2\pi\int_0^{\frac{\pi}{2}}(a\sin^3 t)(3a\cos t\sin t)\,\mathrm{d}t\\ &= 2\pi\int_0^{\frac{\pi}{2}}\cos t\sin^4 t\,\mathrm{d}t\\ &= \frac{6a^2\pi}{5}\left[\sin^5 t\right]_{t=0}^{\frac{\pi}{2}}\\ &= \frac{6a^2\pi}{5}(1-0)\\ &= \underline{\underline{\frac{6a^2\pi}{5}}}.\end{aligned}$$

11. Figure 3 shows a sketch of the curve with equation

$$y=x\,\mathrm{arcosh}\,x,\ 1\leqslant x\leqslant 2.$$

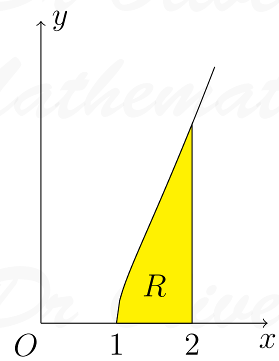


Figure 3: $y = x \operatorname{arcosh} x$

The region R, as shown shaded in the figure, is bounded by the curve, the x-axis, and the $x = 2$. Show that the area of R is

$$\tfrac{7}{4}\ln(2+\sqrt{3}) - \tfrac{\sqrt{3}}{2}.$$

Solution

$$u = \operatorname{arcosh} x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{\sqrt{x^2-1}} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = x \Rightarrow v = \tfrac{1}{2}x^2,$$

$$\theta = \operatorname{arcosh} x \Rightarrow x = \cosh\theta \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = \sinh\theta,$$

and

$$\theta = \operatorname{arcosh} 1 = 0 \text{ and } \theta = \operatorname{arcosh} 2.$$

$$
\begin{aligned}
\int_1^2 x \operatorname{arcosh} x \,\mathrm{d}x &= \left[\tfrac{1}{2}x^2 \operatorname{arcosh} x\right]_{x=1}^2 - \int_1^2 \frac{x^2}{2\sqrt{x^2-1}}\,\mathrm{d}x \\
&= \left(2\operatorname{arcosh} 2 - \tfrac{1}{2}\operatorname{arcosh} 1\right) - \int_0^{\operatorname{arcosh} 2} \frac{\cosh^2\theta}{2\sqrt{\cosh^2\theta - 1}}\sinh\theta\,\mathrm{d}\theta \\
&= 2\operatorname{arcosh} 2 - \int_0^{\operatorname{arcosh} 2} \frac{\sinh\theta\cosh^2\theta}{2\sqrt{\sinh^2\theta}}\,\mathrm{d}\theta \\
&= 2\operatorname{arcosh} 2 - \int_0^{\operatorname{arcosh} 2} \frac{\sinh\theta\cosh^2\theta}{2\sinh\theta}\,\mathrm{d}\theta \\
&= 2\operatorname{arcosh} 2 - \tfrac{1}{2}\int_0^{\operatorname{arcosh} 2} \cosh^2\theta\,\mathrm{d}\theta \\
&= 2\operatorname{arcosh} 2 - \tfrac{1}{4}\int_0^{\operatorname{arcosh} 2} (1+\cosh 2\theta)\,\mathrm{d}\theta \\
&= 2\operatorname{arcosh} 2 - \tfrac{1}{4}\left[\theta + \tfrac{1}{2}\sinh 2\theta\right]_{\theta=0}^{\operatorname{arcosh} 2} \\
&= 2\operatorname{arcosh} 2 - \tfrac{1}{4}\left\{\left(\operatorname{arcosh} 2 + \tfrac{1}{2}\sinh(2\operatorname{arcosh} 2)\right) - (0+0)\right\} \\
&= \tfrac{7}{4}\operatorname{arcosh} 2 - \tfrac{1}{8}\sinh(2\operatorname{arcosh} 2).
\end{aligned}
$$

So, how do we do $\sinh(2\operatorname{arcosh} 2)$? Well,

$$
\begin{aligned}
\sinh(2\operatorname{arcosh} 2) &= 2\sinh(\operatorname{arcosh} 2)\cosh(\operatorname{arcosh} 2) \\
&= 4\sinh(\operatorname{arcosh} 2) \\
&= 4\sinh(\ln(2+\sqrt{3})) \\
&= 4 \times \left[\tfrac{1}{2}\left(\mathrm{e}^{\ln(2+\sqrt{3})} - \mathrm{e}^{-\ln(2+\sqrt{3})}\right)\right] \\
&= 2\left(2+\sqrt{3} - \frac{1}{2+\sqrt{3}}\right) \\
&= 2 \times 2\sqrt{3} \\
&= 4\sqrt{3}.
\end{aligned}
$$

Now,

$$
\begin{aligned}
\int_1^2 x\operatorname{arcosh} x\,\mathrm{d}x &= \tfrac{7}{4}\operatorname{arcosh} 2 - \tfrac{1}{8}\times 4\sqrt{3} \\
&= \tfrac{7}{4}\operatorname{arcosh} 2 - \tfrac{\sqrt{3}}{2} \\
&= \underline{\underline{\tfrac{7}{4}\ln(2+\sqrt{3}) - \tfrac{\sqrt{3}}{2}}}.
\end{aligned}
$$

12. (a) Show that, for $0 < x \leqslant 1$, (3)

$$\ln\left(\frac{1-\sqrt{1-x^2}}{x}\right) = -\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right).$$

Solution

$$\begin{aligned}
\ln\left(\frac{1-\sqrt{1-x^2}}{x}\right) &= \ln\left(\frac{1-\sqrt{1-x^2}}{x} \times \frac{1+\sqrt{1-x^2}}{1+\sqrt{1-x^2}}\right) \\
&= \ln\left(\frac{1-(1-x^2)}{x(1+\sqrt{1-x^2})}\right) \\
&= \ln\left(\frac{x}{1+\sqrt{1-x^2}}\right) \\
&= \underline{\underline{-\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)}}.
\end{aligned}$$

(b) Using the definitions of $\cosh x$ or $\operatorname{sech} x$ in terms of exponentials, for $0 < x \leqslant 1$, (5)

$$\operatorname{arsech} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right).$$

Solution

$$\begin{aligned}
y = \operatorname{arsech} x &\Rightarrow x = \operatorname{sech} y \\
&\Rightarrow x = \frac{2}{\mathrm{e}^y + \mathrm{e}^{-y}} \\
&\Rightarrow x\mathrm{e}^{2y} - 2\mathrm{e}^y + x = 0 \\
&\Rightarrow \mathrm{e}^y = \frac{2+\sqrt{4-4x^2}}{2x} \text{ (since } 0 < x \leqslant 1) \\
&\Rightarrow \mathrm{e}^y = \frac{1+\sqrt{1-x^2}}{x} \\
&\Rightarrow \underline{\underline{y = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)}}.
\end{aligned}$$

13. Evaluate

$$\int_1^4 \frac{1}{\sqrt{x^2-2x+17}}\,\mathrm{d}x,$$

giving your answer as an exact logarthim.

Solution

$$\begin{aligned}\int_1^4 \frac{1}{\sqrt{x^2-2x+17}}\,\mathrm{d}x &= \int_1^4 \frac{1}{\sqrt{(x-1)^2+4^2}}\,\mathrm{d}x\\ &= \left[\operatorname{arsinh}\left(\tfrac{x-1}{4}\right)\right]_{x=1}^4\\ &= \operatorname{arsinh}\tfrac{3}{4} - 0\\ &= \underline{\underline{\ln 2}}.\end{aligned}$$

14. (a) Show that $\operatorname{artanh}(\sin\frac{\pi}{4}) = \ln(1+\sqrt{2})$. (3)

Solution

$$\begin{aligned}\operatorname{artanh}\left(\sin\frac{\pi}{4}\right) &= \operatorname{artanh}\left(\frac{1}{\sqrt{2}}\right)\\ &= \frac{1}{2}\ln\left(\frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}\right)\\ &= \tfrac{1}{2}\ln(3+2\sqrt{2})\\ &= \underline{\underline{\ln(1+\sqrt{2})}}.\end{aligned}$$

(b) Given that $y = \operatorname{artanh}(\sin x)$, show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \sec x$. (2)

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1-\sin^2 x}\times\cos x = \frac{1}{\cos x} = \underline{\underline{\sec x}}.$$

(c) Find the exact value of $\displaystyle\int_0^{\frac{\pi}{4}} \sin x \operatorname{artanh}(\sin x)\,\mathrm{d}x$. (5)

Solution

$$u = \operatorname{artanh}(\sin x) \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \sec x \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \sin x \Rightarrow v = -\cos x$$

and

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \sin x \operatorname{artanh}(\sin x)\,\mathrm{d}x &= \left[-\cos x \operatorname{artanh}(\sin x)\right]_{x=0}^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} 1\,\mathrm{d}x \\ &= \left(-\tfrac{1}{\sqrt{2}}\ln(1+\sqrt{2}) - 0\right) + \left[x\right]_{x=0}^{\frac{\pi}{4}} \\ &= -\tfrac{1}{\sqrt{2}}\ln(1+\sqrt{2}) + \left(\tfrac{\pi}{4} - 0\right) \\ &= \underline{\underline{\tfrac{\pi}{4} - \tfrac{1}{\sqrt{2}}\ln(1+\sqrt{2})}}.\end{aligned}$$

15. Figure 4 shows a sketch of the curve with equation (10)

$$y = x^2 \operatorname{arsinh} x.$$

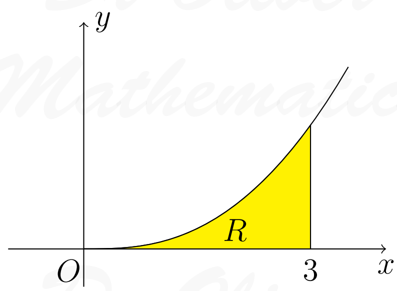


Figure 4: $y = x^2 \operatorname{arsinh} x$

The region R, as shown shaded in the figure, is bounded by the curve, the x-axis, and the $x = 3$. Show that the area of R is

$$9\ln(3+\sqrt{10}) - \tfrac{1}{9}(2+7\sqrt{10}).$$

Solution

$$u = \operatorname{arsinh} x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{\sqrt{x^2+1}} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = x^2 \Rightarrow v = \tfrac{1}{3}x^3,$$

$$u = x^2 + 1 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x \Rightarrow \mathrm{d}u = 2x\,\mathrm{d}x,$$

and

$$x = 0, u = 1 \text{ and } x = 3, u = 10.$$

Now,

$$\begin{aligned}
\int_0^3 x^2 \operatorname{arsinh} x \, \mathrm{d}x &= \left[\tfrac{1}{3}x^3 \operatorname{arsinh} x\right]_{x=0}^3 - \int_0^3 \frac{x^3}{3\sqrt{x^2+1}} \, \mathrm{d}x \\
&= (9 \operatorname{arsinh} 3 - 0) - \tfrac{1}{3} \int_1^{10} \frac{(u-1)^{\frac{3}{2}}}{u^{\frac{1}{2}} \times 2(u-1)^{\frac{1}{2}}} \, \mathrm{d}u \\
&= 9 \operatorname{arsinh} 3 - \tfrac{1}{6} \int_1^{10} \frac{u-1}{u^{\frac{1}{2}}} \, \mathrm{d}u \\
&= 9 \operatorname{arsinh} 3 - \tfrac{1}{6} \int_1^{10} (u^{\frac{1}{2}} - u^{-\frac{1}{2}}) \, \mathrm{d}u \\
&= 9 \operatorname{arsinh} 3 - \tfrac{1}{6} \left[\tfrac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right]_{u=1}^{10} \\
&= 9 \operatorname{arsinh} 3 - \tfrac{1}{6} \left\{(\tfrac{20}{3}\sqrt{10} - 2\sqrt{10}) - (\tfrac{2}{3} - 2)\right\} \\
&= 9 \operatorname{arsinh} 3 - \tfrac{1}{6}(\tfrac{14}{3}\sqrt{10} + \tfrac{4}{3}) \\
&= \underline{\underline{9 \ln(3+\sqrt{10}) - \tfrac{1}{9}(7\sqrt{10}+2)}}.
\end{aligned}$$

16. The curve C has parametric equations

$$x = t - \ln t, y = 4\sqrt{t}, 1 \leqslant t \leqslant 4.$$

(a) Show that the length of C is $3 + \ln 4$. (7)

Solution

$$\begin{aligned}
\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} &= \sqrt{\left(1 - \frac{1}{t}\right)^2 + \left(\frac{2}{t^{\frac{1}{2}}}\right)^2} \\
&= \sqrt{1 - \frac{2}{t} + \frac{1}{t^2} + \frac{4}{t}} \\
&= \sqrt{1 + \frac{2}{t} + \frac{1}{t^2}} \\
&= \sqrt{\left(1 + \tfrac{1}{t}\right)^2} \\
&= 1 + \frac{1}{t}
\end{aligned}$$

and

$$\begin{aligned}\text{length} &= \int_1^4 \left(1 + \frac{1}{t}\right)\,\mathrm{d}t \\ &= [t + \ln|t|]_{t=1}^4 \\ &= (4 + \ln 4) - (1 - 0) \\ &= \underline{\underline{3 + \ln 4}}.\end{aligned}$$

The curve is rotated through 2π radians about the x-axis.

(b) Find the area of the curved surface generated. (4)

Solution

$$\begin{aligned}\text{Area} &= 2\pi \int_1^4 4\sqrt{t}\left(1 + \frac{1}{t}\right)\,\mathrm{d}t \\ &= 2\pi \int_1^4 \left(4t^{\frac{1}{2}} + 4t^{-\frac{1}{2}}\right)\,\mathrm{d}t \\ &= 2\pi \left[\tfrac{8}{3}t^{\frac{3}{2}} + 8t^{\frac{1}{2}}\right]_{t=1}^4 \\ &= 2\pi\left[\left(\tfrac{64}{3} + 16\right) - \left(\tfrac{8}{3} + 8\right)\right] \\ &= \underline{\underline{\tfrac{160\pi}{3}}}.\end{aligned}$$

17. Evaluate $\displaystyle\int_1^3 \frac{1}{\sqrt{x^2 + 4x - 5}}\,\mathrm{d}x$, giving your answer as a natural logarithm. (5)

Solution

$$\begin{aligned}\int_1^3 \frac{1}{\sqrt{x^2 + 4x - 5}}\,\mathrm{d}x &= \int_1^3 \frac{1}{\sqrt{(x+2)^2 - 3^2}}\,\mathrm{d}x \\ &= \left[\operatorname{arcosh}\left(\frac{x+2}{3}\right)\right]_{x=1}^3 \\ &= \operatorname{arcosh}\tfrac{5}{3} - \operatorname{arcosh} 1 \\ &= \underline{\underline{\ln 3}}.\end{aligned}$$

18. The curve C has equation (7)

$$y = \tfrac{1}{4}(2x^2 - \ln x), x > 0.$$

Find the length of C from $x = 0.5$ to $x = 2$, giving your answer in the form $a + b\ln 2$, where a and b are rational numbers.

Solution

$$\begin{aligned}\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} &= \sqrt{1 + \left(x - \frac{1}{4x}\right)^2} \\ &= \sqrt{1 + \left(x^2 - \frac{1}{2} + \frac{1}{16x^2}\right)^2} \\ &= \sqrt{x^2 + \frac{1}{2} + \frac{1}{16x^2}} \\ &= \sqrt{\left(x + \frac{1}{4x}\right)^2} \\ &= x + \frac{1}{4x}\end{aligned}$$

and

$$\begin{aligned}\text{length} &= \int_{0.5}^{2} \left(x + \frac{1}{4x}\right) \mathrm{d}x \\ &= \left[\tfrac{1}{2}x^2 + \tfrac{1}{4}\ln|x|\right]_{x=0.5}^{2} \\ &= (2 + \tfrac{1}{4}\ln 2) - (\tfrac{1}{8} + \tfrac{1}{4}\ln\tfrac{1}{2}) \\ &= \underline{\underline{\tfrac{15}{8} + \tfrac{1}{2}\ln 2}}.\end{aligned}$$

19. Figure 5 shows a sketch of the curve with equation

$$y = \operatorname{arsinh}\sqrt{x},\ x \geqslant 0.$$

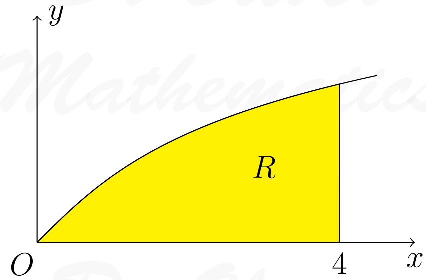


Figure 5: $y = \operatorname{arsinh}\sqrt{x}$

(a) Find the gradient of C at the point where $x = 4$. (3)

Solution

$$\begin{aligned} y = \operatorname{arsinh}\sqrt{x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\sqrt{1 + (\sqrt{x})^2}} \times \tfrac{1}{2}x^{-\frac{1}{2}} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{2\sqrt{x}\sqrt{1+x}} \end{aligned}$$

and

$$x = 4 \Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{5}}{20}}}.$$

The region R, as shown shaded in the figure, is bounded by the curve, the x-axis, and the $x = 4$.

(b) Using the substitution $x = \sinh^2\theta$, or otherwise, show that the area of R is (10)

$$k \ln(2 + \sqrt{5}) - \sqrt{5},$$

where k is a constant to be found.

Solution

$$\begin{aligned} \theta = \operatorname{arsinh}\sqrt{x} \Rightarrow x &= \sinh^2\theta \\ \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} &= 2\sinh\theta\cosh\theta \\ \Rightarrow \mathrm{d}x &= 2\sinh\theta\cosh\theta\,\mathrm{d}\theta, \end{aligned}$$

$$x = 0 \Rightarrow \theta = 0 \text{ and } x = 2 \Rightarrow \theta = \operatorname{arsinh} 2,$$

$$u = \theta \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1 \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \sinh 2\theta \Rightarrow v = \tfrac{1}{2}\cosh 2\theta,$$

and so we have

$$\begin{aligned}\int_0^4 \operatorname{arsinh}\sqrt{x}\,\mathrm{d}x &= \int_0^{\operatorname{arsinh} 2} \theta(2\sinh\theta\cosh\theta)\,\mathrm{d}\theta \\ &= \int_0^{\operatorname{arsinh} 2} \theta\sinh 2\theta\,\mathrm{d}\theta \\ &= \left[\tfrac{1}{2}\theta\cosh 2\theta\right]_{\theta=0}^{\operatorname{arsinh} 2} - \tfrac{1}{2}\int_0^{\operatorname{arsinh} 2} \cosh 2\theta\,\mathrm{d}\theta \\ &= \left[\tfrac{1}{2}\theta\cosh 2\theta - \tfrac{1}{4}\sinh 2\theta\right]_{\theta=0}^{\operatorname{arsinh} 2} \\ &= \left[\tfrac{1}{2}\theta(1+2\sinh^2\theta) - \tfrac{1}{2}\sinh\theta\cosh\theta\right]_{\theta=0}^{\operatorname{arsinh} 2} \\ &= \left[\tfrac{1}{2}\theta(1+2\sinh^2\theta) - \tfrac{1}{2}\sinh\theta\sqrt{1+\sinh^2\theta}\right]_{\theta=0}^{\operatorname{arsinh} 2} \\ &= (\tfrac{9}{2}\operatorname{arsinh} 2 - \tfrac{1}{2}\times 2\times\sqrt{5}) - (0-0) \\ &= \underline{\underline{\tfrac{9}{2}\ln(2+\sqrt{5}) - \sqrt{5}}}.\end{aligned}$$

20. Show that (4)

$$\frac{\mathrm{d}}{\mathrm{d}x}[\ln(\tanh x)] = 2\operatorname{cosech} 2x,\ x > 0.$$

Solution

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}[\ln(\tanh x)] &= \frac{\operatorname{sech}^2 x}{\tanh x} \\ &= \frac{1}{\sinh x\cosh x} \\ &= \frac{2}{2\sinh x\cosh x} \\ &= \frac{2}{\sinh 2x} \\ &= \underline{\underline{2\operatorname{cosech} 2x}}.\end{aligned}$$

21. Show that (7)

$$\int_5^6 \frac{3+x}{\sqrt{x^2-9}}\,\mathrm{d}x = 3\ln\left(\frac{2+\sqrt{3}}{3}\right) + 3\sqrt{3} - 4.$$

Solution

$$\begin{aligned}
\int_5^6 \frac{3+x}{\sqrt{x^2-9}}\,\mathrm{d}x &= \int_5^6 \frac{3}{\sqrt{x^2-9}}\,\mathrm{d}x + \int_5^6 \frac{x}{\sqrt{x^2-9}}\,\mathrm{d}x \\
&= \int_5^6 \frac{3}{\sqrt{x^2-3^2}}\,\mathrm{d}x + \left[\sqrt{x^2-9}\right]_{x=5}^6 \\
&= \left[3\operatorname{arcosh}\tfrac{x}{3} + \sqrt{x^2-9}\right]_{x=5}^6 \\
&= \left[3\ln\left(\frac{x+\sqrt{x^2-9}}{3}\right) + \sqrt{x^2-9}\right]_{x=5}^6 \\
&= (3\ln(6+3\sqrt{3}) + 3\sqrt{3}) - (3\ln 9 + 4) \\
&= \underline{\underline{3\ln(2+\sqrt{3}) + 3\sqrt{3} - 4}}.
\end{aligned}$$

22. The curve C has equation $y = \operatorname{arsinh}(x^3)$, $x \geqslant 0$. The point P on C has x-coordinate $\sqrt{2}$. Show that an equation of the tangent to C at P is (5)

$$y = 2x - 2\sqrt{2} + \ln(3 + 2\sqrt{2}).$$

Solution

Well, $x = \sqrt{2} \Rightarrow y = \operatorname{arsinh}(2\sqrt{2}) = \ln(3+2\sqrt{2})$. Now,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2}{(x^3)^2+1} = \frac{3x^2}{x^6+1} \text{ and } x = \sqrt{2} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2$$

and we have

$$y - \ln(3+2\sqrt{2}) = 2(x - \sqrt{2}) \Rightarrow \underline{\underline{y = 2x - 2\sqrt{2} + \ln(3+2\sqrt{2})}}.$$

23. Figure 6 shows a sketch of the curve with equation

$$y = \tfrac{1}{10}\cosh x \arctan(\sinh x),\ x \geqslant 0.$$

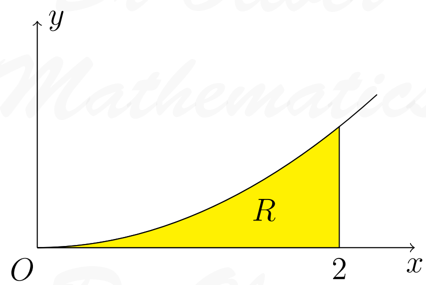


Figure 6: $y = \frac{1}{10}\cosh x \arctan(\sinh x)$

The region R, as shown shaded in the figure, is bounded by the curve, the x-axis, and the $x = 2$.

(a) Find $\displaystyle\int \cosh x \arctan(\sinh x)\,\mathrm{d}x$. (5)

Solution

$$u = \arctan(\sinh x) \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\cosh x}{\sinh^2 x + 1} = \frac{1}{\cosh x},$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \cosh x \Rightarrow v = \sinh x,$$

and

$$\begin{aligned}\int \cosh x \arctan(\sinh x)\,\mathrm{d}x &= \sinh x \arctan(\sinh x) - \int \frac{\sinh x}{\cosh x}\,\mathrm{d}x \\ &= \underline{\underline{\sinh x \arctan(\sinh x) - \ln(\cosh x) + c}}.\end{aligned}$$

(b) Hence show that, to 2 significant figures, the area of R is 0.34. (2)

Solution

$$\begin{aligned}\int_0^2 \tfrac{1}{10}\cosh x \arctan(\sinh x)\,\mathrm{d}x &= \tfrac{1}{10}\left[\sinh x \arctan(\sinh x) - \ln(\cosh x)\right]_{x=0}^2 \\ &= (\sinh 2 \arctan(\sinh 2) - \ln(\cosh 2)) - (0 - 0) \\ &= 0.339\,630\,027\,6 \text{ (FCD)} \\ &= \underline{\underline{0.34 \text{ (2 sf)}}}.\end{aligned}$$

24. The curve C has parametric equations

$$x = 3(t + \sin t), y = 3(1 - \cos t), 0 \leqslant t < \pi.$$

(a) Show that (3)

$$\frac{dy}{dx} = \tan\frac{t}{2}.$$

Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt} \div \frac{dx}{dt} \\ &= \frac{3\sin t}{3 + 3\cos t} \\ &= \frac{\sin t}{1 + \cos t} \\ &= \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{1 + (2\cos^2\frac{t}{2} - 1)} \\ &= \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\cos^2\frac{t}{2}} \\ &= \frac{\sin\frac{t}{2}}{\cos\frac{t}{2}} \\ &= \underline{\underline{\tan\tfrac{t}{2}}}.\end{aligned}$$

The arc length s of C is measured from the origin O.

(b) Show that $s = 12\sin\frac{t}{2}$. (4)

Solution

$$
\begin{aligned}
s &= \int \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}t \\
&= \int \sqrt{(3 + 3\cos t)^2 + (3\sin t)^2}\,\mathrm{d}t \\
&= \int \sqrt{(9 + 18\cos t + 9\cos^2 t) + 9\sin^2 t}\,\mathrm{d}t \\
&= \int \sqrt{18 + 18\cos t}\,\mathrm{d}t \\
&= 3\sqrt{2}\int \sqrt{1 + \cos t}\,\mathrm{d}t \\
&= 3\sqrt{2}\int \sqrt{1 + (2\cos^2 \tfrac{t}{2} - 1)}\,\mathrm{d}t \\
&= 3\sqrt{2}\int \sqrt{2\cos^2 \tfrac{t}{2})}\,\mathrm{d}t \\
&= 6\int \cos\tfrac{t}{2}\,\mathrm{d}t.
\end{aligned}
$$

Now,

$$
\begin{aligned}
s &= 6\int_0^t \cos\tfrac{u}{2}\,\mathrm{d}u \\
&= 6\left[2\sin\tfrac{u}{2}\right]_{u=0}^{t} \\
&= 6(2\sin\tfrac{t}{2} - 0) \\
&= \underline{\underline{12\sin\tfrac{t}{2}}}.
\end{aligned}
$$

The point P lies on C and the arc OP of C has length L. The arc OP is rotated through 2π radians about the x-axis.

(c) Show that the area of the curved surface generated is given by $\dfrac{\pi L^3}{36}$. (7)

Solution

$$
\begin{aligned}
&2\pi \int y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t \\
&= 2\pi \int [3(1-\cos t)][3\sqrt{2}\sqrt{1+\cos t}]\,\mathrm{d}t \\
&= 18\sqrt{2}\pi \int (1-\cos t)\sqrt{1+\cos t}\,\mathrm{d}t \\
&= 18\sqrt{2}\pi \int [1-(1-2\sin^2 \tfrac{t}{2})][\sqrt{2}\cos\tfrac{t}{2}]\,\mathrm{d}t \\
&= 72\pi \int \sin^2\tfrac{t}{2}\cos\tfrac{t}{2}\,\mathrm{d}t
\end{aligned}
$$

and

$$
\begin{aligned}
\text{area} &= 72\pi\left[\tfrac{2}{3}\sin^3\tfrac{u}{2}\right]_{u=0}^{t} \\
&= \tfrac{144\pi}{3}\sin^3\tfrac{t}{2}.
\end{aligned}
$$

Now,

$$\sin\tfrac{t}{2} = \frac{s}{12} = \frac{L}{12}$$

which means

$$\text{area} = \frac{144\pi}{3} \times \frac{L^3}{12^3} = \underline{\underline{\frac{\pi L^3}{36}}}.$$

25. (5)

$$y = (\operatorname{arsinh} 2x)^2.$$

Find the exact value of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at $x = \frac{1}{2}$, giving your answer in the form $a \ln b$, where a and b are real numbers.

Solution

$$
\begin{aligned}
y = (\operatorname{arsinh} 2x)^2 &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2\operatorname{arsinh} 2x \times \frac{1}{\sqrt{(2x)^2+1}} \times 2 \\
&\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4\operatorname{arsinh} 2x}{\sqrt{4x^2+1}}
\end{aligned}
$$

and

$$x = \tfrac{1}{2} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2\sqrt{2}\,\operatorname{arsinh} 1$$

$$\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = 2\sqrt{2}\ln(1+\sqrt{2})}}.$$

26. A curve has parametric equations (9)

$$x = 2t^3, y = 3t^2, 0 \leqslant t \leqslant 1.$$

The curve is rotated through 2π radians about the x-axis.
Prove that the area of the curved surface generated is $\frac{24\pi}{5}(\sqrt{2}+1)$.

Solution

$$\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} = \sqrt{(6t^2)^2 + (6t)^2}$$
$$= \sqrt{36t^4 + 36t^2}$$
$$= 6t\sqrt{t^2+1}$$

and

$$\text{area} = 2\pi \int_0^1 (3t^2)(6t\sqrt{t^2+1})\,\mathrm{d}t$$
$$= 36\pi \int_0^1 t^3\sqrt{t^2+1}\,\mathrm{d}t.$$

Now,

$$u^2 = t^2 + 1 \Rightarrow 2u\,\frac{\mathrm{d}u}{\mathrm{d}t} = 2t\,\mathrm{d}t \Rightarrow u\,\mathrm{d}u = t\,\mathrm{d}t$$

and

$$t = 0 \Rightarrow u = 1 \text{ and } t = 1 \Rightarrow u = \sqrt{2}$$

and this leads to

$$\begin{aligned}
\text{area} &= 36\pi \int_0^1 t^3\sqrt{t^2+1}\,\mathrm{d}t \\
&= 36\pi \int_0^1 t^2\sqrt{t^2+1}\,t\,\mathrm{d}t \\
&= 36\pi \int_1^{\sqrt{2}} (u^2-1)u^2\,\mathrm{d}u \\
&= 36\pi \left[\frac{u^5}{5} - \frac{u^3}{3}\right]_{u=1}^{\sqrt{2}} \\
&= 36\pi \left[\left(\frac{4\sqrt{2}}{5} - \frac{2\sqrt{2}}{3}\right) - \left(\frac{1}{5} - \frac{1}{3}\right)\right] \\
&= 36\pi \left(\frac{2\sqrt{2}}{15} + \frac{2}{15}\right) \\
&= \underline{\underline{\frac{24\pi}{5}(\sqrt{2}+1).}}
\end{aligned}$$

27. Using the substitution $u = \cosh\theta$, find the value of

$$\int_{\ln 2}^{\ln 4} \frac{\cosh\theta + 1}{\sinh\theta(\cosh\theta - 1)^2}\,\mathrm{d}\theta,$$

giving your answer as an exact fraction.

Solution

$$u = \cosh\theta \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}\theta} = \sinh\theta \Rightarrow \mathrm{d}u = \sinh\theta\,\mathrm{d}\theta$$

and

$$\theta = \ln 2 \Rightarrow u = \frac{2+\frac{1}{2}}{2} = \frac{5}{4} \text{ and } \theta = \ln 4 \Rightarrow u = \frac{4+\frac{1}{4}}{2} = \frac{17}{8}.$$

Now,

$$\begin{aligned}\int_{\ln 2}^{\ln 4} \frac{\cosh\theta + 1}{\sinh\theta(\cosh\theta - 1)^2}\,\mathrm{d}\theta &= \int_{\ln 2}^{\ln 4} \frac{\cosh\theta + 1}{\sinh^2\theta(\cosh\theta - 1)^2}\sinh\theta\,\mathrm{d}\theta \\ &= \int_{\ln 2}^{\ln 4} \frac{\cosh\theta + 1}{(\cosh^2\theta - 1)(\cosh\theta - 1)^2}\sinh\theta\,\mathrm{d}\theta \\ &= \int_{\ln 2}^{\ln 4} \frac{\cosh\theta + 1}{(\cosh\theta + 1)(\cosh\theta - 1)^3}\sinh\theta\,\mathrm{d}\theta \\ &= \int_{\ln 2}^{\ln 4} \frac{1}{(\cosh\theta - 1)^3}\sinh\theta\,\mathrm{d}\theta \\ &= \int_{\frac{5}{4}}^{\frac{17}{8}} (u-1)^{-3}\,\mathrm{d}u \\ &= \left[-\tfrac{1}{2}(u-1)^{-2}\right]_{u=\frac{5}{4}}^{\frac{17}{8}} \\ &= -\frac{1}{2}\left(\frac{9}{8}\right)^{-2} + \frac{1}{2}\left(\frac{1}{4}\right)^{-2} \\ &= \underline{\underline{\frac{616}{81}}}.\end{aligned}$$

28. The curve C, with equation $y = \cosh 3x - 4x$, has a minimum point, as shown in Figure 7.

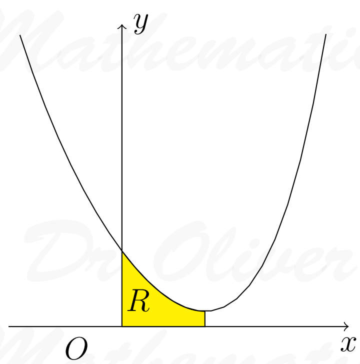


Figure 7: $y = \cosh 3x - 4x$

(a) Use calculus to find the x-coordinate of A. Give your answer in terms of natural logarithm. (5)

Solution

$$\begin{aligned}\frac{dy}{dx} = 0 &\Rightarrow 3\sinh 3x - 4 = 0\\ &\Rightarrow \tfrac{3}{2}\left(e^{3x} - e^{-3x}\right) - 4 = 0\\ &\Rightarrow 3e^{3x} - 3e^{-3x} - 8 = 0\\ &\Rightarrow 3e^{6x} - 8e^{3x} - 3 = 0\\ &\Rightarrow (3e^{3x}+1)(e^{3x}-3) = 0\\ &\Rightarrow e^{3x} = 3\\ &\Rightarrow 3x = \ln 3\\ &\Rightarrow x = \underline{\underline{\tfrac{1}{3}\ln 3}}.\end{aligned}$$

The region R, as shown shaded in the figure, is bounded by the curve, the x-axis, the y-axis, and the line through A parallel to the y-axis.

(b) Show that the area of R is $\frac{2}{9}[2-(\ln 3)^2]$. (6)

Solution

$$\begin{aligned}\text{Area} &= \int_0^{\frac{1}{3}\ln 3} (\cosh 3x - 4x)\,dx\\ &= \left[\tfrac{1}{3}\sinh 3x - 2x^2\right]_{x=0}^{\frac{1}{3}\ln 3}\\ &= \left(\tfrac{1}{3}\sinh(\ln 3) - 2(\tfrac{1}{3}\ln 3)^2\right) - (0-0)\\ &= \tfrac{4}{3} - \tfrac{2}{3}(\ln 3)^2\\ &= \underline{\underline{\tfrac{2}{9}[2-(\ln 3)^2]}}.\end{aligned}$$

29. (a) Using the substitution $x = \dfrac{a}{u}$, or otherwise, find (6)

$$\int \frac{1}{x\sqrt{a^2-x^2}}\,dx.$$

Solution

$$u = \frac{a}{x} \Rightarrow x = \frac{a}{u} \Rightarrow \frac{dx}{du} = -\frac{a}{u^2} \Rightarrow dx = -\frac{a}{u^2}\,du$$

and

$$\begin{aligned}\int \frac{1}{x\sqrt{a^2-x^2}}\,\mathrm{d}x &= \int \frac{1}{\frac{a}{u}\sqrt{a^2-\left(\frac{a}{u}\right)^2}}\left(-\frac{a}{u^2}\right)\mathrm{d}u\\ &= \int \frac{1}{\frac{a^2}{u}\sqrt{1-\left(\frac{1}{u}\right)^2}}\left(-\frac{a}{u^2}\right)\mathrm{d}u\\ &= \int \frac{1}{\frac{a^2}{u^2}\sqrt{u^2-1}}\left(-\frac{a}{u^2}\right)\mathrm{d}u\\ &= \int \frac{u^2}{a^2\sqrt{u^2-1}}\left(-\frac{a}{u^2}\right)\mathrm{d}u\\ &= -\frac{1}{a}\int \frac{1}{\sqrt{u^2-1}}\,\mathrm{d}u\\ &= -\frac{1}{a}\operatorname{arcosh} u + c\\ &= \underline{\underline{-\frac{1}{a}\operatorname{arcosh}\left(\frac{a}{x}\right) + c}}.\end{aligned}$$

(b) Hence find (5)

$$\int_3^4 \frac{1}{x\sqrt{25-x^2}}\,\mathrm{d}x,$$

giving your answer in the form $a\ln b$, where a and b are rational numbers.

Solution

$$\begin{aligned}\int_3^4 \frac{1}{x\sqrt{25-x^2}}\,\mathrm{d}x &= \int_3^4 \frac{1}{x\sqrt{5^2-x^2}}\,\mathrm{d}x\\ &= -\frac{1}{5}\left[\operatorname{arcosh}\left(\frac{5}{x}\right)\right]_{x=3}^{4}\\ &= -\tfrac{1}{5}(\operatorname{arcosh}\tfrac{5}{4} - \operatorname{arcosh}\tfrac{5}{3})\\ &= -\tfrac{1}{5}(\ln 2 - \ln 3)\\ &= -\tfrac{1}{5}\ln\tfrac{2}{3}\\ &= \underline{\underline{\tfrac{1}{5}\ln\tfrac{3}{2}}}.\end{aligned}$$

30. Given that $y = \operatorname{arsinh}(\sqrt{x})$, $x > 0$,

(a) find $\frac{\mathrm{d}y}{\mathrm{d}x}$, giving your answer as a simplified fraction. (3)

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{x+1}} \times \tfrac{1}{2}x^{-\frac{1}{2}} = \underline{\underline{\frac{1}{2\sqrt{x(x+1)}}}}.$$

(b) Hence, or otherwise, find (6)

$$\int_{\frac{1}{4}}^{4} \frac{1}{\sqrt{x(x+1)}}\,\mathrm{d}x,$$

giving your answer in the form $\ln\left(\frac{a+b\sqrt{5}}{2}\right)$, where a and b are integers.

Solution

$$\begin{aligned}
\int_{\frac{1}{4}}^{4} \frac{1}{\sqrt{x(x+1)}}\,\mathrm{d}x &= \left[2\,\mathrm{arsinh}(\sqrt{x})\right]_{x=\frac{1}{4}}^{4} \\
&= 2\,\mathrm{arsinh}\,2 - 2\,\mathrm{arsinh}\,\tfrac{1}{2} \\
&= 2\ln(2+\sqrt{5}) - 2\ln(\tfrac{1+\sqrt{5}}{2}) \\
&= 2\ln(\tfrac{3+\sqrt{5}}{2}) \\
&= \ln(\tfrac{3+\sqrt{5}}{2})^2 \\
&= \underline{\underline{\ln(\tfrac{7+3\sqrt{5}}{2})}}.
\end{aligned}$$

31. A curve, which is part of an ellipse, has parametric equations

$$x = 3\cos\theta, y = 5\sin\theta, 0 \leqslant \theta \leqslant \tfrac{\pi}{2}.$$

The curve is rotated through 2π radians about the x-axis.

(a) Show that the area of the surface generated is given by the integral (6)

$$k\pi \int_0^a \sqrt{16c^2 + 9}\,\mathrm{d}c,$$

where $c = \cos\theta$ and where k and a are constants to be found.

Solution

$$\begin{aligned}
\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2} &= \sqrt{(-3\sin\theta)^2 + (5\cos\theta)^2} \\
&= \sqrt{9\sin^2\theta + 25\cos^2\theta} \\
&= \sqrt{9 + 16\cos^2\theta}
\end{aligned}$$

and

$$\begin{aligned}\text{area} &= 2\pi \int_0^{\frac{\pi}{2}} 5\sin\theta\sqrt{16\cos^2\theta + 9}\,\mathrm{d}\theta \\ &= 10\pi \int_0^{\frac{\pi}{2}} \sin\theta\sqrt{16\cos^2\theta + 9}\,\mathrm{d}\theta.\end{aligned}$$

Now,

$$c = \cos\theta \Rightarrow \frac{\mathrm{d}c}{\mathrm{d}\theta} = -\sin\theta \Rightarrow \mathrm{d}c = -\sin\theta\,\mathrm{d}\theta$$

and then

$$\theta = 0 \Rightarrow c = 1 \text{ and } \theta = \tfrac{\pi}{2} \Rightarrow c = 0.$$

So we have

$$\begin{aligned}\text{area} &= 10\pi \int_0^{\frac{\pi}{2}} \sin\theta\sqrt{16\cos^2\theta + 9}\,\mathrm{d}\theta \\ &= -10\pi \int_0^{\frac{\pi}{2}} (-\sin\theta)\sqrt{16\cos^2\theta + 9}\,\mathrm{d}\theta \\ &= -10\pi \int_1^{0} \sqrt{16c^2 + 9}\,\mathrm{d}c \\ &= \underline{\underline{10\pi \int_0^{1} \sqrt{16c^2 + 9}\,\mathrm{d}c}}.\end{aligned}$$

(b) Using the substitution $\cos\theta = \frac{3}{4}\sinh u$, or otherwise, evaluate the integral, showing all of your working and giving the final answer to 3 significant figures. (5)

Solution

$$c = \tfrac{3}{4}\sinh u \Rightarrow \frac{\mathrm{d}c}{\mathrm{d}u} = \tfrac{3}{4}\cosh u \Rightarrow \mathrm{d}c = \tfrac{3}{4}\cosh u\,\mathrm{d}u$$

and

$$\cos 0 \Rightarrow u = 0 \Rightarrow \cos\tfrac{\pi}{2} \Rightarrow u = \operatorname{arsinh}(\tfrac{4}{3}) = \ln 3.$$

Then

$$\begin{aligned}
R &= 10\pi \int_0^1 \sqrt{16c^2 + 9}\,\mathrm{d}c \\
&= 10\pi \int_0^{\ln 3} \sqrt{9\sinh^2 u + 9}\,(\tfrac{3}{4}\cosh u)\,\mathrm{d}u \\
&= 10\pi \int_0^{\ln 3} \tfrac{9}{4}\cosh^2 u\,\mathrm{d}u \\
&= 10\pi \int_0^{\ln 3} \tfrac{9}{8}(1 + \cosh 2u)\,\mathrm{d}u \\
&= 10\pi \left[\tfrac{9}{8}u + \tfrac{9}{16}\sinh 2u\right]_{u=0}^{\ln 3} \\
&= 10\pi \left\{\tfrac{9}{8}(\tfrac{20}{9} + \ln 3) - (0+0)\right\} \\
&= 117.367\,979\,7 \text{ (FCD)} \\
&= \underline{\underline{117 \text{ (3 sf)}}}.
\end{aligned}$$

32. Use calculus to find the exact value of $\displaystyle\int_{-2}^{1} \frac{1}{x^2 + 4x + 13}\,\mathrm{d}x$. (5)

Solution

$$\begin{aligned}
\int_{-2}^{1} \frac{1}{x^2 + 4x + 13}\,\mathrm{d}x &= \int_{-2}^{1} \frac{1}{(x+2)^2 + 3^2}\,\mathrm{d}x \\
&= \left[\tfrac{1}{3}\arctan\left(\frac{x+2}{3}\right)\right]_{x=-2}^{1} \\
&= \tfrac{1}{3}\arctan 1 - 0 \\
&= \underline{\underline{\tfrac{\pi}{12}}}.
\end{aligned}$$

33. The curve C has equation $y = 2x^3$, $0 \leqslant x \leqslant 2$. (5)
The curve C is rotated through 2π radians about the x-axis.
Using calculus, find the area of the surface generated, giving your answer to 3 significant figures.

Solution

$$\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \sqrt{1+\left(6x^2\right)^2} = \sqrt{1+36x^4}$$

and

$$\begin{aligned}\text{area} &= 2\pi\int_0^2 (2x^3)\sqrt{1+36x^4}\,\mathrm{d}x\\ &= 4\pi\left[\tfrac{1}{216}(1+36x^4)^{\frac{3}{2}}\right]_{x=0}^{2}\\ &= \tfrac{\pi}{54}(577^{\frac{3}{2}}-1)\\ &= 806.284\,845\,5 \text{ (FCD)}\\ &= \underline{\underline{806 \text{ (3 sf)}}}.\end{aligned}$$

34. Show that

(a) $\displaystyle\int_5^8 \frac{1}{x^2-10x+34}\,\mathrm{d}x = k\pi$, giving the value of the fraction k. (5)

Solution

$$\begin{aligned}\int_5^8 \frac{1}{x^2-10x+34}\,\mathrm{d}x &= \int_5^8 \frac{1}{(x-5)^2+3^2}\,\mathrm{d}x\\ &= \left[\tfrac{1}{3}\arctan\left(\frac{x-5}{3}\right)\right]_{x=5}^{8}\\ &= \tfrac{1}{3}\arctan 1 - 0\\ &= \underline{\underline{\tfrac{\pi}{12}}}.\end{aligned}$$

(b) $\displaystyle\int_5^8 \frac{1}{\sqrt{x^2-10x+34}}\,\mathrm{d}x = \ln(A+\sqrt{n})$, giving the values of the integers A and n. (4)

Solution

$$\int_5^8 \frac{1}{\sqrt{x^2 - 10x + 34}}\,\mathrm{d}x = \int_5^8 \frac{1}{\sqrt{(x-5)^2 + 3^2}}\,\mathrm{d}x$$
$$= \left[\operatorname{arsinh}\left(\frac{x-5}{3}\right)\right]_{x=5}^{8}$$
$$= \operatorname{arsinh} 1 - 0$$
$$= \underline{\underline{\ln(1+\sqrt{2})}}.$$

35. The curve C, as shown in Figure 8, has equation (6)

$$y = \tfrac{1}{3}\cosh 3x,\ 0 \leqslant x \leqslant \ln a,$$

where a is a constant and $a > 1$.

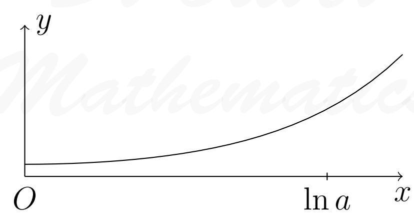


Figure 8: $y = \frac{1}{3}\cosh 3x$

Using calculus, show that the length of curve C is

$$k\left(a^3 - \frac{1}{a^3}\right)$$

and state the value of the constant k.

Solution

$$\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \sqrt{1 + \sinh^2 3x} = \cosh 3x$$

and

$$\begin{aligned}\text{Length of curve} &= \int_0^{\ln a} \cosh 3x \, \mathrm{d}x \\ &= \left[\tfrac{1}{3}\sinh 3x\right]_{x=0}^{\ln a} \\ &= \tfrac{1}{3}(\sinh(3\ln a) - 0) \\ &= \tfrac{1}{3}\sinh(\ln a^3) \\ &= \tfrac{1}{6}\left(\mathrm{e}^{\ln a^3} - \mathrm{e}^{-\ln a^3}\right) \\ &= \underline{\underline{\frac{1}{6}\left(a^3 - \frac{1}{a^3}\right)}}.\end{aligned}$$

36. (a) Differentiate $x \operatorname{arsinh} 2x$ with respect to x. (3)

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}(x \operatorname{arsinh} 2x) = \operatorname{arsinh} 2x + \frac{2x}{\sqrt{(2x)^2+1}} = \underline{\underline{\operatorname{arsinh} 2x + \frac{2x}{\sqrt{4x^2+1}}}}.$$

(b) Hence, or otherwise, find the exact value of (7)

$$\int_0^{\sqrt{2}} x \operatorname{arsinh} 2x \, \mathrm{d}x,$$

giving your answer in the form $A \ln B + C$, where A, B, C are real numbers.

Solution

$$\begin{aligned}\int_0^{\sqrt{2}} x \operatorname{arsinh} 2x \, \mathrm{d}x &= [x \operatorname{arsinh} 2x]_{x=0}^{\sqrt{2}} - \int_0^{\sqrt{2}} \frac{2x}{\sqrt{4x^2+1}} \, \mathrm{d}x \\ &= (\sqrt{2}\operatorname{arsinh} 2\sqrt{2} - 0) - \left[\tfrac{1}{2}(4x^2+1)^{\frac{1}{2}}\right]_{x=0}^{\sqrt{2}} \\ &= \sqrt{2}\operatorname{arsinh} 2\sqrt{2} - (\tfrac{3}{2} - \tfrac{1}{2}) \\ &= \sqrt{2}\operatorname{arsinh} 2\sqrt{2} - 1 \\ &= \underline{\underline{\sqrt{2}\ln(3+2\sqrt{2}) - 1}}.\end{aligned}$$

37.

$$f(x) = 5\cosh x - 4\sinh x.$$

(a) Show that $f(x) = \frac{1}{2}(e^x + 9e^{-x})$. (2)

Solution

$$\begin{aligned} f(x) &= 5\cosh x - 4\sinh x \\ &= \tfrac{5}{2}\left(e^x + e^{-x}\right) - 2\left(e^x - e^{-x}\right) \\ &= \underline{\underline{\tfrac{1}{2}\left(e^x + 9e^{-x}\right)}}. \end{aligned}$$

Hence

(b) solve $f(x) = 5$, (4)

Solution

$$\begin{aligned} f(x) = 5 &\Rightarrow \tfrac{1}{2}\left(e^x + 9e^{-x}\right) = 5 \\ &\Rightarrow e^x + 9e^{-x} = 10 \\ &\Rightarrow e^{2x} - 10e^x + 9 = 0 \\ &\Rightarrow (e^x - 1)(e^x - 9) = 0 \\ &\Rightarrow e^x = 1 \text{ or } e^x = 9 \\ &\Rightarrow \underline{\underline{x = 0}} \text{ or } \underline{\underline{x = \ln 9}} \end{aligned}$$

(c) show that (5)

$$\int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{1}{5\cosh x - 4\sinh x}\,dx = \tfrac{\pi}{18}.$$

Solution

$$\begin{aligned}
\int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{1}{5\cosh x - 4\sinh x}\,\mathrm{d}x &= \int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{2}{\mathrm{e}^x + 9\mathrm{e}^{-x}}\,\mathrm{d}x \\
&= \int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{2\mathrm{e}^x}{\mathrm{e}^{2x} + 9}\,\mathrm{d}x \\
&= \int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{2\mathrm{e}^x}{(\mathrm{e}^x)^2 + 3^2}\,\mathrm{d}x \\
&= \left[\frac{2}{3}\arctan\left(\frac{\mathrm{e}^x}{3}\right)\right]_{x=\frac{1}{2}\ln 3}^{\ln 3} \\
&= \tfrac{2}{3}(\arctan 1 - \arctan \tfrac{1}{\sqrt{3}}) \\
&= \tfrac{2}{3}(\tfrac{\pi}{4} - \tfrac{\pi}{6}) \\
&= \underline{\underline{\tfrac{\pi}{18}}}.
\end{aligned}$$

38. (a) Find (2)

$$\int \frac{1}{\sqrt{4x^2 + 9}}\,\mathrm{d}x.$$

Solution

$$\begin{aligned}
\int \frac{1}{\sqrt{4x^2 + 9}}\,\mathrm{d}x &= \int \frac{1}{\sqrt{(2x)^2 + 3^2}}\,\mathrm{d}x \\
&= \underline{\underline{\tfrac{1}{2}\operatorname{arsinh}\left(\frac{2x}{3}\right) + c}}.
\end{aligned}$$

(b) Use your answer to part (a) to find the exact value of (3)

$$\int_{-3}^{3} \frac{1}{\sqrt{4x^2 + 9}}\,\mathrm{d}x,$$

giving your answer in the form $k\ln(a + b\sqrt{5})$, where a and b are integers and k is a constant.

Solution

$$\int_{-3}^{3} \frac{1}{\sqrt{4x^2+9}}\,\mathrm{d}x = \tfrac{1}{2}\left[\operatorname{arsinh}\left(\frac{2x}{3}\right)\right]_{x=-3}^{3}$$
$$= \tfrac{1}{2}[\operatorname{arsinh} 2 - \operatorname{arsinh}(-2)]$$
$$= \tfrac{1}{2}[\ln(2+\sqrt{5}) - \ln(-2+\sqrt{5})]$$
$$= \underline{\underline{\tfrac{1}{2}\ln(9+4\sqrt{5})}}.$$

39. The curve with parametric equations (7)

$$x = \cosh 2\theta, y = 4\sinh\theta, 0 \leqslant \theta \leqslant 1,$$

is rotated through 2π radians about the x-axis.
Show that the area of of the surface generated is $\lambda(\cosh^3 1 - 1)$, where λ is a constant to be found.

Solution

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 2\sinh 2\theta \text{ and } \frac{\mathrm{d}y}{\mathrm{d}\theta} = 4\cosh\theta$$

and

$$\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2} = \sqrt{(2\sinh 2\theta)^2 + (4\cosh\theta)^2}$$
$$= \sqrt{4\sinh^2 2\theta + 16\cosh^2\theta}$$
$$= \sqrt{4(\cosh^2 2\theta - 1) + 8(1+\cosh 2\theta)}$$
$$= \sqrt{4\cosh^2 2\theta + 8\cosh 2\theta + 4}$$
$$= \sqrt{2(\cosh 2\theta + 1)^2}$$
$$= 2(\cosh 2\theta + 1)$$
$$= 2(2\cosh^2\theta)$$
$$= 4\cosh^2\theta.$$

Now,

$$\begin{aligned}\text{surface generated} &= 2\pi \int_0^1 (4\sinh\theta)(4\cosh^2\theta)\,\mathrm{d}\theta \\ &= 32\pi \int_0^1 \sinh\theta\cosh^2\theta\,\mathrm{d}\theta \\ &= 32\pi\left[\tfrac{1}{3}\cosh^3\theta\right]_{\theta=0}^{1} \\ &= 32\pi(\tfrac{1}{3}\cosh^3\theta - \tfrac{1}{3}) \\ &= \underline{\underline{\tfrac{32\pi}{3}(\cosh^3\theta - 1)}}.\end{aligned}$$

40. Figure 9 shows a sketch of the curve with equation (7)

$$y = 40\operatorname{arcosh} x - 9x,\ x \geqslant 1.$$

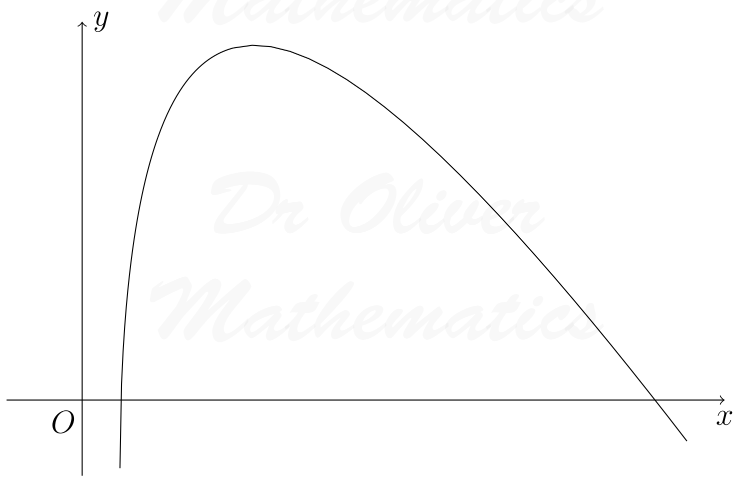


Figure 9: $y = 40\operatorname{arcosh} x - 9x$

Use calculus to find the exact coordinates of the turning point of the curve, giving your answer in the form $\left(\frac{p}{q}, r\ln 3 + s\right)$, where p, q, r, and s are integers.

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow \frac{40}{\sqrt{x^2-1}} - 9 = 0 \\ &\Rightarrow 40 = 9\sqrt{x^2-1} \\ &\Rightarrow 1600 = 81(x^2-1) \\ &\Rightarrow x^2 - 1 = \tfrac{1600}{81} \\ &\Rightarrow x^2 = \tfrac{1681}{81} \\ &\Rightarrow x = \tfrac{41}{9}\end{aligned}$$

and $y = 40 \operatorname{arcosh} \frac{41}{9} - 41 = 40 \ln 9 - 41 = 80 \ln 3 - 41$. The answer is $\underline{\underline{\left(\frac{41}{9}, 80 \ln 3 - 41\right)}}$.

41. Figure 10 shows a sketch of the curve with equations

$$y = 6 \cosh x \text{ and } y = 9 - 2 \sinh x.$$

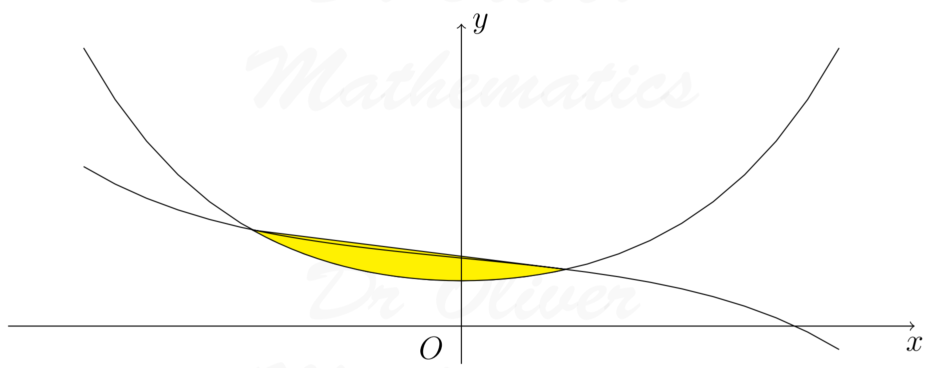


Figure 10: $y = 6 \cosh x$ and $y = 9 - 2 \sinh x$

(a) Using the definitions of $\sinh x$ and $\cosh x$ in terms of e^x, find exact values for the x-coordinates of the two points where the curves intersect. (6)

Solution

$$\begin{aligned}6 \cosh x = 9 - 2 \sinh x &\Rightarrow 3\left(\mathrm{e}^x + \mathrm{e}^{-x}\right) = 9 - \left(\mathrm{e}^x - \mathrm{e}^{-x}\right) \\ &\Rightarrow 4\mathrm{e}^x - 9 + 2\mathrm{e}^{-x} = 0 \\ &\Rightarrow 4\mathrm{e}^{2x} - 9\mathrm{e}^x + 2 = 0 \\ &\Rightarrow (4\mathrm{e}^x - 1)(\mathrm{e}^x - 2) = 0 \\ &\Rightarrow \mathrm{e}^x = \tfrac{1}{4} \text{ or } \mathrm{e}^x = 2 \\ &\Rightarrow \underline{\underline{x = \ln \tfrac{1}{4}}} \text{ or } \underline{\underline{x = \ln 2}}.\end{aligned}$$

The finite region between the two curves is shown shaded in the figure.

(b) Using calculus, find the area of the shaded region, giving your answer in the form $a \ln b + c$, where a, b, and c are integers. (6)

Solution

$$\begin{aligned}\text{Area} &= \int_{\ln \frac{1}{4}}^{\ln 2} (9 - 2\sinh x - 6\cosh x)\,\mathrm{d}x \\ &= \int_{\ln \frac{1}{4}}^{\ln 2} (9 - 4\mathrm{e}^{x} - 2\mathrm{e}^{-x})\,\mathrm{d}x \\ &= \left[9x - 4\mathrm{e}^{x} + 2\mathrm{e}^{-x}\right]_{x=\ln \frac{1}{4}}^{\ln 2} \\ &= (9\ln 2 - 8 + 1) - (9\ln \tfrac{1}{4} - 1 + 8) \\ &= \underline{\underline{9\ln 8 - 14}}.\end{aligned}$$

42. The curve C, shown in Figure 11, has equation

$$y = 2x^{\frac{1}{2}},\ 1 \leqslant x \leqslant 8.$$

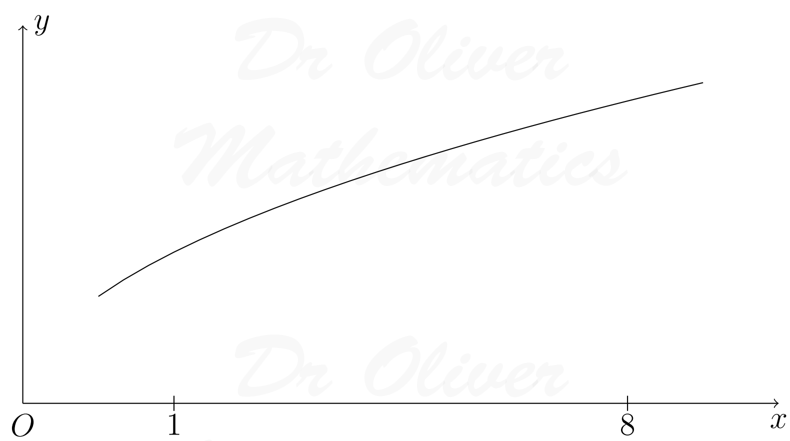


Figure 11: $y = 2x^{\frac{1}{2}}$

(a) Show that the length s of the curve C is given by the equation (2)

$$s = \int_{1}^{8} \sqrt{1 + \frac{1}{x}}\,\mathrm{d}x.$$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x^{\frac{1}{2}}}$$

and we have

$$s = \int_1^8 \sqrt{1 + \frac{1}{x}}\, \mathrm{d}x.$$

(b) Using the substitution $x = \sinh^2 u$, or otherwise, find an exact value for s. Give your answer in the form $a\sqrt{2} + \ln(b + c\sqrt{2})$, where a, b, and c are integers. (9)

Solution

$$u = \operatorname{arsinh} \sqrt{x} \Rightarrow \sinh u = \sqrt{x} \Rightarrow x = \sinh^2 u,$$

$$x = \sinh^2 u \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u} = 2 \sinh u \cosh u \Rightarrow \mathrm{d}x = 2 \sinh u \cosh u \, \mathrm{d}u,$$

and

$$x = 1 \Rightarrow u = \operatorname{arsinh} 1 \text{ and } x = 8 \Rightarrow u = \operatorname{arsinh} 2\sqrt{2}.$$

$$\begin{aligned}
s &= \int_1^8 \sqrt{1 + \frac{1}{x}}\, \mathrm{d}x \\
&= \int_{\operatorname{arsinh} 1}^{\operatorname{arsinh} 2\sqrt{2}} \sqrt{1 + \frac{1}{\sinh^2 u}} 2 \sinh u \cosh u \, \mathrm{d}u \\
&= \int_{\operatorname{arsinh} 1}^{\operatorname{arsinh} 2\sqrt{2}} \sqrt{\frac{\sinh^2 u + 1}{\sinh^2 u}} 2 \sinh u \cosh u \, \mathrm{d}u \\
&= \int_{\operatorname{arsinh} 1}^{\operatorname{arsinh} 2\sqrt{2}} 2 \coth u \sinh u \cosh u \, \mathrm{d}u \\
&= \int_{\operatorname{arsinh} 1}^{\operatorname{arsinh} 2\sqrt{2}} 2 \cosh^2 u \, \mathrm{d}u \\
&= \int_{\operatorname{arsinh} 1}^{\operatorname{arsinh} 2\sqrt{2}} (1 + \cosh 2u) \, \mathrm{d}u \\
&= \left[u + \tfrac{1}{2} \sinh 2u\right]_{u=\operatorname{arsinh} 1}^{\operatorname{arsinh} 2\sqrt{2}} \\
&= \left[\operatorname{arsinh} 2\sqrt{2} + \tfrac{1}{2} \sinh(2 \operatorname{arsinh} 2\sqrt{2})\right] - \left[\operatorname{arsinh} 1 + \tfrac{1}{2} \sinh(2 \operatorname{arsinh} 1)\right] \\
&= \left[\ln(3 + 2\sqrt{2}) + 6\sqrt{2}\right] - \left[\ln(1 + \sqrt{2}) + \sqrt{2}\right] \\
&= \underline{\underline{\ln(1 + \sqrt{2}) + 5\sqrt{2}}}
\end{aligned}$$

and we have, of course,

$$\begin{aligned}
\tfrac{1}{2}\sinh(2\operatorname{arsinh} 2\sqrt{2}) &= \sinh(\operatorname{arsinh} 2\sqrt{2})\cosh(\operatorname{arsinh} 2\sqrt{2}) \\
&= 2\sqrt{2}\cosh[\ln(3+2\sqrt{2})] \\
&= \sqrt{2}\left[\mathrm{e}^{\ln(3+2\sqrt{2})} + \mathrm{e}^{-\ln(3+2\sqrt{2})}\right] \\
&= \sqrt{2}\left(3+2\sqrt{2}+\frac{1}{3+2\sqrt{2}}\right) \\
&= 6\sqrt{2}
\end{aligned}$$

and

$$\begin{aligned}
\tfrac{1}{2}\sinh(2\operatorname{arsinh} 1) &= \sinh(\operatorname{arsinh} 1)\cosh(\operatorname{arsinh} 1) \\
&= \cosh[\ln(1+\sqrt{2})] \\
&= \tfrac{1}{2}\left[\mathrm{e}^{\ln(1+\sqrt{2})} + \mathrm{e}^{-\ln(1+\sqrt{2})}\right] \\
&= \tfrac{1}{2}\left(1+\sqrt{2}+\frac{1}{1+\sqrt{2}}\right) \\
&= \sqrt{2}.
\end{aligned}$$

43. Using calculus, find the exact value of

(a) $\displaystyle\int_1^2 \frac{1}{\sqrt{x^2-2x+3}}\,\mathrm{d}x$, (4)

Solution

$$\begin{aligned}
\int_1^2 \frac{1}{\sqrt{x^2-2x+3}}\,\mathrm{d}x &= \int_1^2 \frac{1}{\sqrt{(x-1)^2+1}}\,\mathrm{d}x \\
&= \left[\operatorname{arsinh}\left(\frac{x-1}{\sqrt{2}}\right)\right]_{x=1}^2 \\
&= \operatorname{arsinh}\tfrac{1}{\sqrt{2}} - 0 \\
&= \underline{\underline{\operatorname{arsinh}\tfrac{1}{\sqrt{2}}}}.
\end{aligned}$$

(b) $\displaystyle\int_0^1 \mathrm{e}^{2x}\sinh x\,\mathrm{d}x$. (4)

Solution

$$\begin{aligned}\int_0^1 \mathrm{e}^{2x}\sinh x\,\mathrm{d}x &= \tfrac{1}{2}\int_0^1 \mathrm{e}^{2x}(\mathrm{e}^x-\mathrm{e}^{-x})\,\mathrm{d}x\\ &= \tfrac{1}{2}\int_0^1 (\mathrm{e}^{3x}-\mathrm{e}^x)\,\mathrm{d}x\\ &= \tfrac{1}{2}\left[\tfrac{1}{3}\mathrm{e}^{3x}-\mathrm{e}^x\right]_{x=0}^1\\ &= \tfrac{1}{2}\left[(\tfrac{1}{3}\mathrm{e}^3-\mathrm{e})-(\tfrac{1}{3}-1)\right]\\ &= \tfrac{1}{2}\left[\tfrac{1}{3}\mathrm{e}^3-\mathrm{e}+\tfrac{2}{3}\right]\\ &= \underline{\underline{\tfrac{1}{6}(\mathrm{e}^3-3\mathrm{e}+2)}}.\end{aligned}$$

44. A circle C with centre O and radius r has cartesian equation $x^2+y^2=r^2$, where r is a positive constant.

(a) Show that $1+\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2=\dfrac{r^2}{r^2-x^2}$. (3)

Solution

$$\begin{aligned}x^2+y^2=r^2 &\Rightarrow 2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}=0\\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{x}{y}\\ &\Rightarrow \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=\left(-\frac{x}{y}\right)^2=\frac{x^2}{y^2}=\frac{x^2}{r^2-x^2}\\ &\Rightarrow 1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=1+\frac{x^2}{r^2-x^2}=\underline{\underline{\frac{r^2}{r^2-x^2}}}.\end{aligned}$$

(b) Show that the surface area of the sphere generated by rotating C through π radians about the x-axis is $4\pi r^2$. (5)

Solution

$$\begin{aligned}
\text{Surface area} &= 2\pi \int_{-r}^{r} y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \\
&= 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2}\sqrt{\frac{r^2}{r^2 - x^2}}\,\mathrm{d}x \\
&= 2\pi r \int_{-r}^{r} 1\,\mathrm{d}x \\
&= 2\pi r\,[x]_{x=-r}^{r} \\
&= 2\pi r[r - (-r)] \\
&= \underline{\underline{4\pi r^2}}.
\end{aligned}$$

(c) Write down the arc length of the arc of the curve $y = \sqrt{1 - x^2}$ from $x = 0$ to $x = 1$. (1)

Solution

We have $r = 1$ and so

$$\text{arc length} = \frac{2 \times \pi \times 1}{4} = \underline{\underline{\tfrac{\pi}{2}}}.$$

45.

$$9x^2 + 6x + 5 \equiv a(x + b)^2 + c.$$

(a) Find the values of the constants a, b, and c. (3)

Solution

$$\begin{aligned}
9x^2 + 6x + 5 &\equiv 9(x^2 + \tfrac{2}{3}x) + 5 \\
&\equiv 9[(x + \tfrac{1}{3})^2 - \tfrac{1}{9}] + 5 \\
&\equiv \underline{\underline{9(x + \tfrac{1}{3})^2 + 4}}.
\end{aligned}$$

Hence, or otherwise, find

(b) $\displaystyle\int \frac{1}{9x^2 + 6x + 5}\,\mathrm{d}x$, (2)

Solution

$$\int \frac{1}{9x^2+6x+5}\,\mathrm{d}x = \int \frac{1}{9(x+\frac{1}{3})^2+2^2}\,\mathrm{d}x$$
$$= \underline{\underline{\frac{1}{6}\arctan\left(\frac{3x+1}{2}\right)+c.}}$$

(c) $\displaystyle\int \frac{1}{\sqrt{9x^2+6x+5}}\,\mathrm{d}x.$ (2)

Solution

$$\int \frac{1}{\sqrt{9x^2+6x+5}}\,\mathrm{d}x = \int \frac{1}{\sqrt{(3(x+\frac{1}{3}))^2+2^2}}\,\mathrm{d}x$$
$$= \underline{\underline{\frac{1}{3}\operatorname{arsinh}\left(\frac{3x+1}{2}\right)+c.}}$$

46. The curve C has equation

$$y = \mathrm{e}^{-x},\ x \in \mathbb{R}.$$

The part of the curve C between $x = 0$ and $x = \ln 3$ is rotated through 2π radians about the x-axis.

(a) Show that the area S of the curved surface generated is given by (3)

$$S = 2\pi \int_0^{\ln 3} \mathrm{e}^{-x}\sqrt{1+\mathrm{e}^{-2x}}\,\mathrm{d}x.$$

Solution

$$\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \sqrt{1+(-\mathrm{e}^{-x})^2} = \sqrt{1+\mathrm{e}^{-2x}}$$

and so we have

$$\underline{\underline{S = 2\pi \int_0^{\ln 3} \mathrm{e}^{-x}\sqrt{1+\mathrm{e}^{-2x}}\,\mathrm{d}x.}}$$

(b) Use the substitution $\mathrm{e}^{-x} = \sinh u$ to show that (5)

$$S = 2\pi \int_{\operatorname{arsinh}\alpha}^{\operatorname{arsinh}\beta} \cosh^2 u \,\mathrm{d}u,$$

where α and β are constants to be determined.

Solution

$$u = \operatorname{arsinh} \mathrm{e}^{-x} \Rightarrow \mathrm{e}^{-x} = \sinh u \Rightarrow -x = \ln(\sinh u) \Rightarrow x = -\ln(\sinh u),$$

$$\frac{\mathrm{d}x}{\mathrm{d}u} = -\coth u \Rightarrow \mathrm{d}x = -\coth u \,\mathrm{d}u,$$

$$x = 0 \Rightarrow u = \operatorname{arsinh} 1 \text{ and } x = \ln 3 \Rightarrow u = \operatorname{arsinh} \tfrac{1}{3}.$$

$$\begin{aligned}
S &= 2\pi \int_0^{\ln 3} \mathrm{e}^{-x}\sqrt{1+\mathrm{e}^{-2x}} \,\mathrm{d}x \\
&= 2\pi \int_{\operatorname{arsinh} 1}^{\operatorname{arsinh}\frac{1}{3}} \sinh u \sqrt{1+\sinh^2 u}(-\coth u) \,\mathrm{d}u \\
&= -2\pi \int_{\operatorname{arsinh} 1}^{\operatorname{arsinh}\frac{1}{3}} \sinh u \cosh u \coth u \,\mathrm{d}u \\
&= -2\pi \int_{\operatorname{arsinh} 1}^{\operatorname{arsinh}\frac{1}{3}} \cosh^2 u \,\mathrm{d}u \\
&= \underline{\underline{2\pi \int_{\operatorname{arsinh}\frac{1}{3}}^{\operatorname{arsinh} 1} \cosh^2 u \,\mathrm{d}u.}}
\end{aligned}$$

(c) Show that (2)

$$2\int \cosh^2 u \,\mathrm{d}u = \tfrac{1}{2}\sinh 2u + u + k,$$

where k is an arbitrary constant.

Solution

$$\begin{aligned}
2\int \cosh^2 u \,\mathrm{d}u &= \int (\cosh 2u + 1) \,\mathrm{d}u \\
&= \underline{\underline{\tfrac{1}{2}\sinh 2u + u + k.}}
\end{aligned}$$

(d) Hence find the value of S, giving your answer to 3 decimal places. (2)

Solution

$$\begin{aligned}
S &= 2\pi \int_0^{\ln 3} \mathrm{e}^{-x}\sqrt{1+\mathrm{e}^{-2x}}\,\mathrm{d}x \\
&= 2\pi \int_{\operatorname{arsinh}\frac{1}{3}}^{\operatorname{arsinh}1} \cosh^2 u\,\mathrm{d}u \\
&= \pi\left[\tfrac{1}{2}\sinh 2u + u\right]_{u=\operatorname{arsinh}\frac{1}{3}}^{\operatorname{arsinh}1} \\
&= \pi[(\tfrac{1}{2}\sinh(2\operatorname{arsinh}1)+\operatorname{arsinh}1) - (\tfrac{1}{2}\sinh(2\operatorname{arsinh}\tfrac{1}{3})+\operatorname{arsinh}\tfrac{1}{3})] \\
&= 5.079\,241\,597 \text{ (FCD)} \\
&= \underline{\underline{5.079 \text{ (3 dp)}}}.
\end{aligned}$$

47. A curve has equation (5)

$$y = \cosh x,\ 1 \leqslant x \leqslant \ln 5.$$

Find the length of this curve. Give your answer in terms of e.

Solution

$$\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \sqrt{1+\sinh^2 x} = \cosh x$$

and we have

$$\begin{aligned}
\text{length of the curve} &= \int_1^{\ln 5} \cosh x\,\mathrm{d}x \\
&= \int_1^{\ln 5} \cosh x\,\mathrm{d}x \\
&= [\sinh x]_{x=1}^{\ln 5} \\
&= \sinh(\ln 5) - \sinh 1 \\
&= \tfrac{12}{5} - \tfrac{1}{2}(\mathrm{e}-\mathrm{e}^{-1}) \\
&= \underline{\underline{\tfrac{12}{5} - \tfrac{1}{2}\mathrm{e} + \tfrac{1}{2}\mathrm{e}^{-1}}}.
\end{aligned}$$

48. The curve C has equation

$$y = \frac{1}{\sqrt{x^2+2x-3}},\ x > 1.$$

(a) Find $\int y \, dx$. (3)

Solution

$$\int \frac{1}{\sqrt{x^2+2x-3}} \, dx = \int \frac{1}{\sqrt{(x+1)^2-2^2}} \, dx$$
$$= \underline{\underline{\operatorname{arcosh}\left(\frac{x+1}{2}\right) + c.}}$$

The region R is bounded by the curve C, the x-axis, and the lines with equations $x = 2$ and $x = 3$. The region R is rotated through 2π radians about the x-axis.

(b) Find the volume of the solid generated. Give your answer in the form $p\pi \ln q$, where p and q are rational numbers to be found. (4)

Solution

$$\text{Volume} = \pi \int_2^3 \frac{1}{(x+1)^2-2^2} \, dx$$
$$= \tfrac{1}{4}\pi \left[\ln\left|\frac{(x+1)-2}{(x+1)+2}\right|\right]_{x=2}^3$$
$$= \tfrac{1}{4}\pi \left[\ln\left|\frac{x-1}{x+3}\right|\right]_{x=2}^3$$
$$= \tfrac{1}{4}\pi(\ln \tfrac{1}{3} - \ln \tfrac{1}{5})$$
$$= \underline{\underline{\tfrac{1}{4}\pi \ln \tfrac{5}{3}}}.$$

49. The curve C has equation (7)

$$y = \frac{x^2}{8} - \ln x, 2 \leqslant x \leqslant 3.$$

Find the length of the curve C giving your answer in the form $p + lnq$, where p and q are rational numbers to be found.

Solution

$$
\begin{aligned}
\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} &= \sqrt{1+\left(\frac{x}{4}-\frac{1}{x}\right)^2} \\
&= \sqrt{1+\left(\frac{x^2}{16}-\frac{1}{2}+\frac{1}{x^2}\right)} \\
&= \sqrt{\frac{x^2}{16}+\frac{1}{2}+\frac{1}{x^2}} \\
&= \sqrt{\left(\frac{x}{4}+\frac{1}{x}\right)^2} \\
&= \frac{x}{4}+\frac{1}{x}
\end{aligned}
$$

and

$$
\begin{aligned}
\text{length} &= \int_2^3 \left(\frac{x}{4}+\frac{1}{x}\right)\,\mathrm{d}x \\
&= \left[\frac{x^2}{8}+\ln|x|\right]_{x=2}^3 \\
&= \left(\tfrac{9}{8}+\ln 3\right)-\left(\tfrac{1}{2}+\ln 2\right) \\
&= \underline{\underline{\tfrac{5}{8}+\ln\tfrac{3}{2}}}.
\end{aligned}
$$

50. (a) Prove that (3)

$$\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{arcoth} x) = \frac{1}{1-x^2}.$$

Solution

$$\begin{aligned}
y = \operatorname{arcoth} x \Rightarrow x &= \coth y \\
\Rightarrow x &= \frac{\cosh y}{\sinh y} \\
\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y} &= \frac{\sinh^2 y - \cosh^2 y}{\sinh^2 y} \\
\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y} &= -\frac{1}{\sinh^2 y} \\
\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y} &= -\operatorname{cosech}^2 y \\
\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y} &= 1 - \coth^2 y \\
\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{1 - \coth^2 y} \\
\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1 - x^2}}}.
\end{aligned}$$

Given that $y = (\operatorname{arcoth} x)^2$,

(b) show that (5)

$$(1 - x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{k}{1 - x^2},$$

where k is a constant to be determined.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2\operatorname{arcoth} x}{1 - x^2}$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{2(1 - x^2) \times \frac{1}{1 - x^2} - 2\operatorname{arcoth} x \times (-2x)}{(1 - x^2)^2} = \frac{2 + 4x\operatorname{arcoth} x}{(1 - x^2)^2}.$$

Now,

$$\begin{aligned}
(1 - x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2 + 4x\operatorname{arcoth} x}{1 - x^2} - \frac{4x\operatorname{arcoth} x}{1 - x^2} \\
&= \underline{\underline{\frac{2}{1 - x^2}}}.
\end{aligned}$$

51. (a) Find, without using a calculator, (5)

$$\int_3^5 \frac{1}{\sqrt{15+2x-x^2}}\,\mathrm{d}x,$$

giving your answer as a multiple of π.

Solution

$$\begin{aligned}\int_3^5 \frac{1}{\sqrt{15+2x-x^2}}\,\mathrm{d}x &= \int_3^5 \frac{1}{\sqrt{4^2-(x-1)^2}}\,\mathrm{d}x \\ &= \left[\arcsin\left(\frac{x-1}{4}\right)\right]_{x=3}^5 \\ &= \arcsin 1 - \arcsin \tfrac{1}{2} \\ &= \frac{\pi}{2} - \frac{\pi}{6} \\ &= \underline{\underline{\frac{\pi}{3}}}.\end{aligned}$$

(b) Show that (3)

$$5\cosh x - 4\sinh x = \frac{\mathrm{e}^{2x}+9}{2\mathrm{e}^x}.$$

Solution

$$\begin{aligned}5\cosh x - 4\sinh x &= \tfrac{5}{2}\left(\mathrm{e}^x+\mathrm{e}^{-x}\right) - 2\left(\mathrm{e}^x-\mathrm{e}^{-x}\right) \\ &= \tfrac{1}{2}\mathrm{e}^x + \tfrac{9}{2}\mathrm{e}^{-x} \\ &= \frac{\mathrm{e}^x+9\mathrm{e}^{-x}}{2} \\ &= \underline{\underline{\frac{\mathrm{e}^{2x}+9}{2\mathrm{e}^x}}}.\end{aligned}$$

(c) Hence, using the substitution $u = \mathrm{e}^x$ or otherwise, find (4)

$$\int \frac{1}{5\cosh x - 4\sinh x}\,\mathrm{d}x.$$

Solution

$$u = \mathrm{e}^x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x \Rightarrow \frac{1}{u}\,\mathrm{d}u = \mathrm{d}x$$

and

$$\begin{aligned}\int \frac{1}{5\cosh x - 4\sinh x}\,\mathrm{d}x &= \int \frac{2\mathrm{e}^x}{\mathrm{e}^{2x}+9}\,\mathrm{d}x \\ &= \int \frac{2u}{u(u^2+9)}\,\mathrm{d}u \\ &= \int \frac{2}{u^2+9}\,\mathrm{d}u \\ &= \frac{2}{3}\arctan(\frac{u}{3}) + c \\ &= \underline{\underline{\frac{2}{3}\arctan\left(\frac{\mathrm{e}^x}{3}\right) + c.}}\end{aligned}$$

52. Given that $y = \operatorname{arsinh}(\tanh x)$, show that (5)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\operatorname{sech}^2 x}{\sqrt{1+\tanh^2 x}}.$$

Solution

$$\begin{aligned}y = \operatorname{arsinh}(\tanh x) &\Rightarrow \sinh y = \tanh x \\ &\Rightarrow \cosh y\,\frac{\mathrm{d}y}{\mathrm{d}x} = \operatorname{sech}^2 x \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\operatorname{sech}^2 x}{\cosh y} \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\operatorname{sech}^2 x}{\sqrt{\cosh^2 x}} \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\operatorname{sech}^2 x}{\sqrt{1+\sinh^2 x}} \\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\operatorname{sech}^2 x}{\sqrt{1+\tanh^2 x}}}}.\end{aligned}$$

53. Use the substitution $x + 2 = u^2$, where $u > 0$, to show that (9)

$$\int_{-1}^{7} \frac{(x+2)^{\frac{1}{2}}}{x+5}\,\mathrm{d}x = a + b\pi\sqrt{3},$$

where a and b are rational numbers to be found.

Solution

You should know by now that $x+2 = u^2$ is an incorrect way to write the substitution (why?). Oh, well.

$$x + 2 = u^2 \Rightarrow 1 = 2u\,\frac{\mathrm{d}u}{\mathrm{d}x} \Rightarrow \mathrm{d}x = 2u\,\mathrm{d}u$$

and

$$x = -1 \Rightarrow u = 1 \text{ and } x = 7 \Rightarrow u = 3.$$

Now,

$$\begin{aligned}
\int_{-1}^{7} \frac{(x+2)^{\frac{1}{2}}}{x+5}\,\mathrm{d}x &= \int_{1}^{3} \frac{(u^2)^{\frac{1}{2}}}{(u^2-2)+5}\,2u\,\mathrm{d}u \\
&= \int_{1}^{3} \frac{2u^2}{u^2+3}\,\mathrm{d}u \\
&= \int_{1}^{3} \frac{2(u^2+3)-6}{u^2+3}\,\mathrm{d}u \\
&= \int_{1}^{3} \left(2 - \frac{6}{u^2+3}\right)\mathrm{d}u \\
&= \left[2u - 2\sqrt{3}\arctan \tfrac{\sqrt{3}}{3}u\right]_{u=1}^{3} \\
&= \left(6 - 2\sqrt{3} \times \tfrac{\pi}{3}\right) - \left(2 - 2\sqrt{3} \times \tfrac{\pi}{6}\right) \\
&= \underline{\underline{4 - \tfrac{1}{3}\pi\sqrt{3}}}.
\end{aligned}$$

54. The curve C has equation

$$y = \ln\left(\frac{\mathrm{e}^x+1}{\mathrm{e}^x-1}\right),\ \ln 2 \leqslant x \leqslant \ln 3.$$

(a) Show that (4)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2\mathrm{e}^x}{\mathrm{e}^{2x}-1}.$$

Solution

$$y = \ln\left(\frac{e^x+1}{e^x-1}\right) \Rightarrow y = \ln(e^x+1) - \ln(e^x-1)$$
$$\Rightarrow \frac{dy}{dx} = \frac{e^x}{e^x+1} - \frac{e^x}{e^x-1}$$
$$\Rightarrow \frac{dy}{dx} = \frac{e^x(e^x-1) - e^x(e^x+1)}{(e^x+1)(e^x-1)}$$
$$\Rightarrow \frac{dy}{dx} = -\frac{2e^x}{e^{2x}-1}.$$

(b) Find the length of the curve C, giving your answer in the form $\ln a$, where a is a rational number. (6)

Solution

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\left(-\frac{2e^x}{e^{2x}-1}\right)^2}$$
$$= \sqrt{1+\frac{4e^{2x}}{(e^{2x}-1)^2}}$$
$$= \sqrt{\frac{(e^{2x}-1)^2+4e^{2x}}{(e^{2x}-1)^2}}$$
$$= \sqrt{\frac{(e^{4x}-2e^{2x}+1)+4e^{2x}}{(e^{2x}-1)^2}}$$
$$= \sqrt{\frac{e^{4x}+2e^{2x}+1}{(e^{2x}-1)^2}}$$
$$= \sqrt{\frac{(e^{2x}+1)^2}{(e^{2x}-1)^2}}$$
$$= \frac{e^{2x}+1}{e^{2x}-1}$$

and

$$\begin{aligned}
\text{length} &= \int_{\ln 2}^{\ln 3} \frac{e^{2x}+1}{e^{2x}-1}\,dx \\
&= \int_{\ln 2}^{\ln 3} \coth x\,dx \\
&= [\ln \sinh x]_{x=\ln 2}^{\ln 3} \\
&= \ln \sinh(\ln 3) - \ln \sinh(\ln 2) \\
&= \ln \tfrac{4}{3} - \ln \tfrac{3}{4} \\
&= \underline{\underline{\ln \tfrac{16}{9}}}.
\end{aligned}$$